

## RIEMANN-FINSLER AND LAGRANGE GERBES AND THE ATIYAH-SINGER THEOREMS

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### Abstract

In this paper, nonholonomic gerbes will be naturally derived for manifolds and vector bundle spaces provided with nonintegrable distributions (in brief, nonholonomic spaces). An important example of such gerbes is related to distributions defining nonlinear connection (N-connection) structures. They geometrically unify and develop the concepts of Riemann-Cartan manifolds and Lagrange-Finsler spaces. The obstruction to the existence of a spin structure on nonholonomic spaces is just the second Stiefel-Whitney class, defined by the cocycle associated to a  $\mathbb{Z}/2$  gerbe, which is called the nonholonomic spin gerbe. The nonholonomic gerbes are canonically endowed with N-connection, Sasaki type metric, canonical linear connection and (for odd dimension spaces) almost complex structures. The study of nonholonomic spin structures and gerbes have both geometric and physical applications. Our aim is to prove the Atiyah-Singer theorems for such nonholonomic spaces.

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## 1 Introduction

Connections and curving on gerbes (abelian and, more recently, nonabelian ones) play an important role in modern differential geometry and mathematical physics. Gerbes enabled with connection structure were introduced as a natural higher-order generalization of abelian bundles with connection provided a new possible framework to generalized gauge theories. They appeared in algebraic geometry [1, 2], and were subsequently developed by Brylinski [3], see a review of results in [4]. Bundles and gerbes and their higher generalizations ( $n$ -gerbes) can be understood both in two equivalent terms of local geometry

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(local functions and forms) and of non-local geometry (holonomies and parallel transports) [5, 6, 7].

The first applications of gerbes formalism were considered for higher Yang–Mills fields and gravity [8] and for a special case of a topological quantum field theory [9, 10]. The approaches were renewed following Hitchin [11] with further applications in physics, for instance, in investigating anomalies [12], new geometrical structures in string theory [13] and Chern–Simons theory [14].

A motivation for noncommutative gerbes [15], related to deformation quantization [16, 17], follows from the noncommutative description of D–branes in the presence of topologically non-trivial background fields. In a more general context, the geometry of commutative and noncommutative gerbes may be connected to the nonholonomic frame method in (non) commutative gauge realizations and generalizations of the Einstein gravity [18], nonholonomic deformations with noncommutative and/or algebroid symmetry [19] and to the geometry of Lagrange–Fedosov nonholonomic manifolds [20]. Here, we note that a manifold is nonholonomic (equivalently, anholonomic) if it is provided with a nonintegrable global distribution. In our works, we restrict the constructions to a subclass of such nonholonomic manifolds, or bundle spaces, when their nonholonomic distribution defines a nonlinear connection (in brief, N–connection) structure. We use the term of N–anholonomic manifold for such spaces. The geometry of N–connections came from the Finsler and Lagrange geometry (see, for instance, details in Ref. [21], and Refs. [22, 23] related to the Ehresmann connection and geometrization of classical mechanics and field theory). Nevertheless, the N–connection structures have to be introduced in general relativity, string theory and on Riemann–Cartan and/or noncommutative spaces and various types of non–Riemannian spaces if generic off–diagonal metrics, nonholonomic frames and generalized connections are introduced into consideration, see discussion and references in [24].

The concept of N–anholonomic spaces unifies a large class of nonholonomic manifolds and bundle spaces, nonholonomic Einstein, non–Riemannian and Lagrange–Finsler geometries which are present in modern gravity and string theory, geometric mechanics and classical field theory and geometric quantization formalism. N–anholonomic spaces are naturally provided with certain canonical N–connection and linear connection structures, Sasaki type metric, almost complex and/or almost symplectic structures induced correspondingly by the Lagrange, or Finsler, fundamental functions, and for gravitational models by the generic off–diagonal metric terms. The N–connection curvatures and Riemannian curvatures are very useful to study the topology of such manifolds.

The study of bundles of spinors on N–anholonomic spaces provides a number of geometric and physical results. For instance, it was possible to give a definition of nonholonomic spinor structures for Finsler spaces [25], to get a spinor interpretation of Lagrange and Hamilton spaces (and their higher order extensions), to define N–anholonomic Dirac operators in connection to noncommutative extensions of Finsler–Lagrange geometry and to construct a number of exact solutions with nonholonomic solitons and spinor interactions [26, 27, 24]. But every compact manifold (being an holonomic or anholonomic one) is not spin. The obstruction to the existence of spin structure, in general, of any nonholonomic spin structure of a nonholonomic space, is the second Stiefel–Whitney class. This class is

also the classifying cocycle associated to a  $\mathbb{Z}/2$  gerbe, which with respect to nonholonomic manifolds is a nonholonomic gerbe. This way one can be constructed a number of new examples of gerbes (Finsler and/or Lagrange ones, parametrizing some higher symmetries of generic off–diagonal solutions in Einstein and string/ brane gravity, with additional noncommutative and/or algebroid symmetries).

The aim of this paper is to study the main geometric properties of nonholonomic gerbes. We shall generalize the Lichnerowicz theorem and prove Atiyah–Singer type theorems for nonholonomic gerbes. For trivial holonomic manifolds, our results will transform into certain similar ones from Refs. [28, 29] but not completely if there are considered ‘non-perturbative’ and nonlinear configurations as exact solutions in gravity models, see details in [30].

We note that the problem of formulating and proof of Atiyah–Singer type theorems for nonholonomic manifolds is not a trivial one. For instance, in Ref. [31], it is advocated the point that it is not possible to define the concept of curvature for general nonholonomic manifolds. Without curvatures, one can not be formulated any types of Atiyah–Singer theorems. In Refs. [31, 32] one proposed such definitions for supermanifolds when supersymmetric structure is treated as nonholonomic distribution. There is a long term history on defining torsions and curvatures for various classes of nonholonomic manifolds (see, for instance, Refs. [35, 36]). More recently, one concluded that such definitions can be given by using the concept of N–connection structure at least for Lagrange–Finsler and Hamilton–Cartan spaces [33, 34]. We note that the problem of definition of curvatures was discussed and solved (also by using the N–connection formalism) in modern approaches to the geometry of noncommutative Riemann–Finsler or Einstein spaces and generalizations [24, 19], Fedosov N–anholonomic manifolds [20], as well in the works on nonholonomic Clifford structures and spinors [25, 26, 27] and generalized Finsler superspaces [37]. The Lichnerowicz type formula and Atiyah–Singer theorems to be proved here for N–anholonomic spaces have strong relations to the mentioned classes of manifolds and supermanifolds.

The article is organized as follows. In section 2 we recall the main results on nonholonomic manifolds provided with N–connection structure, consider some examples of such N–anholonomic spaces (generalized Lagrange/Finsler spaces and Riemann–Cartan manifolds provided with N–connections) and define the concept of nonholonomic gerbe. Section 3 is a study of nonholonomic Clifford gerbes: we consider lifts and nonholonomic vector gerbes and study pre–Hilbertian and scalar structures and define distinguished (by the N–connection structure) linear connections on N–anholonomic gerbes and construct the characteristic classes. Section 4 is devoted to operators and symbols on nonholonomic gerbes. In section 5 we present a  $K$ –theory framework for N–anholonomic manifolds and gerbes. We define elliptic operators and index formulas adapted to the N–connection structure. There are proven the Atiyah–Singer theorems for the canonical d–connection and N–connection structures. The results are applied for a topological study of N–anholonomic spinors and related Dirac operators. Appendix A outlines the basic results on distinguished connections, torsions and curvatures for N–anholonomic manifolds. Appendix B is an introduction into the geometry of nonholonomic spinor structures and N–adapted spin connections.

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## 2 Nonholonomic Manifolds and Gerbes

The aim of this section is to outline some results from the geometry of nonholonomic manifolds provided with N-connection structure and to elaborate the notion of nonholonomic gerbes.

### 2.1 The geometry of N-anholonomic spaces

We consider  $(n + m)$ -dimensional manifold of necessary smoothly class  $\mathbf{V}$  with locally fibred structure. A particular case is that of a vector bundle, when we shall write  $\mathbf{V} = \mathbf{E}$  (where  $\mathbf{E}$  is the total space of a vector bundle  $\pi : \mathbf{E} \rightarrow M$  with the base space  $M$ ). We denote by  $\pi^\top : T\mathbf{V} \rightarrow TM$  the differential of a map  $\pi : V^{n+m} \rightarrow V^n$  defined by fiber preserving morphisms of the tangent bundles  $T\mathbf{V}$  and  $TM$ . The kernel of  $\pi^\top$  is just the vertical subspace  $v\mathbf{V}$  with a related inclusion mapping  $i : v\mathbf{V} \rightarrow T\mathbf{V}$ .

**Definition 2.1.** *A nonlinear connection (N-connection)  $\mathbf{N}$  on a manifold  $\mathbf{V}$  is defined by the splitting on the left of an exact sequence*

$$0 \rightarrow v\mathbf{V} \xrightarrow{i} T\mathbf{V} \rightarrow T\mathbf{V}/v\mathbf{V} \rightarrow 0,$$

*i. e. by a morphism of submanifolds  $\mathbf{N} : T\mathbf{V} \rightarrow v\mathbf{V}$  such that  $\mathbf{N} \circ i$  is the unity in  $v\mathbf{V}$ .*

In an equivalent form, we can say that a N-connection is defined by a splitting to subspaces with a Whitney sum of conventional horizontal (h) subspace,  $(h\mathbf{V})$ , and vertical (v) subspace,  $(v\mathbf{V})$ ,

$$T\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V} \quad (1)$$

where  $h\mathbf{V}$  is isomorphic to  $M$ . In general, a distribution (1) is nonintegrable, i.e. nonholonomic (equivalently, anholonomic). In this case, we deal with nonholonomic manifolds/spaces.

**Definition 2.2.** *A manifold  $\mathbf{V}$  is called N-anholonomic if on the tangent space  $T\mathbf{V}$  it is defined a local (nonintegrable) distribution (1), i.e.  $\mathbf{V}$  is N-anholonomic if it is enabled with a N-connection structure.*

Locally, a N-connection is defined by its coefficients  $N_i^a(u)$ ,

$$\mathbf{N} = N_i^a(u) dx^i \otimes \partial_a$$

where the local coordinates (in general, abstract ones both for holonomic and nonholonomic variables) are split in the form  $u = (x, y)$ , or  $u^\alpha = (x^i, y^a)$ , where  $i, j, k, \dots = 1, 2, \dots, n$  and  $a, b, c, \dots = n + 1, n + 2, \dots, n + m$  when  $\partial_i = \partial/\partial x^i$  and  $\partial_a = \partial/\partial y^a$ . The well known class of linear connections consists on a particular subclass with the coefficients being linear on  $y^a$ , i.e.,  $N_i^a(u) = \Gamma_{bj}^a(x)y^b$ .

A N–connection is characterized by its N–connection curvature (the Nijenhuis tensor)

$$\mathbf{\Omega} = \frac{1}{2}\Omega_{ij}^a dx^i \wedge dx^j \otimes \partial_a,$$

with the N–connection curvature coefficients

$$\Omega_{ij}^a = \delta_{[j} N_{i]}^a = \delta_j N_i^a - \delta_i N_j^a = \partial_j N_i^a - \partial_i N_j^a + N_i^b \partial_b N_j^a - N_j^b \partial_b N_i^a. \quad (2)$$

Any N–connection  $\mathbf{N} = N_i^a(u)$  induces a N–adapted frame (vielbein) structure

$$\mathbf{e}_\nu = (e_i = \partial_i - N_i^a(u)\partial_a, e_a = \partial_a), \quad (3)$$

and the dual frame (coframe) structure

$$\mathbf{e}^\mu = (e^i = dx^i, e^a = dy^a + N_i^a(u)dx^i). \quad (4)$$

The vielbeins (4) satisfy the nonholonomy (equivalently, anholonomy) relations

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = W_{\alpha\beta}^\gamma \mathbf{e}_\gamma \quad (5)$$

with (antisymmetric) nontrivial anholonomy coefficients  $W_{ia}^b = \partial_a N_i^b$  and  $W_{ji}^a = \Omega_{ij}^a$ .<sup>2</sup>

The geometric constructions can be adapted to the N–connection structure:

**Definition 2.3.** *A distinguished connection (d–connection)  $\mathbf{D}$  on a N–anholonomic manifold  $\mathbf{V}$  is a linear connection conserving under parallelism the Whitney sum (1).*

In this work we use boldfaced symbols for the spaces and geometric objects provided/adapted to a N–connection structure. For instance, a vector field  $\mathbf{X} \in T\mathbf{V}$  is expressed  $\mathbf{X} = (X, \text{ } ^-X)$ , or  $\mathbf{X} = X^\alpha \mathbf{e}_\alpha = X^i e_i + X^a e_a$ , where  $X = X^i e_i$  and  $\text{ } ^-X = X^a e_a$  state, respectively, the irreducible (adapted to the N–connection structure) horizontal (h) and vertical (v) components of the vector (which following Refs. [21] is called a distinguished vectors, in brief, d–vector). In a similar fashion, the geometric objects on  $\mathbf{V}$  like tensors, spinors, connections, ... are called respectively d–tensors, d–spinors, d–connections if they are adapted to the N–connection splitting.

One can introduce the d–connection 1–form

$$\mathbf{\Gamma}_\beta^\alpha = \mathbf{\Gamma}_{\beta\gamma}^\alpha \mathbf{e}^\gamma,$$

<sup>2</sup>One preserves a relation to our previous denotations [25, 26] if we consider that  $\mathbf{e}_\nu = (e_i, e_a)$  and  $\mathbf{e}^\mu = (e^i, e^a)$  are, respectively, the former  $\delta_\nu = \delta/\partial u^\nu = (\delta_i, \partial_a)$  and  $\delta^\mu = \delta u^\mu = (d^i, \delta^a)$  when emphasize that operators (3) and (4) define, correspondingly, the “N–elongated” partial derivatives and differentials which are convenient for calculations on N–anholonomic manifolds.

when the N-adapted components of d-connection  $\mathbf{D}_\alpha = (\mathbf{e}_\alpha \rfloor \mathbf{D})$  are computed following formulas

$$\Gamma_{\alpha\beta}^\gamma(u) = (\mathbf{D}_\alpha \mathbf{e}_\beta) \rfloor \mathbf{e}^\gamma, \quad (6)$$

where " $\rfloor$ " denotes the interior product. This allows us to define in standard form the torsion

$$\mathcal{T}^\alpha \doteq \mathbf{D}\mathbf{e}^\alpha = d\mathbf{e}^\alpha + \Gamma_{\beta\gamma}^\alpha \mathbf{e}^\beta \wedge \mathbf{e}^\gamma \quad (7)$$

and curvature

$$\mathcal{R}^\alpha_\beta \doteq \mathbf{D}\Gamma^\alpha_\beta = d\Gamma^\alpha_\beta - \Gamma^\alpha_\beta \wedge \Gamma^\alpha_\gamma. \quad (8)$$

There are certain preferred d-connection structures on N-anholonomic manifolds (see local formulas in Appendix and Refs. [21, 19, 20], for details on computation the components of torsion and curvatures for various classes of d-connections).

## 2.2 Examples of N-anholonomic spaces:

We show how the N-connection geometries can be naturally derived from Lagrange-Finsler geometry and in gravity theories.

### 2.2.1 Lagrange-Finsler geometry

Such geometries are usually modelled on tangent bundles [21] but it is possible to define such structures on general N-anholonomic manifolds, in particular in (pseudo) Riemannian and Riemann-Cartan geometry if nonholonomic frames are introduced into consideration [24, 18]. In the first approach the N-anholonomic manifold  $\mathbf{V}$  is just a tangent bundle  $(TM, \pi, M)$ , where  $M$  is a  $n$ -dimensional base manifold,  $\pi$  is a surjective projection and  $TM$  is the total space. One denotes by  $\widetilde{TM} = TM \setminus \{0\}$  where  $\{0\}$  means the null section of map  $\pi$ .

A differentiable Lagrangian  $L(x, y)$ , i. e. a fundamental Lagrange function, is defined by a map  $L : (x, y) \in TM \rightarrow L(x, y) \in \mathbb{R}$  of class  $\mathcal{C}^\infty$  on  $\widetilde{TM}$  and continuous on the null section  $0 : M \rightarrow TM$  of  $\pi$ . For simplicity, we consider any regular Lagrangian with nondegenerated Hessian

$${}^L g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L(x, y)}{\partial y^i \partial y^j} \quad (9)$$

when  $\text{rank} |g_{ij}| = n$  on  $\widetilde{TM}$  and the left up "L" is an abstract label pointing that the values are defined by the Lagrangian  $L$ .

**Definition 2.4.** A Lagrange space is a pair  $L^n = [M, L(x, y)]$  with  ${}^L g_{ij}(x, y)$  being of constant signature over  $\widetilde{TM}$ .

The notion of Lagrange space was introduced by J. Kern [38] and elaborated in details in Ref. [21] as a natural extension of Finsler geometry.

By straightforward calculations, one can be proved the fundamental results:

1. The Euler–Lagrange equations

$$\frac{d}{d\tau} \left( \frac{\partial L}{\partial y^i} \right) - \frac{\partial L}{\partial x^i} = 0$$

where  $y^i = \frac{dx^i}{d\tau}$  for  $x^i(\tau)$  depending on parameter  $\tau$ , are equivalent to the “nonlinear” geodesic equations

$$\frac{d^2 x^i}{d\tau^2} + 2G^i(x^k, \frac{dx^j}{d\tau}) = 0$$

defining paths of the canonical semispray

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$$

where

$$2G^i(x, y) = \frac{1}{2} {}^L g^{ij} \left( \frac{\partial^2 L}{\partial y^i \partial x^k} y^k - \frac{\partial L}{\partial x^i} \right)$$

with  ${}^L g^{ij}$  being inverse to (9).

2. There exists on  $\widetilde{TM}$  a canonical N–connection

$${}^L N_j^i = \frac{\partial G^i(x, y)}{\partial y^j}, \quad (10)$$

defined by the fundamental Lagrange function  $L(x, y)$ , prescribing nonholonomic frame structures of type (3) and (4),  ${}^L \mathbf{e}_\nu = (e_i, {}^-e_k)$  and  ${}^L \mathbf{e}^\mu = (e^i, {}^-e^k)$ .<sup>3</sup>

3. The canonical N–connection (10), defining  ${}^-e_i$ , induces naturally an almost complex structure  $\mathbf{F} : \chi(\widetilde{TM}) \rightarrow \chi(\widetilde{TM})$ , where  $\chi(\widetilde{TM})$  denotes the module of vector fields on  $\widetilde{TM}$ ,

$$\mathbf{F}(e_i) = {}^-e_i \text{ and } \mathbf{F}({}^-e_i) = -e_i,$$

when

$$\mathbf{F} = {}^-e_i \otimes e^i - e_i \otimes {}^-e^i \quad (11)$$

satisfies the condition  $\mathbf{F}] \mathbf{F} = -\mathbf{I}$ , i. e.  $F^\alpha_\beta F^\beta_\gamma = -\delta^\alpha_\gamma$ , where  $\delta^\alpha_\gamma$  is the Kronecker symbol and “[ ]” denotes the interior product.

4. On  $\widetilde{TM}$ , there is a canonical metric structure

$${}^L \mathbf{g} = {}^L g_{ij}(x, y) e^i \otimes e^j + {}^L g_{ij}(x, y) {}^-e^i \otimes {}^-e^j \quad (12)$$

constructed as a Sasaki type lift from  $M$ .

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<sup>3</sup>On the tangent bundle the indices related to the base space run the same values as those related to fibers: we can use the same symbols but have to distinguish like  ${}^-e_k$  certain irreducible v–components with respect to, (or for) N–adapted bases and co–bases.

5. There is also a canonical d-connection structure  $\widehat{\Gamma}_{\alpha\beta}^\gamma$  defined only by the components of  ${}^L N_j^i$  and  ${}^L g_{ij}$ , i.e. by the coefficients of metric (12) which in its turn is induced by a regular Lagrangian. The d-connection  $\widehat{\Gamma}_{\alpha\beta}^\gamma$  is metric compatible and with vanishing  $h$ - and  $v$ -torsions. Such a d-connection contains also nontrivial torsion components induced by the nonholonomic frame structure, see Proposition 5.2 and formulas (A.3) in Appendix. The canonical d-connection is the "simplest" N-adapted linear connection related by the "non N-adapted" Levi-Civita connection by formulas (A.2).

We can conclude that any regular Lagrange mechanics can be geometrized as an almost Kähler space with N-connection distribution, see [21, 20]. For the Lagrange-Kähler (nonholonomic) spaces, the fundamental geometric structures (semispray, N-connection, almost complex structure and canonical metric on  $\widetilde{TM}$ ) are defined by the fundamental Lagrange function  $L(x, y)$ .

For applications in optics of nonhomogeneous media and gravity (see, for instance, Refs. [21]) one considers metrics of type  $g_{ij} \sim e^{\lambda(x,y)} {}^L g_{ij}(x, y)$  which can not be derived from a mechanical Lagrangian but from an effective "energy" function. In the so-called generalized Lagrange geometry one considers Sasaki type metrics (12) with any general coefficients both for the metric and N-connection.

**Remark 2.1.** *A Finsler space is defined by a fundamental Finsler function  $F(x, y)$ , being homogeneous of type  $F(x, \lambda y) = \lambda F(x, y)$ , for nonzero  $\lambda \in \mathbb{R}$ , may be considered as a particular case of Lagrange geometry when  $L = F^2$ .*

Now we show how N-anholonomic configurations can be defined in gravity theories. In this case, it is convenient to work on a general manifold  $\mathbf{V}$ ,  $\dim \mathbf{V} = n + m$  with global splitting, instead of the tangent bundle  $\widetilde{TM}$ .

### 2.2.2 N-connections and gravity

Let us consider a metric structure on  $\mathbf{V}$  with the coefficients defined with respect to a local coordinate basis  $du^\alpha = (dx^i, dy^a)$ ,

$$\mathbf{g} = \underline{g}_{\alpha\beta}(u) du^\alpha \otimes du^\beta$$

with

$$\underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b h_{ab} & N_j^c h_{ac} \\ N_i^e h_{be} & h_{ab} \end{bmatrix}. \quad (13)$$

In general, such a metric (13) is generic off-diagonal, i.e. it can not be diagonalized by any coordinate transforms. We note that  $N_i^a(u)$  in our approach are any general functions. They may be identified with some gauge potentials in Kaluza-Klein models if the corresponding symmetries and compactifications of coordinates  $y^a$  are considered, see review [39]. Performing a frame transform

$$\mathbf{e}_\alpha = \mathbf{e}_\alpha^\alpha \partial_\alpha \quad \text{and} \quad \mathbf{e}^\beta = \mathbf{e}^\beta_{\underline{\beta}} du^{\underline{\beta}}.$$



with coefficients

$$\mathbf{e}_\alpha^{\underline{a}}(u) = \begin{bmatrix} e_i^{\underline{a}}(u) & N_i^b(u)e_b^{\underline{a}}(u) \\ 0 & e_a^{\underline{a}}(u) \end{bmatrix}, \quad (14)$$

$$\mathbf{e}_\beta^{\underline{a}}(u) = \begin{bmatrix} e_i^{\underline{a}}(u) & -N_k^b(u)e_b^{\underline{a}}(u) \\ 0 & e_a^{\underline{a}}(u) \end{bmatrix}, \quad (15)$$

we write equivalently the metric in the form

$$\mathbf{g} = \mathbf{g}_{\alpha\beta}(u) \mathbf{e}^\alpha \otimes \mathbf{e}^\beta = g_{ij}(u) e^i \otimes e^j + h_{ab}(u) e^a \otimes e^b, \quad (16)$$

where  $g_{ij} \doteq \mathbf{g}(e_i, e_j)$  and  $h_{ab} \doteq \mathbf{g}(e_a, e_b)$  and the vielbeins  $\mathbf{e}_\alpha$  and  $\mathbf{e}^\alpha$  are respectively of type (3) and (4). We can consider a special class of manifolds provided with a global splitting into conventional “horizontal” and “vertical” subspaces (1) induced by the “off-diagonal” terms  $N_i^b(u)$  and prescribed type of nonholonomic frame structure.

If the manifold  $\mathbf{V}$  is (pseudo) Riemannian, there is a unique linear connection (the Levi–Civita connection)  $\nabla$ , which is metric,  $\nabla \mathbf{g} = \mathbf{0}$ , and torsionless,  $\nabla T = 0$ . Nevertheless, the connection  $\nabla$  is not adapted to the nonintegrable distribution induced by  $N_i^b(u)$ . In this case, it is more convenient to work with more general classes of linear connections (for instance, with the canonical d–connection (A.3)) which are N–adapted but contain nontrivial torsion coefficients because of nontrivial nonholonomy coefficients  $W_{\alpha\beta}^\gamma$  (5).

For a splitting of a (pseudo) Riemannian–Cartan space of dimension  $(n+m)$  (we considered also certain (pseudo) Riemannian configurations), the Lagrange and Finsler type geometries were modelled by N–anholonomic structures as exact solutions of gravitational field equations [24, 18].

### 2.3 The notion of nonholonomic gerbes

Let denote by  $\mathbf{S}$  a sheaf of categories on a N–anholonomic manifold  $\mathbf{V}$ , defined by a map of  $\mathbf{U} \rightarrow \mathbf{S}(\mathbf{U})$ , where  $\mathbf{U}$  is an open subset of  $\mathbf{V}$ , with  $\mathbf{S}(\mathbf{U})$ .

**Definition 2.5.** *A sheaf of categories  $\mathbf{S}$  is called a nonholonomic gerbe if there are satisfied the conditions:*

1. *There exists a map  $r_{\mathbf{U}_1\mathbf{U}_2} : \mathbf{S}(\mathbf{U}_1) \rightarrow \mathbf{S}(\mathbf{U}_2)$  such that for superpositions of two such maps  $r_{\mathbf{U}_1\mathbf{U}_2} \circ r_{\mathbf{U}_2\mathbf{U}_3} = r_{\mathbf{U}_1\mathbf{U}_3}$  for any inclusion  $\mathbf{U}_1 \rightarrow \mathbf{U}_2$ .*
2. *It is satisfied the gluing condition for objects, i.e. for a covering family  $\cup_{\hat{i}} \mathbf{U}_{\hat{i}}$  of  $\mathbf{U}$  and objects  $\mathbf{u}_{\hat{i}}$  of  $\mathbf{S}(\mathbf{U}_{\hat{i}})$  for each  $\hat{i}$ , when there are maps of type*

$$q_{\hat{i}\hat{j}} : r_{\mathbf{U}_{\hat{i}} \cap \mathbf{U}_{\hat{j}}, \mathbf{U}_{\hat{j}}}(\mathbf{u}_{\hat{j}}) \rightarrow r_{\mathbf{U}_{\hat{i}} \cap \mathbf{U}_{\hat{j}}, \mathbf{U}_{\hat{i}}}(\mathbf{u}_{\hat{i}})$$

*such that  $q_{\hat{i}\hat{j}} q_{\hat{j}\hat{k}} = q_{\hat{i}\hat{k}}$ , then there exists an object  $\mathbf{u} \in \mathbf{S}(\mathbf{U})$  such that  $r_{\mathbf{U}_{\hat{i}}, \mathbf{U}}(\mathbf{u}) \rightarrow \mathbf{u}_{\hat{i}}$ .*

3. It is satisfied the gluing condition for arrows, i.e. for any two objects  $\mathbf{P}, \mathbf{Q} \in \mathbf{S}(\mathbf{V})$  the map

$$\mathbf{U} \rightarrow \text{Hom}(r_{\mathbf{U}\mathbf{V}}(\mathbf{P}), r_{\mathbf{U}\mathbf{V}}(\mathbf{Q}))$$

is a sheaf.

This Definition is adapted to the  $N$ -connection structure (1) and define similar objects and maps for  $h$ - and  $v$ -subspaces of a  $N$ -anholonomic manifold  $\mathbf{V}$ . That why we use "boldfaced" symbols.

For certain applications is convenient to work with another sheaf  $\mathbf{A}$  called the  $N$ -bund of the nonholonomic gerbe  $\mathbf{S}$ . It is constructed to satisfy the conditions:

- There is a covering  $N$ -adapted family  $(\mathbf{U}_{\hat{i}})_{\hat{i} \in I}$  of  $\mathbf{V}$  such that the category of  $\mathbf{S}(\mathbf{U}_{\hat{i}})$  is not empty for each  $\hat{i}$ .
- For any  $\mathbf{u}_{(1)}, \mathbf{u}_{(2)} \in \mathbf{U} \subset \mathbf{V}$ , there is a covering family  $(\mathbf{U}_{\hat{i}})_{\hat{i} \in I}$  of  $\mathbf{U}$  such that  $r_{\mathbf{U}_{\hat{i}}\mathbf{U}}(\mathbf{u}_{(1)})$  and  $r_{\mathbf{U}_{\hat{i}}\mathbf{U}}(\mathbf{u}_{(2)})$  are isomorphic.
- The  $N$ -bund is introduced as a family of isomorphisms  $\mathbf{A}(\mathbf{V}) \doteq \text{Hom}(\mathbf{u}, \mathbf{u})$ , for each object  $\mathbf{u} \in \mathbf{S}(\mathbf{U})$ , defined by a sheaf  $\mathbf{A}$  in groups, for which every arrow of  $\mathbf{S}(\mathbf{U})$  is invertible and such isomorphisms commute with the restriction maps.

For given families  $(\mathbf{U}_{\hat{i}})_{\hat{i} \in I}$  of  $\mathbf{V}$  and objects  $\mathbf{u}_{\hat{i}}$  of  $\mathbf{S}(\mathbf{U}_{\hat{i}})$ , we denote by  $\mathbf{u}_{\hat{i}_1 \dots \hat{i}_k}^{\hat{i}}$  the element  $r_{(\mathbf{U}_{\hat{i}_1} \cap \dots \cap \mathbf{U}_{\hat{i}_k}, \mathbf{U}_{\hat{j}})}(\mathbf{u}_{\hat{i}})$  and by  $\mathbf{U}_{\hat{i}_1 \dots \hat{i}_k}^{\hat{i}}$  the elements of the intersection  $\mathbf{U}_{\hat{i}_1} \cap \dots \cap \mathbf{U}_{\hat{i}_k}$ . The  $N$ -connection structure distinguishes (d)  $\mathbf{V}$  into  $h$ - and  $v$ -components, i.e. defines a local fiber structure when the geometric objects transform into  $d$ -objects, for instance,  $d$ -vectors,  $d$ -tensors,..... There are two possibilities for further constructions: a) to consider the category of vector bundles over an open set  $\mathbf{U}$  of  $N$ -anholonomic manifold  $\mathbf{V}$ , being the base space or b) to consider such  $N$ -anholonomic vector bundles modelled as  $\mathbf{V} = \mathbf{E}$  with a base  $M$ , where  $\dim M = n$  and  $\dim \mathbf{E} = n + m$ .

**Definition 2.6.**

- a) A  $N$ -anholonomic vector gerbe  $\mathbf{C}_{NQ}$  is defined by the category of vector bundles  $\mathbf{S}(\mathbf{U})$  over  $\mathbf{U} \subset \mathbf{V}$  with typical fiber the vector space  $Q$ .
- b) A nonholonomic gerbe  $\mathbf{C}_{Nd}$  is a  $d$ -vectorial gerbe  $\mathbf{S}(U)$  if and only if for the each open  $U \subset M$  on the  $h$ -subspace  $M$  of  $\mathbf{V}$  the set  $\mathbf{S}(U)$  is a category of  $N$ -anholonomic manifolds with  $h$ -base  $M$ .

In both cases of nonholonomic gerbes a) and b) the maps between  $d$ -objects are isomorphisms of  $N$ -anholonomic bundles/ manifolds adapted to the  $N$ -connection structures.

Let us consider more precisely the case a) (the constructions for the case b) being similar by substituting  $\mathbf{U} \rightarrow U$  and  $\mathbf{V} \rightarrow M$ ). There is a covering family  $(\mathbf{U}_{\hat{\alpha}})_{\hat{\alpha} \in I}$  of  $\mathbf{V}$  and a commutative subgroup  $H$  of the set of linear transforms  $Gl(Q)$ , such that there exit maps

$$\begin{aligned} q'_{\hat{\alpha}\hat{\beta}} &: \mathbf{U}_{\hat{\alpha}} \cap \mathbf{U}_{\hat{\beta}} \times Q \rightarrow \mathbf{U}_{\hat{\alpha}} \cap \mathbf{U}_{\hat{\beta}} \times Q, \\ q'_{\hat{\alpha}\hat{\beta}} &: (\mathbf{u}_{(1)}, \mathbf{u}_{(2)}) \rightarrow (\mathbf{u}_{(1)}, q'_{\hat{\alpha}\hat{\beta}}(\mathbf{u}_{(1)})\mathbf{u}_{(2)}) \end{aligned}$$

when  $c_{\widehat{\alpha}\widehat{\beta}\widehat{\gamma}} = q'_{\widehat{\alpha}\widehat{\beta}}q'_{\widehat{\beta}\widehat{\gamma}}q'_{\widehat{\gamma}\widehat{\alpha}}$  is an  $H$  2–Cech cocycle. Locally, such maps are parametrized by non–explicit functions because of nonholonomic character of manifolds and subspaces under consideration.

### 3 Nonholonomic Clifford Gerbes

Let  $\mathbf{V}$  be an  $N$ –anholonomic manifold of dimension  $\dim \mathbf{V} = n+m$ . We denote by  $O(\mathbf{V})$ , see applications and references in [18], the reduction of linear  $N$ –adapted frames which defines the  $d$ –metric structure (16) of the  $\mathbf{V}$ . The typical fiber of  $O(\mathbf{V})$  is  $O(n+m)$  which with respect to  $N$ –adapted frames splits into  $O(n) \oplus O(m)$ . There is the exact sequence

$$1 \rightarrow \mathbb{Z}/2 \rightarrow Spin(n+m) \rightarrow O(n+m) \rightarrow 1$$

with two  $N$ –distinguished, respectively,  $h$ – and  $v$ –components

$$\begin{aligned} 1 &\rightarrow \mathbb{Z}/2 \rightarrow Spin(n) \rightarrow O(n) \rightarrow 1, \\ 1 &\rightarrow \mathbb{Z}/2 \rightarrow Spin(m) \rightarrow O(m) \rightarrow 1 \end{aligned}$$

where  $Spin(n+m)$  is the universal covering of  $O(n+m)$  splitting into  $Spin(n) \oplus Spin(m)$  distinguished as the universal covering of  $O(n) \oplus O(m)$ . To such sequences, one can be associated a nonholonomic gerbe with band  $\mathbb{Z}/2$  and such that for each open set  $\mathbf{U} \subset \mathbf{V}$  it defined  $Spin_N(\mathbf{U})$  as the category of  $Spin$   $N$ –anholonomic bundles over  $\mathbf{U}$ , such spaces were studied in details in Refs. [24, 25, 26, 27], see also Appendix 6. The classified cocycle of this  $N$ –anholonomic gerbe is defined by the second Stiefel–Whitney class.

In a more general context, the  $N$ –anholonomic gerbe and  $Spin_N(\mathbf{U})$  are associated to a vectorial  $N$ –anholonomic gerbe called the Clifford  $N$ –gerbe (in a similar form we can consider associated Clifford  $d$ –gerbe, for the case b) of Definition 2.6). This way one defines the category  $Cl_N(\mathbf{V})$  which for any open set  $\mathbf{U} \subset \mathbf{V}$ , one have the category of objects being Clifford bundles provided with  $N$ –connection structure associated to the objects of  $Spin_N(\mathbf{U})$ . We can consider such gerbes in terms of transition functions. Let  $q'_{\widehat{\alpha}\widehat{\beta}}$  be the transitions functions of the bundle  $O(\mathbf{V})$ . The  $N$ –connection distinguish them to couples of  $h$ – and  $v$ –transition functions, i.e.  $q'_{\widehat{\alpha}\widehat{\beta}} = (q'_{\widehat{ij}}, q'_{\widehat{ab}})$ . For such  $d$ –functions one can be considered elements  $q_{\widehat{\alpha}\widehat{\beta}} = (q_{\widehat{ij}}, q_{\widehat{ab}})$  acting correspondingly in  $Spin(n+m) = (Spin(n), Spin(m))$ . Such elements act, by left multiplication, correspondingly on  $Cl(\mathbb{R}^{n+m})$  distinguished into  $(Cl(\mathbb{R}^n), Cl(\mathbb{R}^m))$ . We denote by  $s_{\widehat{\alpha}\widehat{\beta}}(\mathbf{u}) = (s_{\widehat{ij}}(\mathbf{u}), s_{\widehat{ab}}(\mathbf{u}))$  the resulting automorphisms on Clifford spaces. We conclude that the Clifford  $N$ –gerbe is defined by maps

$$s_{\widehat{\alpha}\widehat{\beta}} : \mathbf{U}_{\widehat{\alpha}} \cap \mathbf{U}_{\widehat{\beta}} \rightarrow Spin(n+m)$$

distinguished with respect to  $N$ –adapted frames by couples

$$s_{\widehat{ij}} : \mathbf{U}_{\widehat{i}} \cap \mathbf{U}_{\widehat{j}} \rightarrow Spin(n) \text{ and } s_{\widehat{ab}} : \mathbf{U}_{\widehat{a}} \cap \mathbf{U}_{\widehat{b}} \rightarrow Spin(m).$$

For trivial N–connections, such Clifford N–gerbes transform into the usual Clifford gerbes defined in Ref. [28].<sup>4</sup>

### 3.1 Gerbes and lifts associated to d–vector bundles

There are two classes of nonholonomic gerbes defined by lifting problems, respectively, associated to a vector bundle  $\mathbf{E}$  on a N–anholonomic manifold  $\mathbf{V}$  and/or associated just to  $\mathbf{V}$  considering that locally a such space posses a fibered structure distinguished by the N–connection, see corresponding cases a) and b) in Definition 2.6.

#### 3.1.1 Lifts and N–anholonomic vector gerbes

Let us denote by  $Q$  the typical fiber of a vector bundle  $\mathbf{E}$  on  $\mathbf{V}$  with associated principal bundle  $Gl(Q)$ . We suppose that this bundle has a reduction  $\mathbf{E}_K$  for a subgroup  $K \subset Gl(Q)$  and consider a central extension for a group  $G$  when

$$1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1. \quad (17)$$

Such an extension defines a N–anholonomic gerbe  $\mathbf{C}_H$  on  $\mathbf{V}$  when for each open set  $\mathbf{U} \subset \mathbf{V}$  the objects of  $\mathbf{C}_H(\mathbf{U})$  are  $G$ –principal bundles over  $\mathbf{U}$  when the quotient by  $H$  is the restriction of  $\mathbf{E}_K$  to  $\mathbf{U}$ .

We consider the projection  $\pi : G \rightarrow K$  and suppose that for (existing) a representation  $r : G \rightarrow Gl(Q')$  and surjection  $f : Q' \rightarrow Q$  one can be defined a commutative Diagram 1,

$$\begin{array}{ccc} Q' & \xrightarrow{r(s)} & Q' \\ f \downarrow & & \downarrow f \\ Q & \xrightarrow{\pi(s)} & Q \end{array}$$

Figure 1: Diagram 1

For such cases it is defined a N–anholonomic vectorial gerbe  $\mathbf{C}_{H,Q'}$  on  $\mathbf{V}$  when an object of  $\mathbf{C}(\mathbf{U})$  is parametrized  $\mathbf{e}_U \circ r$  where  $\mathbf{e}_U$  is an object of  $\mathbf{C}_H(\mathbf{U})$ . Such constructions are adapted to the N–connection structure on  $\mathbf{U}$ . If  $(\mathbf{U}_{\tilde{\alpha}})_{\tilde{\alpha} \in I}$  is a trivialization of  $\mathbf{E}$ , with transition functions  $q'_{\tilde{\alpha}\tilde{\beta}} = (q'_{ij}, q'_{ab})$ , we can define the maps  $q_{\tilde{\alpha}\tilde{\beta}} : \mathbf{U}_{\tilde{\alpha}} \cap \mathbf{U}_{\tilde{\beta}} \rightarrow G$  over  $q'_{\tilde{\alpha}\tilde{\beta}}$ . This states that  $\mathbf{C}_{H,Q'}$  is defined by  $r(q_{\tilde{\alpha}\tilde{\beta}})$ .

There is a natural scalar product defined on such N–anholonomic gerbes. Its existence follows from the construction of Clifford N–gerbe  $Cl(\mathbf{V})$  because the group  $Spin$  is compact and its action on  $Cl(\mathbb{R}^{n+m})$  preserves a scalar product which is distinguished by the N–connection structure [24, 25, 26, 27]. We can consider this scalar product on each fiber of

<sup>4</sup>We apply the ideas and results developed in that paper in order to investigate N–anholonomic manifolds and gerbes.

an object  $\mathbf{e}_U$  of  $Cl(\mathbf{U})$  and define a Riemannian d–metric

$$\langle, \rangle_{e_U} = \langle, \rangle_{he_U} + \langle, \rangle_{ve_U}$$

distinguished by the N–splitting into h- and v–components. The family of such Riemannian d–metrics defines the Riemannian d–metric on the N–anholonomic gerbe  $Cl(\mathbf{V})$ . There is a canonical such structure defined by the N–connection when  $(g_{ij}, h_{ab})$  in (16) are taken to be some Euclidean ones but  $N_i^a(\mathbf{u})$  are the coefficients for a nontrivial N–connection.

It should be emphasized that the band has to be contained in a compact group in order to preserve the Riemannian d–metric. In result, we can give the

**Definition 3.1.** *A Riemannian d–metric on a N–anholonomic vector gerbe  $\mathbf{C}_{NQ}$  is given by a distinguished scalar product  $\langle, \rangle_{e_U} = (\langle, \rangle_{he_U}, \langle, \rangle_{ve_U})$  on the vector bundle  $\mathbf{e}_U$ , defined for every object of  $\mathbf{C}_{NQ}$  and preserved by morphisms of such objects.*

We can define a global section of a N–anholonomic vector gerbe associated to a 1–Cech N–adapted chain  $q_{\hat{\alpha}\hat{\beta}} = (q_{\hat{i}\hat{j}}, q_{\hat{a}\hat{b}})$  by considering a covering space  $(\mathbf{U}_{\hat{\alpha}})_{\hat{\alpha} \in I}$  of  $\mathbf{V}$  when for each element  $\hat{\alpha}$  of  $I$ , an object  $\mathbf{e}_{\hat{\alpha}} \in \mathbf{C}(\mathbf{U}_{\hat{\alpha}})$ , a section  $z_{\hat{\alpha}}$  of  $\mathbf{e}_{\hat{\alpha}}$  and a family of morphisms  $q_{\hat{\alpha}\hat{\beta}} : \mathbf{e}_{\hat{\beta}}^{\hat{\alpha}} \rightarrow \mathbf{e}_{\hat{\alpha}}^{\hat{\beta}}$  one has  $z_{\hat{\alpha}} = q_{\hat{\alpha}\hat{\beta}}(z_{\hat{\beta}})$ . The family of global sections  $\mathbf{Z}(q_{\hat{\alpha}\hat{\beta}})$  associated to  $(q_{\hat{\alpha}\hat{\beta}})_{\hat{\alpha}, \hat{\beta} \in I}$  defining a d–vector space. If  $\mathbf{V}$  is compact and  $I$  is finite, one can prove that  $\mathbf{Z}(q_{\hat{\alpha}\hat{\beta}})$  is not empty. For such conditions, we can generalize for N–anholonomic vector gerbes the Proposition 5 from Ref. [28],

**Proposition 3.1.** *Let the N–anholonomic vector gerbe  $\mathbf{C}_{NQ}$  is a nonholonomic gerbe associated to the lifting problem defined by the extension (17) and the vector bundle  $E$  and for a representation  $r : G \rightarrow Gl(Q')$  the conditions of the Diagram 1 are satisfied. Then for each G–chain  $q_{\hat{\alpha}\hat{\beta}}$  and each element  $(z_{\hat{\alpha}})_{\hat{\alpha} \in I}$  of the d–vector space of global sections  $\mathbf{Z}(q_{\hat{\alpha}\hat{\beta}})$  it is satisfied the condition that there is a section  $\mathbf{z}$  of  $\mathbf{E}$  such that  $\mathbf{z}|_{U_{\hat{\alpha}}} = f \circ \mathbf{z}_{\hat{\alpha}}$ .*

*Proof.* It is similar to that for the usual vector bundles given in [28] but it should be considered for both h- and v–subspaces of  $\mathbf{V}$  and  $\mathbf{E}$ . In ”non–distinguished” form, we can consider a global section  $(z_{\hat{\alpha}})_{\hat{\alpha} \in I}$  associate to  $\mathbf{Z}(q_{\hat{\alpha}\hat{\beta}})$ . One has  $z_{\hat{\alpha}} = q_{\hat{\alpha}\hat{\beta}}(z_{\hat{\beta}})$  implying that  $f(z_{\hat{\alpha}}) = f(z_{\hat{\beta}})$  on  $\mathbf{U}_{\hat{\alpha}\hat{\beta}}$ . We conclude that the family  $[f(z_{\hat{\alpha}})]_{\hat{\alpha} \in I}$  of local sections  $\mathbf{z}$  of  $\mathbf{E}$ , such that  $\mathbf{z}|_{U_{\hat{\alpha}}} = f \circ \mathbf{z}_{\hat{\alpha}}$  being distinguished in N–adapted sections  $\mathbf{z}|_{U_{\hat{i}}} = f \circ \mathbf{z}_{\hat{i}}$  and  $\mathbf{z}|_{U_{\hat{a}}} = f \circ \mathbf{z}_{\hat{a}}$ . ■ □

For a chain  $z_{\hat{\alpha}\hat{\beta}} = z_{\hat{\alpha}} - q_{\hat{\alpha}\hat{\beta}}(z_{\hat{\beta}})$ , we can construct a 2–cocycle

$$z_{\hat{\beta}\hat{\gamma}} - z_{\hat{\alpha}\hat{\gamma}} + z_{\hat{\alpha}\hat{\beta}} = z_{\hat{\alpha}\hat{\beta}\hat{\gamma}}.$$

Nevertheless, even there are a chain  $q_{\hat{\alpha}\hat{\beta}}$  and a global section  $\mathbf{z} = (\mathbf{z}_{\hat{\alpha}})_{\hat{\alpha} \in I}$  such that  $z_{\hat{\alpha}} = q_{\hat{\alpha}\hat{\beta}}(z_{\hat{\beta}})$  and  $f(z_{\hat{\alpha}}) = \mathbf{z}|_{U_{\hat{\alpha}}}$  it may does not exist a global section for another N–adapted chain. One has to work with the d–vector space  $\mathbf{Z}$  of formal global sections of the

N-anholonomic vector gerbe  $\mathbf{C}_{NQ}$ . The d-vector space is defined by generators  $[\mathbf{z}]$  where  $\mathbf{z}$  is an element of a set of global sections

$$\mathbf{z} \left( q_{\widehat{\alpha\beta}} \right) = \left[ Z \left( q_{\widehat{ij}} \right), Z \left( q_{\widehat{ab}} \right) \right].$$

We can consider that any element of the space  $\mathbf{Z}$  is defined by a formal finite sum of global sections.

### 3.1.2 Lifts and d-vector gerbes

The constructions from the previous section were derived for vector bundles on N-anholonomic manifolds. But such a nonholonomic manifold in its turn has a local fibered structure resulting in definition of a nonholonomic gerbe  $\mathbf{C}_{Nd}$  as a d-vectorial gerbe. We denote by  $Q^m$  the typical fiber which can be associated to a N-anholonomic manifold  $\mathbf{V}$  of dimension  $n + m$ , from the map  $\pi : V^{n+m} \rightarrow V^n$ , see Definition 2.1. We can also associate a principal bundle  $Gl(Q^m)$  supposing that this bundle has a reduction  $\mathbf{V}_K$  for a subgroup  $K \subset Gl(Q^m)$  with a central extension of type (17). For such an extension, we define a d-vector gerbe  $\mathbf{C}_{dH}$  on  $h\mathbf{V}$  when for each open set  $U \subset h\mathbf{V}$  the objects of  $\mathbf{C}_{dH}(U)$  are  $G$ -principal bundles over  $U$  when the quotient by  $H$  is the restriction of  $\mathbf{V}_K$  to  $U$ .

For the projection  $\pi : G \rightarrow K$  and (supposed to exist) representation  $r : G \rightarrow Gl(Q^m)$  and surjection  $f : Q^m \rightarrow Q^m$  one can be defined a commutative Diagram 2,

$$\begin{array}{ccc} Q^m & \xrightarrow{r(s)} & Q^m \\ f \downarrow & & \downarrow f \\ Q^m & \xrightarrow{\pi(s)} & Q^m \end{array}$$

Figure 2: Diagram 2

By such a Diagram, it is defined a d-vectorial gerbe  $\mathbf{C}_{gH,Q'}$  on  $h\mathbf{V}$  when an object of  $\mathbf{C}(U)$  is parametrized  $e_U \propto r$  where  $e_U$  is an object of  $\mathbf{C}_{gH}(U)$ . The constructions are adapted to the N-connection structure on  $U$ . We note that the objects and regions defined with respect to h-subspaces are not boldfaced as those considered for N-anholonomic vector bundles. Stating  $(U_{\widehat{i}})_{\widehat{i} \in I}$  as a trivialization of  $h\mathbf{V}$ , with transition functions  $q'_{\widehat{ij}}$ , we can define the maps  $q'_{\widehat{ii}} : U_{\widehat{i}} \cap U_{\widehat{i}} \rightarrow G$  over  $q'_{\widehat{ij}}$ . This means that  $\mathbf{C}_{gH,Q'}$  is defined by  $r(q'_{\widehat{ij}})$ .

There is also a natural scalar product (a particular case of that for N-anholonomic gerbers) in our case defined by d-vector gerbes. We can consider such a scalar product just for the Clifford N-gerbe  $Cl(h\mathbf{V})$  following the fact that the group  $Spin$  is compact and its action on  $Cl(\mathbb{R}^n)$  preserves a scalar product. We conclude that this scalar product exists for any object  $e_U$  of  $Cl(U)$  and that a d-metric (16) states a splitting  $\langle, \rangle_{e_U} = \langle, \rangle_{he_U} + \langle, \rangle_{ve_U}$ . There are some alternatives: There is a family of Riemannian d-metrics on the d-vector gerbe  $Cl(h\mathbf{V})$  but this is not adapted to the N-connection structure. One

has to apply the concept of d–connection in order to define N–adapted objects. If the d–metric structure is not prescribed, we can introduce a scalar product structure defined by the N–connection when  $(g_{ij}, h_{ab})$  in (16) are taken to be some Euclidean ones but  $N_i^a(\mathbf{u})$  are the coefficients for a nontrivial N–connection.

**Definition 3.2.** *A d–metric (it is connected to a N–anholonomic Riemann–Cartan structure) on a d–vector gerbe  $\mathbf{C}_{Nd}$  is given by a distinguished scalar product  $\langle, \rangle_{e_U} = (\langle, \rangle_{he_U}, \langle, \rangle_{ve_U})$  on the d–vector bundle  $\mathbf{e}_U$ , defined for every object of  $\mathbf{C}_{Nd}$  and preserved by morphisms of such objects.*

Following this Definition, for the d–vector gerbes, one holds the Proposition 3.1 and related results.

### 3.2 Pre–Hilbertian and d–connection structures

For simplicity, hereafter we shall work only with N–anholonomic manifolds. We emphasize that the constructions can be extended to vector bundles  $\mathbf{E}$  on such a nonholonomic manifold  $\mathbf{V}$ . The proofs will be omitted if they are similar to those given for holonomic manifolds and vector spaces [28] but (in our case) adapted to the splitting defined by the N–connection structure. We shall point out the nonholonomic character of the constructions. Such computations may be performed directly by applying ”boldfaced” objects.

#### 3.2.1 Distinguished pre–Hilbertian and scalar structures

Let us consider the two elements  ${}_1z$  and  ${}_2z$  of the d–vector space  $\mathbf{Z}(q_{\hat{\alpha}\hat{\beta}}) = [Z(q_{\hat{i}\hat{j}}), Z(q_{\hat{a}\hat{b}})]$  (we use left low labels which are not indices running values). For a partition of unity  $(\mathbf{U}'_{\hat{\alpha}'}, f_{\hat{\alpha}'})_{\hat{\alpha}' \in I'}$  subordinated to  $(\mathbf{U}_{\hat{\alpha}})_{\hat{\alpha} \in I}$ .<sup>5</sup> Since the support of  $f_{\hat{\alpha}'}$  is a couple of compact subsets of  $\mathbf{U}'_{\hat{\alpha}'}$  distinguished by the N–connection structure, we can consider restrictions of  ${}_1z_{\hat{\alpha}'(\hat{\alpha}'})$  and  ${}_2z_{\hat{\alpha}'(\hat{\alpha}'})$  to  $\mathbf{U}'_{\hat{\alpha}'}$  denoted respectively  ${}_1z_{\hat{\alpha}'\hat{\alpha}'(\hat{\alpha}'})$  and  ${}_2z_{\hat{\alpha}'\hat{\alpha}'(\hat{\alpha}'})$ . We can calculate the value  $\int \langle {}_1z_{\hat{\alpha}'\hat{\alpha}'(\hat{\alpha}')}, {}_2z_{\hat{\alpha}'\hat{\alpha}'(\hat{\alpha}')} \rangle$  which is invariant for any partition. This proofs

**Proposition 3.2.** *The scalar product*

$$\begin{aligned} \langle {}_1z, {}_2z \rangle &= \sum_{\hat{\alpha}'} \int \langle f_{\hat{\alpha}'} {}_1z_{\hat{\alpha}'\hat{\alpha}'(\hat{\alpha}')}, f_{\hat{\alpha}'} {}_2z_{\hat{\alpha}'\hat{\alpha}'(\hat{\alpha}')} \rangle \\ &= \sum_{\hat{i}'} \int \langle f_{\hat{i}'} {}_1z_{\hat{i}\hat{i}(\hat{i}')}, f_{\hat{i}'} {}_2z_{\hat{i}\hat{i}(\hat{i}')} \rangle + \sum_{\hat{a}'} \int \langle f_{\hat{a}'} {}_1z_{\hat{a}\hat{a}(\hat{a}')}, f_{\hat{a}'} {}_2z_{\hat{a}\hat{a}(\hat{a}')} \rangle \end{aligned} \quad (18)$$

defines a pre–Hilbert d–structure of  $(\mathbf{Z}(q_{\hat{\alpha}\hat{\beta}}), \langle, \rangle)$ .

For two formal global sections of d–vector gerbe  $\mathbf{C}_{Nd}$ , we can write

$${}_1z = [z_{\beta_1}] + \dots + [z_{\beta_p}] \quad \text{and} \quad {}_2z = [z_{\gamma_1}] + \dots + [z_{\gamma_p}]$$

<sup>5</sup>this means that for each  $\hat{\alpha}'$  there is an  $\hat{\alpha}(\hat{\alpha}')$  such that  $\mathbf{U}'_{\hat{\alpha}'}$  is a subset of  $\mathbf{U}_{\hat{\alpha}(\hat{\alpha}'})$

where  $z_{\beta_p}$  and  $z_{\gamma_p}$  are global sections. The scalar product on the space  $\mathbf{Z}$ , of formal global sections of the d–vector gerbe  $\mathbf{C}_{Nd}$ , can be defined by the rule

$$\langle [{}_1z], [{}_2z] \rangle = \langle {}_1z, {}_2z \rangle_{\mathbf{Z}(q_{\widehat{\alpha}\widehat{\beta}})}$$

if  ${}_1z$  and  ${}_2z$  are elements of the same set of global sections  $\mathbf{Z}(q_{\widehat{\alpha}\widehat{\beta}})$  and

$$\langle [{}_1z], [{}_2z] \rangle = 0$$

for the elements belonging to different sets of such global sections.

**Proposition 3.3.** *Any element  $\mathbf{z}_{\widehat{\alpha}}$  of the Hilbert completion  $L^2(\mathbf{Z}(q_{\widehat{\alpha}\widehat{\beta}}))$  of the pre–Hilbert d–structure (18) is a N–adapted family of  $L^2$  sections  $\mathbf{z}_{\widehat{\alpha}}$  of  $\mathbf{e}_{\widehat{\alpha}}$  such that  $\mathbf{z}_{\widehat{\alpha}} = q_{\widehat{\alpha}\widehat{\beta}}(z_{\widehat{\beta}})$ .*

*Proof.* The proof is similar to that for the Proposition 7 in Ref. [28] and follows defining a corresponding Cauchy sequence  $(z_{\widehat{\alpha}})_{\widehat{\alpha} \in \mathbb{N}}$  of

$(\mathbf{Z}(q_{\widehat{\alpha}\widehat{\beta}}), \langle, \rangle)$ . For nonholonomic configurations, one uses d–metric structures which can be Riemannian or Riemann–Cartan ones depending on the type of linear connection is considered, a not N–adapted, or N–adapted one. ■ □

We can consider morphisms between d–objects commuting with Laplacian  $\Delta^s$  and define a pre–Hilbertian structure defined by

$$\langle {}_1z, {}_2z \rangle = \int \langle \Delta^s({}_1z), {}_2z \rangle,$$

where  $\Delta^s({}_1z)_{\widehat{\alpha}} = \Delta^s({}_1z_{\widehat{\alpha}})$ . There is a canonical N–adapted Laplacian structure [24, 25, 27] defined on nonholonomic spaces by using the canonical d–connection structure, see Proposition 5.2 in Appendix. We denote by  $H_s(\mathbf{Z}(q_{\widehat{\alpha}\widehat{\beta}}))$  the Hilbert completion of the pre–Hilbert space constructed by using  $\Delta^s$ .

### 3.2.2 D–connections on N–anholonomic gerbes and characteristic classes

The canonical d–connection structure gives rise to a such connection on each d–object  $\mathbf{e}_U$  of  $\mathbf{C}(\mathbf{U})$  and defined a family of d–connections inducing such a structure on the N–anholonomic gerbe  $\mathbf{C}$ . We consider an open covering  $(\mathbf{U}_{\widehat{\alpha}})_{\widehat{\alpha} \in I}$  of  $\mathbf{V}$  and d–objects  $\mathbf{C}(\mathbf{U}_{\widehat{\alpha}})$  as trivial bundles with  $m$ –dimensional fibers. The d–connection  $\Gamma_{\widehat{\alpha}}$  of a d–object  $\mathbf{e}_{\widehat{\alpha}}$  of  $\mathbf{C}(\mathbf{e}_{\widehat{\alpha}})$  is defined by a 1–form with coefficients (A.3) on  $T \mathbf{U}_{\widehat{\alpha}}$ . The curvature of this d–connection is  $\mathcal{R}_{\widehat{\alpha}}$ , see (8).

Having defined the curvature of the N–anholonomic gerbe, it is possible to compute the  $2k$  Chern class of  $\mathbf{e}_{\widehat{\alpha}}$ ,

$$c_{2k}^{\widehat{\alpha}} \sim Tr \left[ \left( \frac{i}{2\pi} \mathcal{R}_{\widehat{\alpha}} \right)^k \right],$$



where  $Tr$  denotes the trace operation, which is invariant for transforms  $\mathbf{e}_{\hat{\alpha}} \rightarrow \mathbf{e}'_{\hat{\alpha}}$ . In a more general case, we can compute the sum

$$c(\mathbf{C}) = c_1(\mathbf{V}) + \dots + c_2(\mathbf{V})$$

for the total Chern form of an  $N$ -anholonomic gerbe. This form define locally the total Chern character

$$ch(\mathbf{C})_{\mathbf{U}_{\hat{\alpha}}} = Tr \left[ \exp \left( \frac{i}{2\pi} \mathcal{R} \right) \right]. \quad (19)$$

It should be noted that the formula (19) is defined by a  $d$ -metric (16) and its canonical  $d$ -connection (A.2) which correspond respectively to the Riamannian metric and the Levi–Civita connection. The notion of connection is not well-defined for general vector gerbes but the existence of Riemannian structures gives a such possibility. In the case of  $N$ -anholonomic frame, even a  $d$ -metric structure is not stated, we can derive a canonical  $d$ -connection configuration by considering a formal  $d$ -metric with  $g_{ij}$  and  $h_{ab}$  taking diagonal Euclidean values and computing a curvature tensor  $\mathcal{R}^{[N]}$  by contracting the  $N$ -connection coefficients  $N_i^a$  and theirs derivatives. As a matter of principle, we can take the  $N$ -connection curvature  $\Omega$  (2) instead of  $\mathcal{R}^{[N]}$  but in this case we shall deal with metric noncommpatible  $d$ -connections. Finally, we not that we need at leas to Chern characters, one for the  $d$ -connection structure and another one for the  $N$ -connection structure in order to give a topological characteristic of  $N$ -anholonomic gerbes.

## 4 Operators and Symbols on Nonholonomic Gerbes

On  $N$ -anholonomic manifolds we deal with geometrical objects distinguished by a  $N$ -connection structure. The aim of this section is to analyze pseudo-differential operators  $\mathbf{D}_{\bar{\alpha}}$  on such spaces.

### 4.1 Operators on $N$ -anholonomic spaces

In local form, the geometric constructions adapted to a  $N$ -connection are for open sets of couples  $(\mathbb{R}^n, \mathbb{R}^m)$ , or for  $\mathbb{R}^{n+m}$ . Let us consider an open set  $\mathbf{U} \subset \mathbb{R}^{n+m}$  and denote by  $Z^r(\mathbf{U})$  the set of smooth functions  $p(\mathbf{v}, \mathbf{u})$  defined on  $\mathbf{U} \times \mathbb{R}^{n+m}$  satisfying the conditions that for any compact  $\mathbf{U}' \subset \mathbf{U}$  and every multi-indices  $\alpha$  and  $\beta$  one has

$$\left\| \mathbf{D}^{\bar{\alpha}} \mathbf{D}^{\bar{\beta}} p(\mathbf{v}, \mathbf{u}) \right\| < C_{\bar{\alpha}, \bar{\beta}, \mathbf{U}'} (1 + \|u\|)^{r - |\bar{\alpha}|},$$

for  $C_{\bar{\alpha}, \bar{\beta}, \mathbf{U}'} = const.$

**Definition 4.1.** A map  $\hat{p}$  of two smooth functions  $k(\mathbf{U})$  and  $k'(\mathbf{U})$  with compact support defined on  $\mathbf{U}$ ,  $\hat{p}: k(\mathbf{U}) \rightarrow k'(\mathbf{U})$ , such that locally

$$\hat{p}(f) = \int p(\mathbf{v}, \mathbf{u}) \hat{f}(\mathbf{u}) e^{i\langle \mathbf{v}, \mathbf{u} \rangle} \delta \mathbf{u},$$

where  $\hat{f}$  is the Fourier transform of function  $f$ , is called to be a pseudo-differential distinguished operator, in brief *pdd-operator*.

Now we extend the concept of *pdd*-operator for a  $N$ -anholonomic manifold  $\mathbf{V}$  endowed with  $d$ -metric structure (16). In this case,  $k(\mathbf{U})$  and  $k'(\mathbf{U})$  are smooth sections, with compact support, of  $\mathbf{V}$  provided with local fibered structure. A map  $\widehat{p}$  is defined for a covering family  $(\mathbf{U}_{\widehat{\alpha}})_{\widehat{\alpha} \in I}$  satisfying the conditions:

1. Any restriction of  $\mathbf{V}$  to  $\mathbf{U}_{\widehat{\alpha}}$  is trivial.
2. The map  $\widehat{p}_{\widehat{\alpha}} : k(\mathbf{U}_{\widehat{\alpha}} \times Q^m) \rightarrow k'(\mathbf{U}'_{\widehat{\alpha}} \times Q^m)$ , where the vector space  $Q^m$  is isomorphic to  $v\mathbf{V}$ ,  $\dim(v\mathbf{V}) = m$ , defines the restriction of  $\widehat{p}$  to  $\mathbf{U}_{\widehat{\alpha}}$ . There is a horizontal component of the map,  $\widehat{p}_{\widehat{\alpha}} : k(U_{\widehat{\alpha}} \times Q^m) \rightarrow k'(U'_{\widehat{\alpha}} \times Q^m)$
3. For any section  $z'$  over  $h\mathbf{U}_{\widehat{\alpha}} = U_{\widehat{\alpha}}$  and  $z = (z_1, \dots, z_m) = \phi_a(z')$ , we can define

$$t_b = \sum_{a=1}^{a=m} \int p_{ab}(x^i, v^k) \widehat{z}_a(v^l) e^{i \langle x, v \rangle} \delta v^j$$

and  $\widehat{p}_a(z') = \psi_a^{-1}(t_b)$ , where the carts  $\phi_a$  and  $\psi_a$  are such that

$$\phi_a(U_{\widehat{\alpha}} \times Q^m) = \psi_a(U'_{\widehat{\alpha}} \times Q^m) \simeq h\mathbf{U} \times \mathbb{R}^m$$

and the map  $\widehat{p}_a$  is defined by a matrix  $p_{ab}$  defining an operator of degree  $r$ .

4. The values  $t_b, z_a, \phi_a, \psi_a$  and  $p_{ab}$  can be extended to corresponding distinguished objects

$$\begin{aligned} t_b &\rightarrow t_{\alpha} = (t_i, t_b), z_a \rightarrow z_{\alpha} = (z_i, z_a), \\ \phi_a &\rightarrow \phi_{\alpha} = (\phi_i, \phi_a), \psi_a \rightarrow \psi_{\alpha} = (\psi_i, \psi_a), p_{ab} \rightarrow p_{\alpha\beta}. \end{aligned}$$

**Definition 4.2.** A map  $\widehat{p}$  satisfying the conditions 1-4 defines a *pdd*-operator on  $N$ -anholonomic manifold  $\mathbf{V}$  provided with  $d$ -metric structure (16). For Euclidean values for  $h$ - and  $v$ -components of  $d$ -metric, with respect to  $N$ -adapted frames, one gets a *pdd*-operator generated by the  $N$ -connection structure.

The Definition 4.2 can be similarly formulated for  $N$ -anholonomic vector bundles  $\mathbf{E} \rightarrow \mathbf{V}$ . We denote by  ${}^{loc}H_s(\mathbf{V}, \mathbf{E})$ , with  $s$  being a positive integer, the space of distributions sections  $\underline{\mathbf{u}}$  of  $\mathbf{E}$  such that  $\mathbf{D}(\underline{\mathbf{u}})$  is a  ${}^{loc}L^2$  section, where  $\mathbf{D}$  is any differential  $d$ -operator of order less than  $s$ . The subset of elements of  ${}^{loc}H_s(\mathbf{V}, \mathbf{E})$  with compact support is written  ${}^{comp}H_s(\mathbf{V}, \mathbf{E})$ . The space  ${}^{loc}H_{-s}(\mathbf{V}, \mathbf{E})$  is defined to be the dual space of  ${}^{comp}H_s(\mathbf{V}, \mathbf{E})$  and the space  ${}^{comp}H_{-s}(\mathbf{V}, \mathbf{E})$  is defined to be the dual space of  ${}^{loc}H_s(\mathbf{V}, \mathbf{E})$ .

**Definition 4.3.** The Sobolev canonical  $d$ -space  $H_s$  is an Hilbert space provided with the norm

$$\left\| \int \langle \widehat{\Delta}^s \underline{\mathbf{u}}, \underline{\mathbf{u}} \rangle \right\|^{1/2}$$

defined by the Laplace operator  $\widehat{\Delta}^s \doteq \widehat{\mathbf{D}}_{\alpha} \widehat{\mathbf{D}}^{\alpha}$  of the canonical  $d$ -connection structure (A.2). Every  $d$ -operator  $\widehat{p}$  of order less than  $r$  can be extended to a continuous morphism  $H_s \rightarrow H_{s-r}$ .

We can generalize the last two Definitions for N–anholonomic gerbes:

**Definition 4.4.** A  $d$ –operator  $\mathbf{D}$  of degree  $r$  on  $N$ –anholonomic gerbe  $\mathbf{C}$  provided with  $d$ –metric and canonical  $d$ –connection structures is defined by a family of operators  $\mathbf{D}_e$  of degree  $r$  defined on an object  $\mathbf{e}$  of the category  $\mathbf{C}(\mathbf{U})$  when for each morphism  $\varphi : \mathbf{e} \rightarrow f$  one holds  $\mathbf{D}_f \varphi^\# = \varphi^\# \mathbf{D}_e$ .

In this Definition the map  $\varphi^\#$  transforms a section  $z$  to  $\varphi(z)$  and it is supposed that  $\mathbf{D}_e$  is invariant under N–adapted automorphisms of  $\mathbf{e}$ . It is also assumed to be a continuous operator as a map

$$\mathbf{D}_e : \text{comp} H_s(\mathbf{U}, \mathbf{e}) \rightarrow \text{loc} H_{s-r}(\mathbf{U}, \mathbf{e}).$$

For a global distributional section  $z$  as an element of  $H_s(\mathbf{Z}_{q_{\alpha\beta}})$ , we can write  $q_{\hat{\alpha}\hat{\beta}} \left( \mathbf{D}_{e_{\hat{\alpha}}^{\hat{\alpha}}} (e_{\hat{\beta}}^{\hat{\alpha}}) \right) = \left( \mathbf{D}_{e_{\hat{\alpha}}^{\hat{\alpha}}} (e_{\hat{\alpha}}^{\hat{\beta}}) \right)$ . This proves

**Proposition 4.1.** Any  $d$ –operator  $\mathbf{D}$  of degree  $r$  defined on an  $N$ –anholonomic gerbe  $\mathbf{C}$  provided with  $d$ –metric and canonical  $d$ –connection structures induces two maps

$$\mathbf{D}_{\mathbf{Z}(q_{\hat{\alpha}\hat{\beta}})} : H_s \left( \mathbf{Z} \left( q_{\hat{\alpha}\hat{\beta}} \right) \right) \rightarrow H_{s-r} \left( \mathbf{Z} \left( q_{\hat{\alpha}\hat{\beta}} \right) \right) \quad \text{and} \quad \mathbf{D}_{\mathbf{Z}} : H_s(\mathbf{Z}) \rightarrow H_{s-r}(\mathbf{Z}).$$

In this paper, we shall consider only  $pdd$ –operators preserving  $C^\infty$  sections.

## 4.2 The symbols of nonholonomic operators

Let us consider a  $pdd$ –operator  $\hat{p}_{\hat{\alpha}}(f) = \int p_{\hat{\alpha}}(\mathbf{v}, \mathbf{u}) \hat{f}(\mathbf{u}) e^{i\langle \mathbf{v}, \mathbf{u} \rangle} \delta \mathbf{u}$  defined for a restriction of  $\hat{p}$  to  $\mathbf{U}_{\hat{\alpha}}$  from a covering  $(\mathbf{U}_{\hat{\alpha}})_{\hat{\alpha} \in I}$  of an open  $\mathbf{U} \subset \mathbf{V}$ .

**Definition 4.5.** The operator  $\hat{p}_{\hat{\alpha}}$  is of degree  $r$  with the symbol  $\sigma(\mathbf{p})$  if there exist the limit  $\sigma(\mathbf{p}|_{\mathbf{U}_{\hat{\alpha}}}) = \lim_{\lambda \rightarrow \infty} (p_{\hat{\alpha}}(\mathbf{v}, \lambda \mathbf{u}) / \lambda^r)$ .

This definition can be extended for a sphere bundle  $S\mathbf{V}$  of the cotangent bundle  $T^*\mathbf{V}$  of  $\mathbf{V}$  and for  $\pi^*\mathbf{E}$  being the pull–back of the vector bundle  $\mathbf{E}$  on  $\mathbf{V}$  to  $T^*\mathbf{V}$ . The symbols defined by the matrix  $p_{\alpha\beta}$  define a map  $\sigma : \pi^*\mathbf{E} \rightarrow \pi^*\mathbf{E}$ . This map also induces a map  $\sigma_S : \pi_S^*\mathbf{E} \rightarrow \pi_S^*\mathbf{E}$  if we consider the projection  $\pi_S : S\mathbf{V} \rightarrow \mathbf{V}$ .

Now we analyze the symbols of operators on a  $N$ –anholonomic vector gerbe  $\mathbf{C}_{NQ}$  defined on  $\mathbf{V}$  endowed with the operator  $D$  of degree  $r$ . For each object  $\mathbf{e}$  of  $\mathbf{C}(\mathbf{U})$ , it is possible to pull back the bundle  $\mathbf{e}$  by the projection map  $\pi_{SU} : S\mathbf{U} \rightarrow \mathbf{U}$  to a bundle  $\pi_{SU}^* \mathbf{e}$  over  $S\mathbf{U}$ . This nonholonomic bundle is the restriction of the co–sphere bundle defined by a fixed  $d$ –metric on  $T^*\mathbf{V}$ . We can define a category consisting from the family  $\mathbf{C}_S(\mathbf{U})$  with elements  $\pi_{SU}^* e$  and baps of such elements induced by maps between elements of  $\mathbf{C}(\mathbf{U})$ . In result, we can consider that the distinguished by  $N$ –connection map  $\mathbf{U} \rightarrow C_S(\mathbf{U})$  is a  $N$ –anholonomic gerbe with the same band as for  $\mathbf{C}$ . For an object  $e$ , it is possible to define the symbol  $\sigma_{D_e} : \pi_{SU}^* e \rightarrow \pi_{SU}^* e$ .

**Proposition 4.2.** For any sequence  $f_k$  of elements of  $H_s \left( \mathbf{Z} \left( q_{\hat{\alpha}\hat{\beta}} \right) \right)$  and a constant  $f_{[0]}$  such that  $\|f_k\|_s < f_{[0]}$ , there is a subsequence  $f_{k'}$  converging in  $H_{s'}$  for any  $s > s'$ .

The proof follows from the so-called Relich Lemma (Proposition 5) in Ref. [28]: we have only to consider it both for the so-called h- and v-subspaces. For simplicity, in this subsection, we outline only some basic properties of d-operators for N-anholonomic gerbes which are distinguished by the N-connection structure: First, the space  $Op(\mathbf{C})$  of continuous linear N-adapted maps of  $H_s\left(\mathbf{Z}\left(q_{\hat{\alpha}\hat{\beta}}\right)\right)$  is a Banach space. Secondly, the last Proposition states the possibility to define  $O^r$  the completion of the pseudo-differential operators in  $OP(\mathbf{C})$  of order  $r$  and to extend the symbol  $\sigma$  to this completion. Finally, the kernel of such an extension of the symbol to  $O^r$  contains only compact operators.

## 5 K-Theory and the N-Adapted Index

This section is devoted to K-theory groups  $K_0$  and  $K_1$  associated to symbols of d-operators on N-anholonomic gerbes as elements of  $K_0(T^*\mathbf{V})$ .

### 5.1 K-theory groups $K_0$ and $K_1$ and N-anholonomic spaces

We show how some basic results from K-theory (see, for instance, Ref. [40]) can be applied for nonholonomic manifolds.

#### 5.1.1 Basic definitions

Let us denote by  $A_n$  the vector space of  $n \times n$  complex matrices, consider natural injections  $A_n \rightarrow A_{n'}$  for  $n \leq n'$  and denote by  $A_\infty$  the inductive limit of the vector space  $A_n, n \in \mathbb{N}$ . For a ring  $B$  and two idempotents  $a'$  and  $b'$  of  $B_\infty = B \otimes A_\infty$ , one says that  $a \sim b$  if and only if there exists elements  $a', b' \in A_\infty$  such that  $a = a'b'$  and  $b = b'a'$ . Let us denote by  $[a]$  the class of  $a$  and by  $Idem(B_\infty)$  the set of equivalence classes. Representing, respectively,  $[a]$  and  $[b]$  by elements of  $B \otimes A_n$  and  $B \otimes A_{n'}$ , we can define an idempotent of  $B \otimes A_{n+n'}$  represented by  $[a + b] = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ . The semi-group  $Idem(B_\infty)$  provided with the operation  $[a] + [b] = [a + b]$  is denoted by  $K_0(B)$ .

One can extend the construction for a compact N-manifold  $\mathbf{V}$  and a the set of complex valued functions  $\mathbb{C}(\mathbf{V})$  on  $\mathbf{V}$ . For compact manifolds, it is possible to consider a trivial bundle isomorphic to  $\mathbf{V} \times \mathbb{C}^r$  and identify a vector bundle over  $\mathbf{V}$  to an idempotent of  $\mathbb{C}(\mathbf{U}) \otimes A_r$ , for  $\mathbf{U} \subset \mathbf{V}$ , which is also an idempotent of  $\mathbb{C}(\mathbf{U})_\infty$ . This allows us to identify  $K_0(\mathbf{V})$  to  $K_0(\mathbb{C}(\mathbf{U}))$ . Such a group for N-anholonomic manifold is a distinguished one, i.e. d-groups, into two different components, respectively for the h-subspace and the v-subspaces of  $\mathbf{V}$ .

Now we define the  $K_1$  group: Let  $Al_r(B)$  is the group of invertible elements contained in the matrix group  $A_r(B)$ . For  $r' \leq r$ , there is a canonical inclusion map  $Al_{r'}(B) \rightarrow Al_r(B)$ . The group  $Al_\infty(B)$  denotes the inductive limit of the groups  $Al_r(B)$  and  $Al_\infty(B)_{con}$  is the respective connected component. In result, the group  $K_1(B)$  is the quotient  $Al_\infty(B)/Al_\infty(B)_{con}$ . For a compact N-anholonomic manifold  $\mathbf{V}$ , one defines  $K_1(\mathbf{V})$  by  $K_1(\mathbb{C}(\mathbf{U}))$ .

### 5.1.2 N–anholonomic elliptic operators and indices

We consider a N–anholonomic gerbe  $\mathbf{C}$  on manifold  $\mathbf{V}$  and an elliptic operator  $D$  of degree  $r$  on  $\mathbf{C}$ , inducing a distinguished morphism

$$D : L^2 \left( \mathbf{Z} \left( q_{\widehat{\alpha}\widehat{\beta}} \right) \right) \rightarrow H_{2-r} \left( \mathbf{Z} \left( q_{\widehat{\alpha}\widehat{\beta}} \right) \right).$$

Let  $(1 - \Delta)^{-r'}$  be an operator of degree  $-r$ , for instance, we can take  $\Delta$  to be the Laplace operator  $\widehat{\Delta}^s \doteq \widehat{\mathbf{D}}_\alpha \widehat{\mathbf{D}}^\alpha$  of the canonical d–connection structure (A.2). It is possible to define the symbol  $\sigma(D) = (1 - \Delta)^{-r'} D$  which is an operator with image in  $\mathbf{Z} \left( q_{\widehat{\alpha}\widehat{\beta}} \right)$  being an invertible morphism.

For an exact sequence

$$0 \rightarrow B_1 \rightarrow B_2 \rightarrow B_3 \rightarrow 0$$

of  $C^*$ –algebras, one has the following exact sequence in  $K$ –theory

$$K_1(B_1) \rightarrow K_1(B_2) \rightarrow K_1(B_3) \rightarrow K_0(B_1) \rightarrow K_0(B_2) \rightarrow K_0(B_3).$$

Let us consider  $O(\mathcal{H})$ , the space of continuous operators on an Hilbert space  $\mathcal{H}$ , and denote by  $\mathcal{K}$  the subspace of compact continuous operators. There is the following exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow O(\mathcal{H}) \rightarrow O(\mathcal{H})/\mathcal{K} = \mathbf{Ca} \rightarrow 0$$

when  $K_0(\mathcal{K}) = \mathbb{Z}$ . Now, it is possible to introduce the class  $[\sigma(D')]$ , for  $D' = (1 - \Delta)^{-r'} D$ , of  $K_1(\mathbf{Ca})$ .

**Definition 5.1.** *The image of  $[\sigma(D')]$  in  $K_0(\mathcal{K})$  depends only on the symbol of operator  $D$  and define the index of this operator.*

We consider finite covering families  $\mathbf{U}_{\widehat{\alpha}}$  of  $\mathbf{V}$  when  $\mathbf{C}(\mathbf{U}_{\widehat{\alpha}})$  are trivial bundles. One holds

**Proposition 5.1.** *One exists a class  $[\sigma_D]$  in  $K_1(T^*\mathbf{V})$  associated to the symbol of an elliptic operator  $D$  of degree  $r$  on N–anholonomic gerbe  $\mathbf{C}$  on  $\mathbf{V}$ .*

*Proof.* A similar result is proven in [28] for the Riemannian gerbes. We do not repeat those constructions in distinguished form for h- and v–components but note two important differences: In the N–anholonomic case there are N–connections, d–metrics and d–connections. In result, one can follow two ways: to define the class for the canonical d–connection and/or to derive the class from the N–connection structure and related curvature of N–connection.  $\square$

We denote by  $X^*\mathbf{V}$  (with the fibers isomorphic to the unit ball defined by the N–connection) the compactification of  $T^*\mathbf{V}$ . The sphere N–anholonomic bundle  $S^*\mathbf{V}$  is identified to  $X^*\mathbf{V}/T^*\mathbf{V}$ . In result, one can define the exact sequence

$$0 \rightarrow \mathbf{C}(T^*\mathbf{V}) \rightarrow \mathbf{C}(X^*\mathbf{V}) \rightarrow \mathbf{C}(S^*\mathbf{V}) \rightarrow 0$$

resulting to the following exact sequence

$$\begin{aligned} K_1 [\mathbf{C}(S^*\mathbf{V}) \otimes A_r] &\rightarrow K_0 [\mathbf{C}(T^*\mathbf{V}) \otimes A_r] \\ &\rightarrow K_0 [\mathbf{C}(X^*\mathbf{V}) \otimes A_r] \rightarrow K_1 [\mathbf{C}(S^*\mathbf{V}) \otimes A_r] \rightarrow 0. \end{aligned} \quad (20)$$

The last sequence allows us to consider the boundary operator  $\delta([\sigma'_D])$  as an element of  $K_0 [\mathbf{C}(T^*\mathbf{V}) \otimes A_r]$  being isomorphic to  $K_0 [T^*\mathbf{V}]$  where

$$[\sigma'_D] \in K_1 [\mathbf{C}(S^*\mathbf{V}) \otimes A_r] \simeq K_1 [\mathbf{C}(S^*\mathbf{V})]$$

for any N-anholonomic component of the sequence (20). This concludes that the index of a d-operator  $D$  depends only on the class of boundary operator  $\delta([\sigma'_D])$ . For N-anholonomic configurations, such constructions are possible both the canonical d-connection and if it is not defined by a d-metric, one can re-define the constructions just for the N-connection and related metric compatible and N-adapted linear connection and resulting curvatures.

It should be noted that the class  $[\sigma_D]$  is not unique. For N-anholonomic spaces, we can define such classes, for instance, by using the canonical d-connection or following d-metrics and d-connections derived from the N-connection structure.

## 5.2 The index formulas and applications

The results stated in previous subsections allow us to deduce Atiyah–Singer type theorems for N-anholonomic gerbes (in general form, for any their explicit realizations like Lagrange, or Finsler, gerbes and Riemann–Cartan gerbes provided with N-connection structure). Such theorems may have a number of applications in modern noncommutative geometry and physics. We shall consider the topic related to Dirac d-operators and N-anholonomic gerbes.

### 5.2.1 The index formulas for d-operators and gerbes

The Chern character of the cotangent bundle  $T^*M$  induces a well known isomorphism  $K_0(T^*M) \otimes \mathbb{R} \rightarrow {}^{ev}H_c(M, \mathbb{R})$  when for elements  $u^* \subset T^*M$  and  $t \in \mathbb{R}$  one has the map  $u^* \otimes t \rightarrow tch(u^*)$ . The constructions may be generalized for N-anholonomic gerbes, see (19), with additional possibilities related to the N-connection and d-connection structures.

We denote by  $d^*Vect(Ind)$  the subspace of  $K_0(\mathbf{V}^*)$ , with N-anholonomic manifold  $\mathbf{V}^*$  constructed to have local coordinates  ${}^*u^\alpha = (x^i, {}^*y_a)$ , with  ${}^*y_a$  being dual to  $y^a$ , where  $u^\alpha = (x^i, y^a)$  are local coordinates on  $\mathbf{V}$ . We consider the symbol  $\sigma_p$  of a d-operator  $\hat{p}$  on corresponding to  $\mathbf{V}^*$  N-anholonomic gerbe. The mentioned subspace is also a subspace of  ${}^{ev}H_c(\mathbf{V}, \mathbb{R})$ . In result, the map  $d^*Vect(\sigma_p) \rightarrow \mathbb{R}$  given by  $ch([\sigma'_p]) \rightarrow ind(\hat{p})$  can be extended to a linear map  ${}^{ev}H_c(\mathbf{V}, \mathbb{R}) \rightarrow \mathbb{R}$ .

One can be performed similar constructions starting from the N-anholonomic space  $T^*\mathbf{V}$  and using the distinguished isomorphism

$$K_0(T^*\mathbf{V}) \otimes \mathbb{R} \rightarrow {}^{ev}H_c(\mathbf{V}, \mathbb{R}).$$

In this case, we introduce  $dVect(Ind)$  as the subspace of  $K_0(T^*\mathbf{V})$  generated by  $\sigma_p$  related to  $T^*\mathbf{V}$ . Here we also note that the symbol operator  $\sigma_p$  and related maps can be introduced

for d–metric and canonical d–connection structures or for a ”pure” N–connection structure. So, there are two classes of symbols  $\sigma_p$  (both in the case related to the constructions with  $\mathbf{V}^*$  and to the case for constructions with  $T^*\mathbf{V}$ ), i. e. four variants of relations Chern character – index of d–operator and corresponding extensions to linear maps.

The above presented considerations consist in a proof (of four Atiyah–Singer type theorems):

**Theorem 5.1.** *The Poincare duality of N–anholonomic gerbes implies the existence of classes  $t_d^*(\mathbf{V}), t_V^*(\mathbf{V})$  and  $t_d(\mathbf{V}), t_V(\mathbf{V})$  for which, respectively,*

$$\begin{aligned} \text{Ind}_d^*(\hat{p}) &= \int_{\mathbf{V}^*} \text{ch}([\sigma'_p]) \wedge t_d^*(\mathbf{V}), \text{ch}([\sigma'_p]) \text{ related to } d\text{-metric,} \\ \text{Ind}_N^*(\hat{p}) &= \int_{\mathbf{V}^*} \text{ch}([\sigma'_p]) \wedge t_N^*(\mathbf{V}), \text{ch}([\sigma'_p]) \text{ related to } N\text{-connection} \end{aligned}$$

and

$$\begin{aligned} \text{Ind}_d(\hat{p}) &= \int_{T^*\mathbf{V}} \text{ch}([\sigma'_p]) \wedge t_d(\mathbf{V}), \text{ch}([\sigma'_p]) \text{ related to } d\text{-metric,} \\ \text{Ind}_N(\hat{p}) &= \int_{T^*\mathbf{V}} \text{ch}([\sigma'_p]) \wedge t_N(\mathbf{V}), \text{ch}([\sigma'_p]) \text{ related to } N\text{-connection.} \end{aligned}$$

The index formulas from this Theorem present topological characteristics for Lagrange (in particular, Finsler) spaces and gerbes, see (10) and (12), of nonholonomic Riemann–Cartan spaces, see ansatz (16), considered for constructing exact solutions in modern gravity [19].

### 5.2.2 N–anholonomic spinors and the Dirac operator

The theory and methods developed in this paper have a number of motivations following from applications to the theory of nonholonomic Clifford structures and Dirac operators on N–anholonomic manifolds [25, 26, 27, 24]. In Appendix 6, there are given the necessary results on N–anholonomic spinor structures and spin d–connections.

#### The Dirac d–operator:

We consider a vector bundle  $\mathbf{E}$  on an N–anholonomic manifold  $\mathbf{V}$  (with two compatible N–connections defined as h– and v–splittings of  $T\mathbf{E}$  and  $T\mathbf{V}$ ). A d–connection

$$\mathcal{D} : \Gamma^\infty(\mathbf{E}) \rightarrow \Gamma^\infty(\mathbf{E}) \otimes \Omega^1(\mathbf{V})$$

preserves by parallelism splitting of the tangent total and base spaces and satisfy the Leibniz condition

$$\mathcal{D}(f\sigma) = f(\mathcal{D}\sigma) + \delta f \otimes \sigma$$

for any  $f \in C^\infty(\mathbf{V})$ , and  $\sigma \in \Gamma^\infty(\mathbf{E})$  and  $\delta$  defining an N–adapted exterior calculus by using N–elongated operators (3) and (4) which emphasize d–forms instead of usual forms on  $\mathbf{V}$ , with the coefficients taking values in  $\mathbf{E}$ .

The metricity and Leibniz conditions for  $\mathcal{D}$  are written respectively

$$\mathbf{g}(\mathcal{D}\mathbf{X}, \mathbf{Y}) + \mathbf{g}(\mathbf{X}, \mathcal{D}\mathbf{Y}) = \delta[\mathbf{g}(\mathbf{X}, \mathbf{Y})], \quad (21)$$

for any  $\mathbf{X}, \mathbf{Y} \in \chi(\mathbf{V})$ , and

$$\mathcal{D}(\sigma\beta) \doteq \mathcal{D}(\sigma)\beta + \sigma\mathcal{D}(\beta), \quad (22)$$

for any  $\sigma, \beta \in \Gamma^\infty(\mathbf{E})$ .

For local computations, we may define the corresponding coefficients of the geometric d-objects and write

$$\mathcal{D}\sigma_{\dot{\beta}} \doteq \Gamma_{\dot{\beta}\mu}^{\dot{\alpha}} \sigma_{\dot{\alpha}} \otimes \delta u^\mu = \Gamma_{\dot{\beta}i}^{\dot{\alpha}} \sigma_{\dot{\alpha}} \otimes dx^i + \Gamma_{\dot{\beta}a}^{\dot{\alpha}} \sigma_{\dot{\alpha}} \otimes \delta y^a,$$

where fiber "acute" indices, in their turn, may split  $\dot{\alpha} \doteq (i, \hat{a})$  if any N-connection structure is defined on  $T\mathbf{E}$ . For some particular constructions of particular interest, we can take  $\mathbf{E} = T^*\mathbf{V}$  and/or any Clifford d-algebra  $\mathbf{E} = \mathbf{ICl}(\mathbf{V})$  with a corresponding treating of "acute" indices to of d-tensor and/or d-spinor type as well when the d-operator  $\mathcal{D}$  transforms into respective d-connection  $\mathbf{D}$  and spin d-connections  $\widehat{\nabla}^{\mathbf{S}}$  (A.18),  $\widehat{\nabla}^{SL}$  (A.19).... All such, adapted to the N-connections, computations are similar for both N-anholonomic (co) vector and spinor bundles.

The respective actions of the Clifford d-algebra and Clifford-Lagrange algebra can be transformed into maps  $\Gamma^\infty(\mathbf{Sp}) \otimes \Gamma(\mathbf{ICl}(\mathbf{V}))$  to  $\Gamma^\infty(\mathbf{Sp})$  by considering maps of type (A.8) and (A.14)

$$\widehat{\mathbf{c}}(\check{\psi} \otimes \mathbf{a}) \doteq \mathbf{c}(\mathbf{a})\check{\psi} \text{ and } \widehat{\mathbf{c}}(\psi \otimes a) \doteq c(a)\psi.$$

**Definition 5.2.** *The Dirac d-operator (Dirac-Lagrange operator) on a spin N-anholonomic manifold  $(\mathbf{V}, \mathbf{Sp}, J)$  where  $J : \mathbf{Sp} \rightarrow \mathbf{Sp}$  is the antilinear bijection, is defined*

$$\begin{aligned} \mathbf{ID} &\doteq -i (\widehat{\mathbf{c}} \circ \nabla^{\mathbf{S}}) \\ &= (\mathbf{ID} = -i (\widehat{\mathbf{c}} \circ \nabla^{\mathbf{S}}), \quad -\mathbf{ID} = -i (\widehat{\mathbf{c}} \circ -\nabla^{\mathbf{S}})) \end{aligned} \quad (23)$$

Such N-adapted Dirac d-operators are called canonical and denoted  $\widehat{\mathbf{ID}} = (\widehat{\mathbf{ID}}, -\widehat{\mathbf{ID}})$  if they are defined for the canonical d-connection (A.3) and respective spin d-connection (A.18) ((A.19)).

Now we can formulate the (see Proof of Theorem 6.1 [24])

**Theorem 5.2.** *Let  $(\mathbf{V}, \mathbf{Sp}, J)$  be a spin N-anholonomic manifold ( spin Lagrange space). There is the canonical Dirac d-operator (Dirac-Lagrange operator) defined by the almost Hermitian spin d-operator*

$$\widehat{\nabla}^{\mathbf{S}} : \Gamma^\infty(\mathbf{Sp}) \rightarrow \Gamma^\infty(\mathbf{Sp}) \otimes \Omega^1(\mathbf{V})$$

commuting with  $J$  and satisfying the conditions

$$(\widehat{\nabla}^{\mathbf{S}}\check{\psi} \mid \check{\phi}) + (\check{\psi} \mid \widehat{\nabla}^{\mathbf{S}}\check{\phi}) = \delta(\check{\psi} \mid \check{\phi})$$

and

$$\widehat{\nabla}^{\mathbf{S}}(\mathbf{c}(\mathbf{a})\check{\psi}) = \mathbf{c}(\widehat{\mathbf{D}}\mathbf{a})\check{\psi} + \mathbf{c}(\mathbf{a})\widehat{\nabla}^{\mathbf{S}}\check{\psi}$$

for  $\mathbf{a} \in \mathbf{ICl}(\mathbf{V})$  and  $\check{\psi} \in \Gamma^\infty(\mathbf{Sp})$  determined by the metricity (21) and Leibnitz (22) conditions.



**The Clifford N–gerbe and the Dirac operator:**

We consider the Clifford N–gerbe  $Cl_N(\mathbf{V})$  on a N–anholonomic manifold  $\mathbf{V}$ , see section 3, provided with d–connection structure (16). For an opening covering  $(\mathbf{U}_{\hat{\alpha}})_{\hat{\alpha} \in I}$  of  $\mathbf{V}$ , the canonical d–connection (A.2) is extended on all family of  $\mathbf{U}_{\hat{\alpha}}$  as a corresponding family of N–adapted  $so(n+m)$  forms  $\hat{\Gamma}_{\hat{\alpha}}$  on  $\mathbf{U}_{\hat{\alpha}}$  satisfying

$$\hat{\Gamma}_{\hat{\alpha}} = ad(q_{\hat{\beta}\hat{\alpha}})^{-1}\hat{\Gamma}_{\hat{\beta}} + (q_{\hat{\beta}\hat{\alpha}})^{-1}\delta(q_{\hat{\beta}\hat{\alpha}})$$

where  $\delta(q_{\hat{\beta}\hat{\alpha}})$  is computed by using the N–elongated partial derivatives (3) and (4). This d–connection induces the covariant derivative  $\delta + \hat{\Gamma}_{\hat{\alpha}}$ . Fixing an orthogonal basis  $\mathbf{e}_{\hat{\alpha}}$ , to which (3) are transformed in general, on  $T\mathbf{U}_{\hat{\alpha}}$ , we can write

$$\hat{\Gamma}_{\hat{\alpha}} = \left( \hat{\Gamma}_{\hat{i}} = \sum_{\hat{k}=1}^{\hat{k}=n} \hat{\Gamma}_{\hat{i}\hat{k}} e_{\hat{k}}, \hat{\Gamma}_{\hat{a}} = \sum_{\hat{b}=n+1}^{\hat{b}=n+m} \hat{\Gamma}_{\hat{a}\hat{b}} e_{\hat{b}} \right).$$

In local form, the d–spinor covariant derivative was investigated in Refs. [25, 26, 27, 24]. We can extend it to a covering family  $(\mathbf{U}_{\hat{\alpha}})_{\hat{\alpha} \in I}$  of  $\mathbf{V}$  by introducing the d–object  $\varrho_{\hat{\beta}\hat{\alpha}} = -\frac{1}{4} \hat{\Gamma}_{\hat{\beta}\hat{\alpha}} = -\varrho_{\hat{\alpha}\hat{\beta}}$ , see formula (A.18) in Appendix.

Let  $\mathbf{e}_U$  be an object of  $Cl_N(\mathbf{U})$ . For a trivialization  $(\mathbf{U}_{\hat{\alpha}})_{\hat{\alpha} \in I}$ , we can generalize for N–anholonomic gerbes the Definition 5.2 and write

$$\mathbf{ID}_{\mathbf{e}_U} \doteq \sum_{\hat{\alpha}=1}^{\hat{\alpha}=n+m} \mathbf{e}_{\hat{\alpha}} \mathbf{ID} = \left( \sum_{\hat{i}=1}^{\hat{i}=n} e_{\hat{i}} \mathbf{ID}, \sum_{\hat{a}=n+1}^{\hat{a}=n+m} e_{\hat{a}} \mathbf{ID} \right)$$

where  $\mathbf{ID}$  is defined locally by (23). On any such object one holds a distinguished variant of the Lichnerowicz–Weitzenbock formula (defined by the d–connection and/or N–connection structure)

$$D^2 = \mathbf{ID}^* \mathbf{ID} + \frac{1}{4} \overleftarrow{\mathbf{R}}$$

where  $\mathbf{ID}^* \mathbf{ID}$  is the Laplacian and  $\overleftarrow{\mathbf{R}}$  is the scalar curvature (A.6) of the corresponding d–connection (the definition of curvature of a general nonholonomic manifold is not a trivial task [31, 32] but for N–anholonomic manifolds this follows from a usual N–adapted tensor and differential calculus [33, 34, 35, 36] (see the supersymmetric variant in [37]) like that presented in Appendix 5.2.2.

Such global d–spinors are canonically d–harmonic if  $\mathbf{ID}_{\mathbf{e}_{\hat{\alpha}}}(\overleftarrow{\mathbf{R}}_{\hat{\alpha}}) = 0$  for each  $\overleftarrow{\mathbf{R}}_{\hat{\alpha}}$ . If a d–metric (16) is not provided, we can define a formal canonical d–connection  ${}^N \hat{\Gamma}_{\hat{\beta}\hat{\gamma}}^{\alpha}$  computed by formulas (A.3) with  $g_{ij}$  and  $h_{ab}$  taken for Euclidean spaces (this metric compatible canonical d–connection is defined only by the N–connection coefficients). Introducing  ${}^N \hat{\Gamma}_{\hat{\beta}\hat{\gamma}}^{\alpha}$  into (A.4) and (A.6), one computes respectively the curvature  ${}^N \mathcal{R}_{\hat{\beta}}^{\alpha}$  and scalar curvature  ${}^N \overleftarrow{\mathbf{R}}$ .

For the Riemannian gerbes derived for compact Riemannian manifolds  $V$  with strictly positive curvature, it is known the result that the topological class  $\tau(V)$  associated to the

index formula for operators on the  $Cl(V)$  gerbe is zero [28]. One holds a similar result for  $N$ -anholonomic manifolds and gerbes  $Cl(\mathbf{V})$  but in terms of  $d$ -metrics and canonical  $d$ -connections defining  $\widehat{\tau}(\mathbf{V}) = 0$ . We can compute the class  ${}^N\widehat{\tau}(\mathbf{V}) = 0$  even a  $d$ -metric is not given but its  ${}^N\widehat{\mathbf{R}}$  is strictly positive and the  $N$ -anholonomic manifold is compact.

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## The Canonical $d$ -Connection

A  $d$ -connection splits into  $h$ - and  $v$ -covariant derivatives,  $\mathbf{D} = D + {}^-D$ , where  $D_k = (L_{jk}^i, L_{bk}^a)$  and  ${}^-D_c = (C_{jk}^i, C_{bc}^a)$  are correspondingly introduced as  $h$ - and  $v$ -parametrizations of (6),

$$L_{jk}^i = (\mathbf{D}_k e_j) \rfloor e^i, \quad L_{bk}^a = (\mathbf{D}_k e_b) \rfloor e^a, \quad C_{jc}^i = (\mathbf{D}_c e_j) \rfloor e^i, \quad C_{bc}^a = (\mathbf{D}_c e_b) \rfloor e^a.$$

The components  $\Gamma_{\alpha\beta}^\gamma = (L_{jk}^i, L_{bk}^a, C_{jc}^i, C_{bc}^a)$  completely define a  $d$ -connection  $\mathbf{D}$  on a  $N$ -anholonomic manifold  $\mathbf{V}$ .

The simplest way to perform a covariant calculus by applying  $d$ -connections is to use  $N$ -adapted differential forms like  $\mathbf{\Gamma}_\beta^\alpha = \Gamma_{\beta\gamma}^\alpha \mathbf{e}^\gamma$  with the coefficients defined with respect to (4) and (3).

**Theorem 5.3.** *The torsion  $\mathcal{T}^\alpha(\gamma)$  of a  $d$ -connection has the irreducible  $h$ -  $v$ - components ( $d$ -torsions) with  $N$ -adapted coefficients*

$$\begin{aligned} T_{jk}^i &= L_{jk}^i - L_{kj}^i, \quad T_{ja}^i = -T_{aj}^i = C_{ja}^i, \quad T_{ji}^a = \Omega_{ji}^a, \\ T_{bi}^a &= T_{ib}^a = \frac{\partial N_i^a}{\partial y^b} - L_{bi}^a, \quad T_{bc}^a = C_{bc}^a - C_{cb}^a. \end{aligned} \quad (\text{A.1})$$

*Proof.* By a straightforward calculation, we can verify the formulas.  $\blacksquare$   $\square$

The Levi-Civita linear connection  $\nabla = \{\nabla \mathbf{\Gamma}_{\beta\gamma}^\alpha\}$ , with vanishing both torsion and nonmetricity, is not adapted to the global splitting (1).

One holds:

**Proposition 5.2.** *There is a preferred, canonical  $d$ -connection structure,  $\widehat{\mathbf{D}}$ , on  $N$ -anholonomic manifold  $\mathbf{V}$  constructed only from the metric and  $N$ -connection coefficients  $[g_{ij}, h_{ab}, N_i^a]$  and satisfying the conditions  $\widehat{\mathbf{D}}\mathbf{g} = 0$  and  $\widehat{T}_{jk}^i = 0$  and  $\widehat{T}_{bc}^a = 0$ .*

*Proof.* By straightforward calculations with respect to the  $N$ -adapted bases (4) and (3), we can verify that the connection

$$\widehat{\mathbf{\Gamma}}_{\beta\gamma}^\alpha = \nabla \mathbf{\Gamma}_{\beta\gamma}^\alpha + \widehat{\mathbf{P}}_{\beta\gamma}^\alpha \quad (\text{A.2})$$

with the deformation  $d$ -tensor <sup>6</sup>

$$\widehat{\mathbf{P}}_{\beta\gamma}^\alpha = (P_{jk}^i = 0, P_{bk}^a = e_b(N_k^a), P_{jc}^i = -\frac{1}{2}g^{ik}\Omega_{kj}^a h_{ca}, P_{bc}^a = 0)$$

<sup>6</sup> $\widehat{\mathbf{P}}_{\beta\gamma}^\alpha$  is a tensor field of type (1,2). As is well known, the sum of a linear connection and a tensor field of type (1,2) is a new linear connection.

satisfies the conditions of this Proposition. It should be noted that, in general, the components  $\widehat{T}_{ja}^i$ ,  $\widehat{T}_{ji}^a$  and  $\widehat{T}_{bi}^a$  are not zero. This is an anholonomic frame (or, equivalently, off-diagonal metric) effect. ■  $\square$

With respect to the N–adapted frames, the coefficients  $\widehat{\Gamma}_{\alpha\beta}^\gamma = \left(\widehat{L}_{jk}^i, \widehat{L}_{bk}^a, \widehat{C}_{jc}^i, \widehat{C}_{bc}^a\right)$  are computed:

$$\begin{aligned}\widehat{L}_{jk}^i &= \frac{1}{2}g^{ir} (e_k g_{jr} + e_j g_{kr} - e_r g_{jk}), \\ \widehat{L}_{bk}^a &= e_b(N_k^a) + \frac{1}{2}h^{ac} \left(e_k h_{bc} - h_{dc} e_b N_k^d - h_{db} e_c N_k^d\right), \\ \widehat{C}_{jc}^i &= \frac{1}{2}g^{ik} e_c g_{jk}, \quad \widehat{C}_{bc}^a = \frac{1}{2}h^{ad} (e_c h_{bd} + e_c h_{cd} - e_d h_{bc}).\end{aligned}\tag{A.3}$$

For the canonical d–connection, there are satisfied the conditions of vanishing of torsion on the h–subspace and v–subspace, i.e.,  $\widehat{T}_{jk}^i = \widehat{T}_{bc}^a = 0$ .

The curvature of a d–connection  $\mathbf{D}$  on an N–anholonomic manifold is defined by the usual formula

$$\mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} \doteq \mathbf{D}_X \mathbf{D}_Y \mathbf{Z} - \mathbf{D}_Y \mathbf{D}_X \mathbf{Z} - \mathbf{D}_{[X, Y]}\mathbf{Z}.$$

By straightforward calculations, we can prove:

**Theorem 5.4.** *The curvature  $\mathcal{R}_\beta^\alpha \doteq \mathbf{D}\Gamma_\beta^\alpha = d\Gamma_\beta^\alpha - \Gamma_\beta^\gamma \wedge \Gamma_\gamma^\alpha$  of a d–connection has the irreducible h– v– components (d–curvatures) of  $\mathbf{R}_{\beta\gamma\delta}^\alpha$ ,*

$$\begin{aligned}R_{hjk}^i &= e_k L_{hj}^i - e_j L_{hk}^i + L_{hj}^m L_{mk}^i - L_{hk}^m L_{mj}^i - C_{ha}^i \Omega_{kj}^a, \\ R_{bjk}^a &= e_k L_{bj}^a - e_j L_{bk}^a + L_{bj}^c L_{ck}^a - L_{bk}^c L_{cj}^a - C_{bc}^a \Omega_{kj}^c, \\ R_{jka}^i &= e_a L_{jk}^i - D_k C_{ja}^i + C_{jb}^i T_{ka}^b, \\ R_{bka}^c &= e_a L_{bk}^c - D_k C_{ba}^c + C_{bd}^c T_{ka}^d, \\ R_{jbc}^i &= e_c C_{jb}^i - e_b C_{jc}^i + C_{jb}^h C_{hc}^i - C_{jc}^h C_{hb}^i, \\ R_{bcd}^a &= e_d C_{bc}^a - e_c C_{bd}^a + C_{bc}^e C_{ed}^a - C_{bd}^e C_{ec}^a.\end{aligned}\tag{A.4}$$

Contracting respectively the components of (A.4), one proves

**Corollary 5.1.** *The Ricci d–tensor  $\mathbf{R}_{\alpha\beta} \doteq \mathbf{R}_{\alpha\beta}^\tau$  has the irreducible h– v–components*

$$R_{ij} \doteq R_{ijk}^k, \quad R_{ia} \doteq -R_{ika}^k, \quad R_{ai} \doteq R_{aib}^b, \quad R_{ab} \doteq R_{abc}^c,\tag{A.5}$$

for a N–holonomic manifold  $\mathbf{V}$ .

**Corollary 5.2.** *The scalar curvature of a d–connection is*

$$\overleftarrow{\mathbf{R}} \doteq \mathbf{g}^{\alpha\beta} \mathbf{R}_{\alpha\beta} = g^{ij} R_{ij} + h^{ab} R_{ab},\tag{A.6}$$

defined by the "pure" h– and v–components of (A.5).

## 6 Nonholonomic Spinors and Spin Connections

We outline the necessary results on spinor structures and N-connections [25, 26, 27, 24].

Let us consider a manifold  $M$  of dimension  $n$ . We define the algebra of Dirac's gamma matrices (in brief, h-gamma matrices defined by self-adjoints matrices  $A_k(\mathbb{C})$  where  $k = 2^{n/2}$  is the dimension of the irreducible representation of the set gamma matrices, defining the Clifford structure  $\mathbf{ICl}(M)$ , for even dimensions, or of  $\mathbf{ICl}(M)^+$  for odd dimensions) from the relation

$$\gamma^{\hat{i}}\gamma^{\hat{j}} + \gamma^{\hat{j}}\gamma^{\hat{i}} = 2\delta^{\hat{i}\hat{j}} \mathbf{I}. \quad (\text{A.7})$$

We can consider the action of  $dx^{\hat{i}} \in \mathbf{ICl}(M)$  on a spinor  $\psi \in Sp$  via representations

$$c(dx^{\hat{i}}) \doteq \gamma^{\hat{i}} \text{ and } c(dx^{\hat{i}})\psi \doteq \gamma^{\hat{i}}\psi \equiv e^{\hat{i}}_{\hat{i}} \gamma^{\hat{i}}\psi. \quad (\text{A.8})$$

For any tangent bundle  $TM$  and/or N-anholonomic manifold  $\mathbf{V}$  possessing a local (in any point) or global fibered structure  $F$  (being isomorphic to a real vector space of dimension  $m$ ) and, in general, enabled with a N-connection structure, we can introduce similar definitions of the gamma matrices following algebraic relations and metric structures on fiber subspaces,

$$e^{\hat{a}} \doteq e^{\hat{a}}_{\underline{a}}(x, y) e^{\underline{a}} \text{ and } e^{\underline{a}} \doteq e^{\underline{a}}_{\hat{a}}(x, y) e^{\hat{a}}, \quad (\text{A.9})$$

where

$$g^{\underline{a}\underline{b}}(x, y) e^{\hat{a}}_{\underline{a}}(x, y)e^{\hat{b}}_{\underline{b}}(x, y) = \delta^{\hat{a}\hat{b}} \text{ and } g^{\underline{a}\underline{b}}(x, y) e^{\underline{a}}_{\hat{a}}(x, y)e^{\underline{b}}_{\hat{b}}(x, y) = h^{\hat{a}\hat{b}}(x, y).$$

In a similar form, we define the algebra of Dirac's matrices related to typical fibers (in brief, v-gamma matrices defined by self-adjoint matrices  $M'_k(\mathbb{C})$  where  $k' = 2^{m/2}$  is the dimension of the irreducible representation of  $\mathbf{ICl}(F)$  for even dimensions, or of  $\mathbf{ICl}(F)^+$  for odd dimensions, of the typical fiber) from the relation

$$\gamma^{\hat{a}}\gamma^{\hat{b}} + \gamma^{\hat{b}}\gamma^{\hat{a}} = 2\delta^{\hat{a}\hat{b}} \mathbf{I}. \quad (\text{A.10})$$

The action of  $dy^{\hat{a}} \in \mathbf{ICl}(F)$  on a spinor  ${}^-\psi \in {}^-Sp$  is considered via representations

$${}^-c(dy^{\hat{a}}) \doteq \gamma^{\hat{a}} \text{ and } {}^-c(dy^{\hat{a}}) {}^-\psi \doteq \gamma^{\hat{a}} {}^-\psi \equiv e^{\hat{a}}_{\hat{a}} \gamma^{\hat{a}} {}^-\psi. \quad (\text{A.11})$$

A more general gamma matrix calculus with distinguished gamma matrices (in brief, d-gamma matrices) can be elaborated for N-anholonomic manifolds  $\mathbf{V}$  provided with d-metric structure  $\mathbf{g} = [g, {}^-g]$  and for d-spinors  $\check{\psi} \doteq (\psi, {}^-\psi) \in \mathbf{Sp} \doteq (Sp, {}^-Sp)$ , which are usual spinors but adapted locally to the N-connection structure, i. e. they are defined with respect to N-elongated bases (3) and (4). Firstly, we should write in a unified form, related to a d-metric (16), the formulas (A.9),

$$e^{\hat{\alpha}} \doteq e^{\hat{\alpha}}_{\underline{\alpha}}(u) e^{\underline{\alpha}} \text{ and } e^{\underline{\alpha}} \doteq e^{\underline{\alpha}}_{\hat{\alpha}}(u) e^{\hat{\alpha}}, \quad (\text{A.12})$$

where

$$g^{\underline{\alpha}\underline{\beta}}(u) e^{\hat{\alpha}}_{\underline{\alpha}}(u)e^{\hat{\beta}}_{\underline{\beta}}(u) = \delta^{\hat{\alpha}\hat{\beta}} \text{ and } g^{\underline{\alpha}\underline{\beta}}(u) e^{\underline{\alpha}}_{\hat{\alpha}}(u)e^{\underline{\beta}}_{\hat{\beta}}(u) = g^{\alpha\beta}(u).$$

The second step, is to consider d–gamma matrix relations (unifying (A.7) and (A.10))

$$\gamma^{\hat{\alpha}}\gamma^{\hat{\beta}} + \gamma^{\hat{\beta}}\gamma^{\hat{\alpha}} = 2\delta^{\hat{\alpha}\hat{\beta}} \mathbb{I}, \quad (\text{A.13})$$

with the action of  $du^\alpha \in \mathbf{ICl}(\mathbf{V})$  on a d–spinor  $\check{\psi} \in \mathbf{Sp}$  resulting in distinguished irreducible representations (unifying (A.8) and (A.11))

$$\mathbf{c}(du^{\hat{\alpha}}) \doteq \gamma^{\hat{\alpha}} \text{ and } \mathbf{c} = (du^\alpha) \check{\psi} \doteq \gamma^\alpha \check{\psi} \equiv e_{\hat{\alpha}}^\alpha \gamma^{\hat{\alpha}} \check{\psi} \quad (\text{A.14})$$

which allows us to write

$$\gamma^\alpha(u)\gamma^\beta(u) + \gamma^\beta(u)\gamma^\alpha(u) = 2g^{\alpha\beta}(u) \mathbb{I}. \quad (\text{A.15})$$

In the canonical representation we can write in irreducible form  $\check{\gamma} \doteq \gamma \oplus \bar{\gamma}$  and  $\check{\psi} \doteq \psi \oplus \bar{\psi}$ , for instance, by using block type of h– and v–matrices, or, writing alternatively as couples of gamma and/or h– and v–spinor objects written in N–adapted form,

$$\gamma^\alpha \doteq (\gamma^i, \gamma^a) \text{ and } \check{\psi} \doteq (\psi, \bar{\psi}). \quad (\text{A.16})$$

The decomposition (A.15) holds with respect to a N–adapted vielbein (3). We also note that for a spinor calculus, the indices of spinor objects should be treated as abstract spinorial ones possessing certain reducible, or irreducible, properties depending on the space dimension. For simplicity, we shall consider that spinors like  $\check{\psi}, \psi, \bar{\psi}$  and all type of gamma objects can be enabled with corresponding spinor indices running certain values which are different from the usual coordinate space indices. In a ”rough” but brief form we can use the same indices  $i, j, \dots, a, b, \dots, \alpha, \beta, \dots$  both for d–spinor and d–tensor objects.

The spin connection  ${}^S\nabla$  for the Riemannian manifolds is induced by the Levi–Civita connection  ${}^\nabla\Gamma$ ,

$${}^S\nabla \doteq d - \frac{1}{4} {}^\nabla\Gamma^i_{jk} \gamma_i \gamma^j dx^k. \quad (\text{A.17})$$

On N–anholonomic spaces, it is possible to define spin connections which are N–adapted by replacing the Levi–Civita connection by any d–connection.

**Definition 6.1.** *The canonical spin d–connection is defined by the canonical d–connection (A.2) as*

$$\mathbf{s}\hat{\nabla} \doteq \delta - \frac{1}{4} \hat{\Gamma}^\alpha_{\beta\mu} \gamma_\alpha \gamma^\beta \delta u^\mu, \quad (\text{A.18})$$

where the absolute differential  $\delta$  acts in N–adapted form resulting in 1–forms decomposed with respect to N–elongated differentials like  $\delta u^\mu = (dx^i, \delta y^a)$  (4).

We note that the canonical spin d–connection  $\mathbf{s}\hat{\nabla}$  is metric compatible and contains nontrivial d–torsion coefficients induced by the N–anholonomy relations (see the formulas (A.1) proved for arbitrary d–connection). It is possible to introduce more general spin d–connections  $\mathbf{S}\mathbf{D}$  by using the same formula (A.18) but for arbitrary metric compatible d–connection  $\Gamma^\alpha_{\beta\mu}$ .

**Proposition 6.1.** *On Lagrange spaces, there is a canonical spin d–connection (the canonical spin–Lagrange connection),*

$${}^{SL}\widehat{\nabla} \doteq \delta - \frac{1}{4} {}^L\mathbf{\Gamma}_{\beta\mu}^{\alpha} \gamma_{\alpha} \gamma^{\beta} \delta u^{\mu}, \quad (\text{A.19})$$

where  $\delta u^{\mu} = (dx^i, \delta y^k = dy^k + {}^LN_i^k dx^i)$ .

We emphasize that even regular Lagrangians of classical mechanics without spin particles induce in a canonical (but nonholonomic) form certain classes of spin d–connections like (A.19).

For the spaces provided with generic off–diagonal metric structure (13) (in particular, for such Riemannian manifolds) resulting in equivalent N–anholonomic manifolds, it is possible to prove a result being similar to Proposition 6.1:

**Remark 6.1.** *There is a canonical spin d–connection (A.18) induced by the off–diagonal metric coefficients with nontrivial  $N_i^a$  and associated nonholonomic frames in gravity theories.*

The N–connection structure also states a global h– and v–splitting of spin d–connection operators, for instance,

$${}^{SL}\widehat{\nabla} \doteq \delta - \frac{1}{4} {}^L\widehat{L}^i_{jk} \gamma_i \gamma^j dx^k - \frac{1}{4} {}^L\widehat{C}^a_{bc} \gamma_a \gamma^b \delta y^c. \quad (\text{A.20})$$

So, any spin d–connection is a d–operator with conventional splitting of action like  $\nabla^{\mathbf{S}} \equiv (\nabla^{\mathbf{S}}, -\nabla^{(\mathbf{S})})$ , or  $\nabla^{(SL)} \equiv (\nabla^{SL}, -\nabla^{SL})$ . For instance, for  $\widehat{\nabla}^{SL} \equiv (\widehat{\nabla}^{SL}, -\widehat{\nabla}^{SL})$ , the operators  $\widehat{\nabla}^{SL}$  and  $-\widehat{\nabla}^{SL}$  act respectively on a h–spinor  $\psi$  as

$$\widehat{\nabla}^{SL} \psi \doteq dx^i \frac{\delta \psi}{dx^i} - dx^k \frac{1}{4} {}^L\widehat{L}^i_{jk} \gamma_i \gamma^j \psi \quad (\text{A.21})$$

and

$$-\widehat{\nabla}^{SL} \psi \doteq \delta y^a \frac{\partial \psi}{dy^a} - \delta y^c \frac{1}{4} {}^L\widehat{C}^a_{bc} \gamma_a \gamma^b \psi$$

being defined by the canonical d–connection (A.2).

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