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GENERALIZED QUATERNIONIC STRUCTURES ON THE TOTAL SPACE OF A COMPLEX FINSLER SPACE

Annamária SZÁSZ [1]

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Abstract

In this note our goal is to introduce a generalized quaternionic structures, on the total space of a complex Finsler space. Some important properties of this structures are emphasized. A special approach is devoted to the commutative almost quaternionic connections.

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1 Introduction

Let (M, F) be a complex Finsler manifold, i.e., M is a smooth manifold and F is a Finsler metric on M. In this paper, we introduce the following metric G on $T'M$ (cf. Section 3):

$$G(z,\eta) = g_{i\bar{j}}(z,\eta)\mathrm{d}z^i \otimes \mathrm{d}\bar{z}^j + a(L)g_{i\bar{j}}(z,\eta)\delta\eta^i \otimes \delta\bar{\eta}^j, \tag{1.1}$$

where $a : \mathrm{Im}(L) \subset \mathbb{R}_+ \to \mathbb{R}_+$, and $L := F^2$.

We define an almost hyper-complex structure (G, J_1, J_2), on the complexified holomorphic tangent bundle $T'M$ of a complex manifold M, where J_1 is the natural complex structure and J_2 is an almost complex structure defined by the help of $a = a(L)$. We demonstrate that $(T'M, J_1, J_2, J_3)$ is a commutative quaternion structure, [Mu2], where $J_3 = J_1 \circ J_2$.

In the rest of the § 4 we are concerned with the integrability conditions of the structures. How J_1 is the natural complex structure, his Nijenhuis tensor field is vanishing, but the case of J_2 is more complicated. Theorem 3.3. is the main result of this section, and tells when the (J_1, J_2) structure is integrable.

[1]Transilvania University of Braşov, Faculty of Mathematics and Informatics Iuliu Maniu 50, Braşov 500091, Romania, e-mail: am.szasz@yahoo.com

In Section 4 we evaluate under what conditions $(T'M, G, J_1, J_2)$ an almost hyper-Hermitian is almost hyper-Kählerian or at least hyper-Kählerian structure. This conditions are in the Proposition 4.1., Theorem 4.2. and Theorem 4.3.

The last part ot this paper is about the construction of a metric compatible linear connection with the commutative quaternion structure (J_1, J_2).

2 Preliminaries

Let M be a complex manifold, $\dim_C M = n$ and (z^k) local complex coordinates in a chart (U, φ). The holomorphic tangent bundle $T'M$ has a natural structure of complex manifold, $\dim_C T'M = 2n$, and the induced coordinates in a local chart in $u \in T'M$ are $u = (z^k, \eta^k)$. The changes of local coordinates in u are given by:

$$\begin{aligned} \frac{\partial}{\partial z^k} &= \frac{\partial z'^h}{\partial z^k}\frac{\partial}{\partial z'^h} + \frac{\partial^2 z'^h}{\partial z^j \partial z^k}\frac{\partial}{\partial \eta'^h}; \\ \frac{\partial}{\partial \eta^k} &= \frac{\partial z'^h}{\partial z^k}\frac{\partial}{\partial \eta'^h}. \end{aligned} \tag{2.2}$$

Consider the sections of the complexified tangent bundle of $T'M$. Let $VT'M \subset T'(T'M)$ be the vertical bundle, locally spanned by $\{\frac{\partial}{\partial \eta^k}\}$, and $VT''M$ its conjugate. The idea of complex nonlinear connection, briefly $(c.n.c.)$, is an instrument in 'linearization' of the geometry of $T'M$ manifold. A $(c.n.c.)$ is a supplementary complex subbundle to $VT'M$ in $T'(T'M)$, i.e. $T'(T'M) = HT'M \oplus VT'M$. The horizontal distribution $H_u T'M$ is locally spanned by

$$\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}, \tag{2.3}$$

where $N_k^j(z, \eta)$ are the coefficients of the $(c.n.c.)$. The pair $\{\delta_k := \frac{\delta}{\delta z^k}, \dot{\partial}_k := \frac{\partial}{\partial \eta^k}\}$ will be called the adapted frame of the $(c.n.c.)$ which obey to the change rules $\delta_k = \frac{\partial z'^j}{\partial z^k}\delta'_j$ and $\dot{\partial}_k = \frac{\partial z'^j}{\partial z^k}\dot{\partial}'_j$. By conjugation everywhere we obtain an adapted frame $\{\delta_{\bar{k}}, \dot{\partial}_{\bar{k}}\}$ on $T''_u(T'M)$. The dual adapted bases are $\{dz^k, \delta\eta^k\}$ and $\{d\bar{z}^k, \delta\bar{\eta}^k\}$.

The action of natural complex structure on $T_C(T'M)$ is

$$J(\partial_k) = i\partial_k; \quad J(\dot{\partial}_k) = i\dot{\partial}_k; \quad J(\partial_{\bar{k}}) = -i\partial_{\bar{k}}; \quad J(\dot{\partial}_{\bar{k}}) = i\dot{\partial}_{\bar{k}} \tag{2.4}$$

wich in view of (2.3) yields

$$J(\delta_k) = i\delta_k; \quad J(\dot{\partial}_k) = i\dot{\partial}_k; \quad J(\delta_{\bar{k}}) = -i\delta_{\bar{k}}; \quad J(\dot{\partial}_{\bar{k}}) = i\dot{\partial}_{\bar{k}} \tag{2.5}$$

and hence $H(T'M)$ and $\overline{H(T'M)}$ are J invariant.

The base manifold of a complex Finsler space is $T'M$ and the main objects of this geometry operate on the section of the complexified tangent bundle $T_C(T'M)$, which itself is decomposed into horizontal, vertical and their conjugates subbundles by a complex nonlinear connection N, uniquely determined by the complex Finsler function, [3], [5].

Definition 2.1. *A complex Finsler metric on M is a continuous function $F : T'M \to \mathbb{R}_+$ satisfying:*

(a) $L := F^2$ *is smooth on* $\widetilde{T'M} := TM \setminus \{0\}$.

(b) $F(z,\eta) \geq 0$, *the equality holds if and only if* $\eta = 0$;

(c) $F(z,\lambda\eta) = |\lambda|\, F(z,\eta)$ *for* $\forall \lambda \in \mathbb{C}$;

(d) the Hermitian matrix

$$g_{i\bar{j}}(z,\eta) = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$$

is positive-definite on $\widetilde{T'M}$.

Definition 2.2. *The pair (M, F) is called a complex Finsler space.*

The assertion (c) says that L is positively homogeneous with respect to the complex norm, i.e. $L(z,\lambda\eta) = \lambda\bar{\lambda}L(z,\eta)$ for any $\lambda \in \mathbb{C}$. The assertion (d) allows us to define a Hermitian metric structure on $T'M$, because $g_{i\bar{j}}$ is a d-tensor complex nondegenerate, called in [5] as *the fundamental metric tensor* of the complex Finsler space (M, F), with the inverse $g^{\bar{j}i}$, and $g^{\bar{j}i}g_{i\bar{k}} = \delta^{\bar{j}}_{\bar{k}}$.

The homogenity condition of the complex Finsler metric allows us to enumerate some important results. Applying the Euler's Theorem for $L = F^2$, we have:

Proposition 2.1. *The complex Finsler metric satisfies the conditions*

(a) $\frac{\partial L}{\partial \eta^k}\eta^k = L$; $\frac{\partial L}{\partial \bar{\eta}^k}\bar{\eta}^k = L$;

(b) $g_{i\bar{j}}\eta^i = \frac{\partial L}{\partial \bar{\eta}^j}$; $g_{i\bar{j}}\bar{\eta}^j = \frac{\partial L}{\partial \eta^i}$; $L = g_{i\bar{j}}\eta^i\bar{\eta}^j$;

(c) $\frac{\partial g_{i\bar{j}}}{\partial \eta^k}\eta^k = 0$; $\frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k}\bar{\eta}^k = 0$; $\frac{\partial g_{i\bar{j}}}{\partial \eta^k}\eta^i = 0$;

(d) $g_{ij}\eta^i = 0$; $\frac{\partial g_{i\bar{j}}}{\partial \eta^k}\bar{\eta}^j = g_{ik}$, *where* $g_{ij} = \frac{\partial^2 L}{\partial \eta^i \partial \eta^j}$.

A fundamental problem in a complex Finsler space remains that of determinating the $(c.n.c)$ function only on complex Finsler metric F. A well-known solution is provided by the complex Chern-Finsler connection, [3]. Determined from the technique of good vertical connection, it is proved that the *Chern-Finsler connection* is a unique $N-(c.l.c)$ of $(1,0)$-type. With the notations in [5], the Chern-Finsler connection is: $\overset{CF}{D\Gamma} = (\overset{CF}{N^i_j}, \overset{CF}{L^i_{jk}}, \overset{CF}{C^i_{jk}})$, where

$$\overset{CF}{N^i_j} = g^{\bar{m}i}\frac{\partial g_{l\bar{m}}}{\partial z^j}\eta^l = g^{\bar{m}i}\frac{\partial^2 L}{\partial z^j \partial \bar{\eta}^m}; \quad \overset{CF}{L^i_{jk}} = g^{\bar{m}i}\frac{\delta g_{j\bar{m}}}{\delta z^k}; \quad \overset{CF}{C^i_{jk}} = g^{\bar{m}i}\frac{\partial g_{j\bar{m}}}{\partial \eta^k}, \tag{2.6}$$

and $\overset{CF}{L^{\bar{i}}_{\bar{j}k}} = \overset{CF}{C^{\bar{i}}_{\bar{j}k}} = 0$.

With a straightforward computation we obtained that $\overset{CF}{L^i_{jk}} = \dot{\partial}_j \overset{CF}{N^i_k}$.

Observation 2.1. *As a direct consequance of (2.6) it results*

$$\delta_k L = \delta_{\bar{k}} L = \delta_{\bar{k}} \left(\frac{\partial L}{\partial \eta^j} \right) = 0 \tag{2.7}$$

Observation 2.2. *We will use the notation for the Chern-Finsler connection whithout the indexes* ^{CF}.

Locally, in adapted frame fields of the (*c.n.c*) Chern-Finsler N, the components of the Lie brackets are:

$$\begin{aligned} [\delta_j, \delta_k] &= (\delta_k N^i_j - \delta_j N^i_k)\dot{\partial}_i = 0; \\ [\delta_j, \delta_{\bar{k}}] &= (\delta_{\bar{k}} N^i_j)\dot{\partial}_i - (\delta_j N^{\bar{i}}_{\bar{k}})\dot{\partial}_{\bar{i}}; \\ \left[\delta_j, \dot{\partial}_k\right] &= (\dot{\partial}_k N^i_j)\dot{\partial}_i; \\ \left[\delta_j, \dot{\partial}_{\bar{k}}\right] &= (\dot{\partial}_{\bar{k}} N^i_j)\dot{\partial}_i; \\ \left[\dot{\partial}_j, \dot{\partial}_k\right] &= 0; \ \left[\dot{\partial}_j, \dot{\partial}_{\bar{k}}\right] = 0; \end{aligned} \tag{2.8}$$

A simple computation get:

$$(\dot{\partial}_{\bar{k}} N^i_j) g_{i\bar{m}} = (\dot{\partial}_{\bar{m}} N^i_j) g_{i\bar{k}}. \tag{2.9}$$

Now we can add that the nonzero torsion of the complex Chern-Finsler connection are only:

$$T^l_{jk} = L^l_{jk} - L^l_{kj} = \dot{\partial}_j N^l_k - \dot{\partial}_k N^l_j; \quad Q^l_{jk} = C^l_{jk}; \quad \Theta^l_{j\bar{k}} = \delta_{\bar{k}} N^l_j; \quad \rho^l_{j\bar{k}} = \dot{\partial}_{\bar{k}} N^l_j \tag{2.10}$$

In the terminology of Abate and Patrizio, [3], the complex Finsler space (M, F) is strongly Kähler iff $T^i_{jk} = 0$, Kähler iff $T^i_{jk}\eta^k = 0$, and weakly Kähler iff $g_{i\bar{l}} T^i_{jk}\eta^k\bar{\eta}^l = 0$. In [4] is proved that the strongly Kähler and the Kähler notions actually coincide.

3 Integrabilty of (J_1, J_2, J_3) structure

Consider a *generalized Sasaki metric* G on $T'M$ given by

$$G(z, \eta) = g_{i\bar{j}}(z, \eta) \mathrm{d}z^i \otimes \mathrm{d}\bar{z}^j + a(L) g_{i\bar{j}}(z, \eta) \delta\eta^i \otimes \delta\bar{\eta}^j, \tag{3.11}$$

where $a : \mathrm{Im}(L) \subset \mathbb{R}_+ \to \mathbb{R}_+$.

Let J_1 the natural complex structure on $T'M$), and J_2 an other almost complex structure on $T'M$ defined by:

$$\begin{aligned} J_1(\delta_k) = i\delta_k \quad &; \quad J_2(\delta_k) = \frac{1}{\sqrt{a}}\dot{\partial}_k \\ J_1(\delta_{\bar{k}}) = -i\delta_{\bar{k}} \quad &; \quad J_2(\delta_{\bar{k}}) = \frac{1}{\sqrt{a}}\dot{\partial}_{\bar{k}} \\ J_1(\dot{\partial}_k) = i\dot{\partial}_k \quad &; \quad J_2(\dot{\partial}_k) = -\sqrt{a}\delta_k \\ J_1(\dot{\partial}_{\bar{k}}) = -i\dot{\partial}_{\bar{k}} \quad &; \quad J_2(\dot{\partial}_{\bar{k}}) = -\sqrt{a}\delta_{\bar{k}}. \end{aligned} \tag{3.12}$$

We will denote whith $J_3 := J_1 \circ J_2$. The following relations are true:

$$\begin{aligned} J_1^2 &= J_2^2 = -I, \quad J_3^2 = I \\ J_1 J_2 &= J_2 J_1 = J_3 \\ J_1 J_3 &= J_3 J_1 = -J_2 \\ J_2 J_3 &= J_3 J_2 = -J_1 \end{aligned} \tag{3.13}$$

Using (3.12) and (3.13), according to [6], we obtain the next theorem:

Theorem 3.1. $(T'M, J_1, J_2, J_3)$ *is a commutative quaternion structure.*

Now we shall study the integrability problem for the obtained almost commutative quaternion structure. The integrability conditions for such a structure are expressed with the help of various Nijenhuis tensor fields obtained from the tensor fields $J_1, J_2, J_3 = J_1 J_2$. For a tensor field K of type (1,1) on a given manifold, we can consider its Nijenhuis tensor field N_K defined by

$$N_K(X,Y) = [KX, KY] - K[X, KY] - K[KX, Y] + K^2[X,Y],$$

where X, Y are vector fields on the given manifold. For two tensor fields K, L of type (1,1) on the given manifold, we can consider the corresponding Nijenhuis tensor field $N_{K,L}$ defined by

$$\begin{aligned} N_{K,L}(X,Y) &= [KX, LY] + [LX, KY] - K([X, LY] + [LX, Y]) - \\ &\quad -L([KX, Y] + [X, KY]) + (KL + LK)[X,Y]. \end{aligned}$$

The almost commutative quaternion structure defined by (J_1, J_2, J_3) is integrable if $N_1 = 0$, $N_2 = 0$, where N_1, N_2 are the Nijenhuis tensor fields of J_1, J_2. Equaivalently, the structure is integrable if $N_1 + N_2 + N_3 = 0$, or if $N_{12} = 0$, where N_3 is the Hijenhuis tensor field of $J_3 = J_1 J_2$ and N_{12} is the NIjenhuis tensor field of J_1, J_2.

Since J_1 is the natural complex structure, then it is integrable, i.e.

$$N_1 = 0.$$

Remains the study of N_2. With the help of the Lie brackets (2.8) we have obtained:

$$\begin{aligned}
N_2[\delta_j, \delta_k] &= \frac{a'}{2a^2}\left(\frac{\partial L}{\partial \eta^j}\delta_k^l - \frac{\partial L}{\partial \eta^k}\delta_j^l\right)\dot{\partial}_l + (L^i_{kj} - L^i_{jk})\delta_l \\
N_2(\delta_j, \delta_{\bar{k}}) &= \frac{a'}{2a^2}\left(\frac{\partial L}{\partial \bar{\eta}^k}\dot{\partial}_j - \frac{\partial L}{\partial \eta^j}\dot{\partial}_{\bar{k}}\right) - (\delta_{\bar{k}} N_j^l)\dot{\partial}_l + (\delta_j N_{\bar{k}}^{\bar{l}})\dot{\partial}_{\bar{l}} - (\dot{\partial}_j N_{\bar{k}}^{\bar{l}})\delta_{\bar{l}} + (\dot{\partial}_{\bar{k}} N_j^l)\delta_l \\
N_2(\delta_j, \dot{\partial}_k) &= \frac{a'}{2a}\left(\frac{\partial L}{\partial \eta^k}\delta_j - \frac{\partial L}{\partial \eta^j}\delta_k\right) + (\dot{\partial}_j N_k^l)\dot{\partial}_l - (\dot{\partial}_k N_j^l)\dot{\partial}_l \\
N_2(\delta_j, \dot{\partial}_{\bar{k}}) &= \frac{a'}{2a}\left(\frac{\partial L}{\partial \bar{\eta}^k}\delta_j - \frac{\partial L}{\partial \eta^j}\delta_{\bar{k}}\right) + a\left((\delta_j N_{\bar{k}}^{\bar{l}})\delta_{\bar{l}} - (\delta_{\bar{k}} N_j^l)\delta_l\right) + (\dot{\partial}_j N_{\bar{k}}^{\bar{l}})\dot{\partial}_{\bar{j}} - (\dot{\partial}_{\bar{k}} N_j^l)\dot{\partial}_l \\
N_2(\dot{\partial}_j, \dot{\partial}_k) &= \frac{a'}{2a}\left(\frac{\partial L}{\partial \eta^j}\dot{\partial}_k - \frac{\partial L}{\partial \eta^k}\dot{\partial}_j\right) + a\left((\dot{\partial}_j N_k^l)\delta_l - (\dot{\partial}_k N_j^l)\delta_l\right) \\
N_2(\dot{\partial}_j, \dot{\partial}_{\bar{k}}) &= \frac{a'}{2a}\left(\frac{\partial L}{\partial \eta^j}\dot{\partial}_k - \frac{\partial L}{\partial \bar{\eta}^k}\dot{\partial}_j\right) + a\left((\delta_{\bar{k}} N_j^l)\dot{\partial}_l - (\delta_j N_{\bar{k}}^{\bar{l}})\dot{\partial}_{\bar{l}} + (\dot{\partial}_j N_{\bar{k}}^{\bar{l}})\delta_{\bar{l}} - (\dot{\partial}_{\bar{k}} N_j^l)\delta_l\right)
\end{aligned}$$

We have replaced the expressions for the torsions (2.10) in the above relations, and is obtain:

$$\begin{aligned}
N_2(\delta_j,\delta_k) &= \frac{a'}{2a^2}\left(\frac{\partial L}{\partial\eta^j}\delta^l_k-\frac{\partial L}{\partial\eta^k}\delta^l_j\right)\dot\partial_l-T^l_{jk}\delta_l\\
N_2(\delta_j,\delta_{\bar k}) &= \frac{a'}{2a^2}\left(\frac{\partial L}{\partial\bar\eta^k}\dot\partial_j-\frac{\partial L}{\partial\eta^j}\dot\partial_{\bar k}\right)-\rho^{\bar l}_{\bar jk}\delta_{\bar l}+\rho^l_{j\bar k}\delta_l-\Theta^l_{j\bar k}\dot\partial_l+\Theta^{\bar l}_{\bar kj}\dot\partial_{\bar l}\\
N_2(\delta_j,\dot\partial_k) &= \frac{a'}{2a}\left(\frac{\partial L}{\partial\eta^k}\delta^l_j-\frac{\partial L}{\partial\eta^j}\delta^l_k\right)\delta_l+T^l_{jk}\dot\partial_l\\
N_2(\delta_j,\dot\partial_{\bar k}) &= \frac{a'}{2a}\left(\frac{\partial L}{\partial\bar\eta^k}\delta_j-\frac{\partial L}{\partial\eta^j}\delta_{\bar k}\right)+a\left(\Theta^{\bar l}_{\bar kj}\delta_{\bar l}-\Theta^l_{j\bar k}\delta_l\right)+\rho^{\bar l}_{\bar kj}\dot\partial_{\bar l}-\rho^l_{j\bar k}\dot\partial_l\\
N_2(\dot\partial_j,\dot\partial_k) &= \frac{a'}{2a}\left(\frac{\partial L}{\partial\eta^j}\delta^l_k-\frac{\partial L}{\partial\eta^k}\delta^l_k\right)\dot\partial_l+aT^l_{jk}\delta_l\\
N_2(\dot\partial_j,\dot\partial_{\bar k}) &= \frac{a'}{2a}\left(\frac{\partial L}{\partial\eta^j}\dot\partial_{\bar k}-\frac{\partial L}{\partial\bar\eta^k}\dot\partial_j\right)+a\left(\Theta^l_{j\bar k}\dot\partial_l-\Theta^{\bar l}_{\bar kj}\dot\partial_{\bar l}+\rho^{\bar l}_{\bar kj}\delta_{\bar l}-\rho^l_{j\bar k}\delta_l\right)
\end{aligned}$$

From the linear independence of the base fields it results that $(T'M,G,J_2)$ is complex if and only if:

$$\begin{aligned}
a'\left(\frac{\partial L}{\partial\eta^j}\delta^l_k-\frac{\partial L}{\partial\eta^k}\delta^l_j\right) &= 0 \text{ and } \quad T^l_{jk}=0\\
\Theta^l_{j\bar k} &= 0 \text{ and } \rho^l_{j\bar k}=0,
\end{aligned} \tag{3.14}$$

and their conjugates.

Theorem 3.2. *The manifold $(T'M,G,J_2)$ is complex if and only if (M,F) is Kähler, the torsions $\Theta^l_{j\bar k}$ and $\rho^l_{j\bar k}$ are zero and*

$$a'\left(\frac{\partial L}{\partial\eta^j}\delta^l_k-\frac{\partial L}{\partial\eta^k}\delta^l_j\right)=0. \tag{3.15}$$

Corollary 3.1. *$(T'M,G,J_2)$ is a complex manifold if and only if (M,F) is a generalized complex Berwald space and $\Theta^i_{j\bar k}=0$.*

Observation 3.1. *The notion of generalized complex Berwald space is described in [AM].*

We have seen, that J_1 is integrable, J_2 is integrable when the conditions in the Theorem 3.2. are fullfield, then the integrability condition for the (J_1,J_2,J_3) quaternion structure are in the next theorem:

Theorem 3.3. *The commutative quaternion structure (J_1,J_2,J_3) is intergable if and only if (M,F) is a generalized complex Berwald space and $\Theta^i_{j\bar k}=0$.*

4 Hyper-Kähler Structures on $T'M$

The structure defined in (3.13) is one hypercomplex four dimensional. Moreover, $(T'M, G, J_1, J_2)$ has an almost hyper-Kählerian structure if the following conditions are satisfied:

(a) $(T'M, G, J_1, J_2)$ is an almost hyper-Hermitian manifold;

(b) The fundamental 4-form Ω is closed.

For the point (a) we shall be interested in the conditions under wich the metric G is almost Hermitian with respect to the almost complex structures $J_1,\ J_2$, considered in (3.12), i.e.

$$G(J_1X, J_1Y) = G(X, Y) \quad G(J_2X, J_2Y) = G(X, Y),\ \forall X, Y \in T_C(T'M).$$

We have

Proposition 4.1. *$(T'M, G, J_1)$ $(T'M, G, J_2)$ and $(T'M, G, J_3)$ are almost Hermitian manifolds, i.e. $G(JX, JY) = G(X, Y)$ $\forall X, Y$.*

Proof. For J_1 the condition $G(J_1X, J_1Y) = G(X, Y)$ is verified imediatly, and for J_2 it's enough to verify for the elements of the adapted frame $\left\{\delta_k, \dot\partial_k, \delta_{\bar k}, \dot\partial_{\bar k}\right\}$ the above relations. The nonzero values of $G(J_2X, J_2Y)$ are

$$\begin{aligned} G(J_2\delta_j, J_2\delta_{\bar k}) &= G(\frac{1}{\sqrt{a}}\dot\partial_j, \frac{1}{\sqrt{a}}\dot\partial_{\bar k}) = \frac{1}{a}G(\dot\partial_j, \dot\partial_{\bar k}) = \\ &= \frac{1}{a}\cdot a g_{j\bar k} = g_{j\bar k} = G(\delta_j, \delta_{\bar k}) \\ G(J_2\dot\partial_j, J_2\dot\partial_{\bar k}) &= G(-\sqrt{a}\delta_j, -\sqrt{a}\delta_{\bar k}) = aG(\delta_j, \delta_{\bar k}) = \\ &= a\cdot g_{j\bar k} = G(\dot\partial_j, \dot\partial_{\bar k}), \end{aligned}$$

■

For the almost hyper-Hermitian manifold $(T'M, G, J_1, J_2)$ the fundamental 2−forms ϕ_1, ϕ_2 are defined by

$$\phi_1(X, Y) = G(X, J_1Y), \quad \phi_2(X, Y) = G(X, J_2Y),$$

where X, Y are vector fields on sections of $T_C(T'M)$.

Since we have a third almost complex structure $J_3 = J_1J_2$ which is almost Hermitian with respect to G, we can consider a third 2−form ϕ_3 defined by $\phi_3(X, Y) = G(X, J_3Y)$, next we have the fundamental 4-form Ω, defined by

$$\Omega = \phi_1 \wedge \phi_1 + \phi_2 \wedge \phi_2 + \phi_3 \wedge \phi_3.$$

The almost hyper-Hermitian manifold $(T'M, G, J_1, J_2)$ is almost hyper-Kählerian if the fundamental 4-form Ω is closed, i.e. $\mathrm{d}\Omega = 0$. The condition for Ω to be closed is equivalent to the conditions for ϕ_1, ϕ_2 (and hence for ϕ_3 too) to be closed, i.e. $\mathrm{d}\phi_1 = 0, \mathrm{d}\phi_2 = 0$. In our case, it is more convenient to study the conditions under which the 2-forms ϕ_1, ϕ_2 are closed.

The expressions of ϕ_1, ϕ_2 in adapted local frames are

$$\phi_1(z,\eta) = -ig_{j\bar{k}}dz^i \wedge d\bar{z}^j - ia(L)g_{j\bar{k}}\delta\eta^j \wedge \delta\bar{\eta}^k. \tag{4.16}$$

$$\phi_2(z,\eta) = -\sqrt{a(L)}g_{j\bar{k}}\mathrm{d}z^j \wedge \delta\bar{\eta}^k + \sqrt{a(L)}g_{j\bar{k}}\delta\eta^j \wedge \mathrm{d}\bar{z}^k. \tag{4.17}$$

With a straightforward computation using properties of the Chern-Finsler (*c.n.c.*) results

$$\begin{aligned}
\mathrm{d}\phi_1 \;\; = \;\; & -i\Big\{\delta_i g_{j\bar{k}}\mathrm{d}z^i \wedge \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k + \delta_{\bar{\imath}} g_{j\bar{k}}\mathrm{d}\bar{z}^i \wedge \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k + \\
+ \;\; & \dot{\partial}_i(ag_{j\bar{k}})\delta\eta^j \wedge \delta\bar{\eta}^k \wedge \delta\eta^i + \dot{\partial}_{\bar{\imath}}(ag_{j\bar{k}})\delta\eta^j \wedge \delta\bar{\eta}^k \wedge \delta\bar{\eta}^i + \\
+ \;\; & \Big[\dot{\partial}_i g_{j\bar{k}}\delta\eta^i \wedge \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k + ag_{j\bar{k}}\delta_h\overline{(N_l^k)}\delta\eta^j \wedge \mathrm{d}\bar{z}^l \wedge \mathrm{d}z^h\Big] + \\
+ \;\; & \Big[\dot{\partial}_{\bar{\imath}} g_{j\bar{k}}\delta\bar{\eta}^i \wedge \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k - ag_{j\bar{k}}\delta_{\bar{h}}(N_l^j)\delta\bar{\eta}^k \wedge \mathrm{d}z^l \wedge \mathrm{d}\bar{z}^h\Big] + \\
+ \;\; & \Big[\delta_i(ag_{j\bar{k}})\mathrm{d}z^i \wedge \delta\eta^j \wedge \delta\bar{\eta}^k - ag_{j\bar{k}}\dot{\partial}_h(N_l^j)\mathrm{d}z^l \wedge \delta\eta^h \wedge \delta\bar{\eta}^k\Big] + \\
+ \;\; & \Big[\delta_{\bar{\imath}}(ag_{j\bar{k}})\mathrm{d}\bar{z}^i \wedge \delta\eta^j \wedge \delta\bar{\eta}^k + ag_{j\bar{k}}\dot{\partial}_{\bar{h}}\overline{(N_l^k)}\delta\eta^j \wedge \mathrm{d}\bar{z}^l \wedge \delta\bar{\eta}^h\Big] - \\
- \;\; & ag_{j\bar{k}}\dot{\partial}_{\bar{h}}(N_l^j)\mathrm{d}z^l \wedge \delta\bar{\eta}^h \wedge \delta\bar{\eta}^k + ag_{j\bar{k}}\dot{\partial}_h\overline{(N_l^k)}\delta\eta^j \wedge \mathrm{d}\bar{z}^l \wedge \delta\eta^h\Big\} = \\
= \;\; & -i\Big\{\frac{1}{2}(\delta_i g_{j\bar{k}} - \delta_j g_{i\bar{k}})\mathrm{d}z^i \wedge \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k + \frac{1}{2}(\delta_{\bar{\imath}} g_{j\bar{k}} - \delta_{\bar{\jmath}} g_{i\bar{k}})\mathrm{d}\bar{z}^i \wedge \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k + \\
& a'\frac{L}{\partial\eta^i}g_{j\bar{k}}\delta\eta^j \wedge \delta\bar{\eta}^k \wedge \delta\eta^i + a'\frac{L}{\partial\bar{\eta}^i}g_{j\bar{k}}\delta\eta^j \wedge \delta\bar{\eta}^k \wedge \delta\bar{\eta}^i + \\
& \Big[\dot{\partial}_i g_{j\bar{k}} - ag_{i\bar{l}}\delta_j\overline{(N_k^l)}\Big]\mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k \wedge \delta\eta^i + \Big[\dot{\partial}_{\bar{\imath}} g_{j\bar{k}} - ag_{l\bar{h}}\delta_{\bar{k}}(N_h^l)\Big]\mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k \wedge \delta\bar{\eta}^i\Big\}
\end{aligned}$$

So we have deduced

Theorem 4.1. *The manifold* $(T'M, G, J_1)$ *is Kähler if and only if:*

$$\begin{aligned}
\delta_i g_{j\bar{k}} &= \delta_j g_{i\bar{k}} \\
a'(L) = 0 &\Leftrightarrow a(L) = c \in \mathbb{R} \\
g^{\bar{l}i}\dot{\partial}_i g_{j\bar{k}} &= a\delta_j\overline{(N_k^l)}
\end{aligned} \tag{4.18}$$

and their conjugates.

Analogous for $\mathrm{d}\phi_2$ we have:

$$
\begin{aligned}
\mathrm{d}\phi_2 \;=\;& -\left(\frac{a'}{2\sqrt{a}}(\delta_i L)g_{j\bar{k}} + \sqrt{a}\delta_i g_{j\bar{k}}\right)\mathrm{d}z^j \wedge \delta\bar{\eta}^k \wedge \mathrm{d}z^i + \\
&+\left(\frac{a'}{2\sqrt{a}}(\delta_i L)g_{j\bar{k}} + \sqrt{a}\delta_i g_{j\bar{k}}\right)\delta\eta^j \wedge \mathrm{d}\bar{z}^k \wedge \mathrm{d}z^i - \\
&-\left(\frac{a'}{2\sqrt{a}}(\delta_{\bar{i}} L)g_{j\bar{k}} + \sqrt{a}\delta_{\bar{i}} g_{j\bar{k}}\right)\mathrm{d}z^j \wedge \delta\bar{\eta}^k \wedge \mathrm{d}\bar{z}^i + \\
&+\left(\frac{a'}{2\sqrt{a}}(\delta_{\bar{i}} L)g_{j\bar{k}} + \sqrt{a}\delta_{\bar{i}} g_{j\bar{k}}\right)\delta\eta^j \wedge \mathrm{d}\bar{z}^k \wedge \mathrm{d}\bar{z}^i - \\
&-\left(\frac{a'}{2\sqrt{a}}(\dot{\partial}_i L)g_{j\bar{k}} + \sqrt{a}\dot{\partial}_i g_{j\bar{k}}\right)\mathrm{d}z^j \wedge \delta\bar{\eta}^k \wedge \delta\eta^i + \\
&+\left(\frac{a'}{2\sqrt{a}}(\dot{\partial}_i L)g_{j\bar{k}} + \sqrt{a}\dot{\partial}_i g_{j\bar{k}}\right)\delta\eta^j \wedge \mathrm{d}\bar{z}^k \wedge \delta\eta^i - \\
&-\left(\frac{a'}{2\sqrt{a}}(\dot{\partial}_{\bar{i}} L)g_{j\bar{k}} + \sqrt{a}\dot{\partial}_{\bar{i}} g_{j\bar{k}}\right)\mathrm{d}z^j \wedge \delta\bar{\eta}^k \wedge \delta\bar{\eta}^i + \\
&+\left(\frac{a'}{2\sqrt{a}}(\dot{\partial}_{\bar{i}} L)g_{j\bar{k}} + \sqrt{a}\dot{\partial}_{\bar{i}} g_{j\bar{k}}\right)\delta\eta^j \wedge \mathrm{d}\bar{z}^k \wedge \delta\bar{\eta}^i + \\
&+\sqrt{a}g_{j\bar{k}}\delta_h\overline{(N_l^k)}\mathrm{d}z^j \wedge \mathrm{d}\bar{z}^l \wedge \mathrm{d}z^h + \sqrt{a}g_{j\bar{k}}\dot{\partial}_{\bar{h}}\overline{(N_l^k)}\mathrm{d}z^j \wedge \mathrm{d}\bar{z}^l \wedge \delta\bar{\eta}^h \\
&+\sqrt{a}g_{j\bar{k}}\dot{\partial}_h\overline{(N_l^k)}\mathrm{d}z^j \wedge \mathrm{d}\bar{z}^l \wedge \delta\eta^h \\
&-\sqrt{a}g_{j\bar{k}}\delta_{\bar{h}}(N_l^j)\mathrm{d}z^l \wedge \mathrm{d}\bar{z}^h \wedge \mathrm{d}\bar{z}^k - \sqrt{a}g_{j\bar{k}}\dot{\partial}_h(N_l^j)\mathrm{d}z^l \wedge \delta\eta^h \wedge \mathrm{d}\bar{z}^k \\
&-\sqrt{a}g_{j\bar{k}}\dot{\partial}_{\bar{h}}(N_l^j)\mathrm{d}z^l \wedge \delta\bar{\eta}^h \wedge \mathrm{d}\bar{z}^k.
\end{aligned}
\tag{4.19}
$$

Theorem 4.2. *The almost complex manifold $(T'M, G, J_2)$ is almost Kähler if and only if one of the next condition sets are fullfield:*

$$a = 0 \;\; \textit{and} \;\; \dot{\partial}_i g_{j\bar{k}} = 0; \tag{4.20}$$

or

$$\delta_i g_{j\bar{k}} = \delta_j g_{i\bar{k}},\;\; \Theta_{\bar{l}h}^{\bar{k}} = 0,\;\; L_{ki}^l g_{l\bar{j}} = -\dot{\partial}_k\left(N_{\bar{j}}^{\bar{l}}\right)g_{i\bar{l}},\;\; \frac{a'}{2a}(\dot{\partial}_i L)g_{j\bar{k}} = -\dot{\partial}_i g_{j\bar{k}} \;\;, \tag{4.21}$$

and their conjugates.

Using the integrability conditions for J_2 in Theorem 3.2., we obtain:

Theorem 4.3. *The manifold $(T'M, G, J_2)$ is Kähler if and only if, one of the next condition sets are fullfield:*

$$a = 0 \;\; \textit{and } G \textit{ is purely Hermitian}; \tag{4.22}$$

or

$$L_{ki}^l g_{l\bar{j}} = 0,\;\; \frac{a'}{2a}(\dot{\partial}_i L)g_{j\bar{k}} = -\dot{\partial}_i g_{j\bar{k}}, \tag{4.23}$$

and their conjugates.

Corollary 4.1. *The structure* $(T'M, G, J_1, J_2, J_3)$ *is Hyper-Kählerian if and only if* (M, F) *is a complex Berwald manifold with* $\Theta^i_{j\bar{k}} = 0$, $a' = 0$, G *is purely Hermitian, and or* $a = 0$ *or* $L^l_{ki} g_{l\bar{j}} = 0$.

5 Metric compatible linear connection with the commutative quaternion structure

Further we will deal with linear connections compatible with a commutative quaternion metric structure.

Definition 5.1. *A linear connection* D *on* $T'M$ *is called metric connection commutative quaterion if:*

$$DJ_i = 0,\ i = 1, 2, 3; \quad and \quad DG = 0. \tag{5.24}$$

The general family of the linear connections D compatible with the metric G, according to [6], is

$$D_X Y = \check{D}_X Y + \frac{1}{2} g^{-1} (\check{D}_X g)_Y, \tag{5.25}$$

where $\check{D}$ is an arbitrary linear connection.
Let us consider the connection transformations:

$$\check{D}_X Y \xrightarrow{\mathcal{T}_1} D^1_X Y = \check{D}_X Y + \frac{1}{2} J_1 \check{D}_X (J_1 Y) \tag{5.26}$$

$$\check{D}_X Y \xrightarrow{\mathcal{T}_2} D^2_X Y = \check{D}_X Y + \frac{1}{2} J_2 \check{D}_X (J_2 Y) \tag{5.27}$$

$$\check{D}_X Y \xrightarrow{\mathcal{T}_3} D^3_X Y = \check{D}_X Y - \frac{1}{2} J_3 \check{D}_X (J_3 Y) \tag{5.28}$$

$$\check{D}_X Y \xrightarrow{\mathcal{T}_4} D^4_X Y = \check{D}_X Y + \frac{1}{2} (\check{D}_X g)_Y \tag{5.29}$$

where $(\check{D}_X g)_Y$ is a 1-form defined as follows $(\check{D}_X g)_Y Z = (\check{D}_X g)(Y, Z)$. Obviously $D^i_X J_i = 0,\ i = 1, 2, 3,\ X \in T_C(T'M)$.

Then, according to [6], we consider the commutative quaternion connection:

$$\tilde{D}_X Y = \frac{1}{4} \left\{ \check{D}_X Y - J_1(\check{D}_X J_1 Y) - J_2(\check{D}_X J_2 Y) + J_3(\check{D}_X J_3 Y) \right\} \tag{5.30}$$

where $\check{D}$ is an arbitrary linear connection.

Proposition 5.1. *The following relation is true:*

$$\tilde{D} D^4 = D^4 \tilde{D},$$

where $\tilde{D} D^4$ *(respectively* $D^4 \tilde{D}$*) is a connection obtained from* $\tilde{D}$ *(respectively* D^4*) by replacing* $\check{D}$ *with* D^4 *(respectively* $\tilde{D}$*).*

Proof.

$$
\begin{aligned}
(\tilde{D}D^4)_X Y &= \frac{1}{4}\left\{D^4_X Y - J_1(D^4_X J_1 Y) - J_2(D^4_X J_2 Y) + J_3(D^4_X J_3 Y)\right\} = \\
&= \frac{1}{4}\Big\{\check{D}_X Y + \frac{1}{2}G^{-1}(\check{D}_X G)_Y - J_1(\check{D}_X(J_1 Y) + \frac{1}{2}G^{-1}(\check{D}_X G)_{(J_1 Y)}) - \\
&- J_2(\check{D}_X(J_2 Y) + \frac{1}{2}G^{-1}(\check{D}_X G)_{(J_2 Y)}) + J_3(\check{D}_X(J_3 Y) + \frac{1}{2}G^{-1}(\check{D}_X G)_{(J_3 Y)})\Big\} = \\
&= \frac{1}{4}\left\{\check{D}_X Y - J_1(\check{D}_X(J_1 Y) - J_2(\check{D}_X(J_2 Y) + J_3(\check{D}_X(J_3 Y)\right\} + \\
&+ \frac{1}{8}\left\{G^{-1}(\check{D}_X G)_Y - J_1 G^{-1}(\check{D}_X G)_{(J_1 Y)} - J_2 G^{-1}(\check{D}_X G)_{(J_2 Y)} + J_3 G^{-1}(\check{D}_X G)_{(J_3 Y)}\right\}
\end{aligned}
\tag{5.31}
$$

On the other hand:

$$
\begin{aligned}
(D^4\tilde{D})_X Y &= \tilde{D}_X Y + \frac{1}{2}G^{-1}(\tilde{D}_X G)_Y = \\
&= \frac{1}{4}\left\{\check{D}_X Y - J_1(\check{D}_X J_1 Y) - J_2(\check{D}_X J_2 Y) + J_3(\check{D}_X J_3 Y)\right\} + \\
&+ \frac{1}{8}G^{-1}\left\{(\check{D}_X G)_Y - J_1(\check{D}_X G)_{J_1 Y} - J_2(\check{D}_X G)_{J_2 Y} + J_3(\check{D}_X G)_{J_3 Y}\right\} = \\
&= \frac{1}{4}\left\{\check{D}_X Y - J_1(\check{D}_X J_1 Y) - J_2(\check{D}_X J_2 Y) + J_3(\check{D}_X J_3 Y)\right\} + \\
&+ \frac{1}{8}\left\{G^{-1}(\check{D}_X G)_Y - G^{-1}J_1(\check{D}_X G)_{J_1 Y} - G^{-1}J_2(\check{D}_X G)_{J_2 Y} + G^{-1}J_3(\check{D}_X G)_{J_3 Y}\right\},
\end{aligned}
\tag{5.32}
$$

where $\tilde{D}_X G_Y Z = \tilde{D}_X G(Y, Z)$. Therefore $\tilde{D}D^4 = D^4\tilde{D}$. ∎

Theorem 5.1. *The following linear connection:*

$$D_X Y = (\tilde{D}D^4)_X Y, \qquad \mathcal{X}, Y \in T_C(T'M)$$

or equivalently

$$
\begin{aligned}
D_X Y &= \frac{1}{4}\left\{\check{D}_X Y - J_1(\check{D}_X J_1 Y) - J_2(\check{D}_X J_2 Y) + J_3(\check{D}_X J_3 Y)\right\} + \\
&+ \frac{1}{8}\left\{G^{-1}(\check{D}_X G)_Y - G^{-1}J_1(\check{D}_X G)_{J_1 Y} - G^{-1}J_2(\check{D}_X G)_{J_2 Y} + G^{-1}J_3(\check{D}_X G)_{J_3 Y}\right\}
\end{aligned}
\tag{5.33}
$$

is a metric commutative quaternion connection, where $\check{D}$ an arbitrary linear connection on $T'M$.

Proof. $D_X J_i = 0,\ i = 1, 2, 3$, because D is obtained from $\tilde{D}$ (that is commutative quaternion) by replacing the arbitrary connection with D^4.

Similary, $D_X G = 0$ because, based on Proposition 5.1., $D = D^4\tilde{D}$, i.e. D is obtained from the metric connection D^4 by replacing the arbitrary connection with $\tilde{D}$. ∎

Theorem 5.2. *If ∇ is the Levi-Civita connection defined by the metric G, then the connection*

$$\hat{D}_X Y = \frac{1}{4}\left\{\nabla_X Y - J_1(\nabla_X J_1 Y) - J_2(\nabla_X J_2 Y) + J_3(\nabla_X J_3 Y)\right\} \tag{5.34}$$

has properties

(a) $\hat{D}_X G = 0$, $\hat{D}_X J_i = 0$, $i = 1, 2, 3$, $X \in T_C(T'M)$;

(b) $\hat{D}$ *is uniquily determined by the metric commutative quaternion structure.*

Proof. Both results from (5.33) considering $\nabla_X G = 0$. ■

The local expression of the Levi-Civita connection defined by the metric G will be studied in a forthcoming paper.

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