Bulletin of the *Transilvania* University of Braşov • Vol 5(54), No. 1 - 2012
Series III: Mathematics, Informatics, Physics, 71-84

THE BERWALD-MOOR METRIC IN NILPOTENT DIRAC SPINOR SPACE

Peter ROWLANDS[1]

Communicated to the conference:
Finsler Extensions of Relativity Theory, Braşov, Romania, Aug. 20 - Sept. 4, 2011

Abstract

The nilpotent version of the Dirac equation can be constructed on the basis of the algebra of a double vector space or complexified double quaternions. This algebra is isomorphic to the standard gamma matrix algebra, with 64 units which can be produced by just 5 generators. The H4 algebra used in the Berwald-Moor metric is a distinct subalgebra of this 64-part algebra. The creation of the 5 generators requires the rotation symmetry of one of the two component vector spaces to be preserved while the other is broken. It is convenient to identify the respective spaces as an observable real space and an unobservable 'vacuum' space, with corresponding physical properties. In combination the 5 generators produce a nilpotent structure which can be identified as a fermionic wavefunction or solution of the Dirac equation. The spinors required to generate the 4 components of the wavefunction can be derived from first principles and have exactly the same form as the four components of the Berwald-Moor metric. They also incorporate the units of the H4 algebra in an identical way. The spinors produce a zero product which can be interpreted in terms of a fermionic singularity arising from the distortion introduced into the vacuum (or spinor) space by the application of a nilpotent condition.

2000 *Mathematics Subject Classification:* 53B40, 53B50, 81Q05, 81R05, 81R25.
Key words: Berwald-Moor metric, Dirac spinor space, nilpotent quantum mechanics, fermion singularit.

1 A dual vector space

We need to begin by describing a number of significant algebras. The four quaternion units, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, 1, follow the well-known multiplication rules:

$$\boldsymbol{i}^2 = k\boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{i}\boldsymbol{j}\boldsymbol{k} = -1 \tag{1}$$

[1]Department of Physics, University of Liverpool, Oliver Lodge Laboratory, Liverpool, L69 7ZE, UK, e-mail:- p.rowlands@liverpool.ac.uk

$$\boldsymbol{i}\boldsymbol{i} = -\boldsymbol{j}\boldsymbol{i} = \boldsymbol{k} \tag{2}$$

$$\boldsymbol{j}\boldsymbol{k} = -\boldsymbol{k}\boldsymbol{j} = \boldsymbol{i} \tag{3}$$

$$\boldsymbol{k}\boldsymbol{i} = -\boldsymbol{i}\boldsymbol{k} = \boldsymbol{j}. \tag{4}$$

The multivariate vector units, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, 1, are effectively complexified quaternions $(i\boldsymbol{i}) = \mathbf{i}$, $(i\boldsymbol{j}) = \mathbf{j}$, $(i\boldsymbol{k}) = \mathbf{k}$, $(i1) = i$, and follow the multiplication rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{1} \tag{5}$$

$$\mathbf{ij} = -\mathbf{ji} = i\mathbf{k} \tag{6}$$

$$\mathbf{jk} = -\mathbf{kk} = i\mathbf{i} \tag{7}$$

$$\mathbf{ki} = -\mathbf{ik} = i\mathbf{j}. \tag{8}$$

They are isomorphic to Pauli matrices. If we complexify this algebra, we revert to quaternions, so $(i\mathbf{i}) = \boldsymbol{i}$, $(i\mathbf{j}) = \boldsymbol{j}$, $(i\mathbf{k}) = \boldsymbol{k}$, etc. Multivariate vectors differ from ordinary vectors in having a full (algebraic) product:

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + i\mathbf{a} \times \mathbf{b} \tag{9}$$

from which all the rules concerning unit vector multiplication may be derived. Terms like $i\mathbf{i}$, $i\mathbf{j}$, $i\mathbf{k}$ are pseudovectors (e.g. area, angular momentum) and i is a pseudoscalar (e.g. volume). The units $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ define a complete Clifford algebra of 3D space:

$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	*vector*		
$i\mathbf{i}$	$i\mathbf{j}$	$i\mathbf{k}$	*bivector*	*pseudovector*	*quaternion*
i			*trivector*	*pseudoscalar*	
1			*scalar*		

Pseudovectors and pseudoscalars give us areas and volumes, etc. The intrinsic complexification produces a kind of doubling of the elements. Let us suppose we have another such algebra, isomorphic with the first:

$\mathbf{I}$	$\mathbf{J}$	$\mathbf{K}$	*vector*		
$i\mathbf{I}$	$i\mathbf{J}$	$i\mathbf{K}$	*bivector*	*pseudovector*	*quaternion*
i			*trivector*	*pseudoscalar*	
1			*scalar*		

If we combine these two algebras commutatively in a tensor product, or alternatively take the algebraic product of the eight base units, 1, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, i, $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$, we obtain 64 terms, which are + and − versions of:

$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	$i\mathbf{i}$	$i\mathbf{j}$	$i\mathbf{k}$	i	1
$\mathbf{I}$	$\mathbf{J}$	$\mathbf{K}$	$i\mathbf{I}$	$i\mathbf{J}$	$i\mathbf{K}$		
$\mathbf{iI}$	$\mathbf{jI}$	$\mathbf{kI}$	$i\mathbf{iI}$	$i\mathbf{jI}$	$i\mathbf{kI}$		
$\mathbf{iJ}$	$\mathbf{jJ}$	$\mathbf{kJ}$	$i\mathbf{iJ}$	$i\mathbf{jJ}$	$i\mathbf{kJ}$		
$\mathbf{iK}$	$\mathbf{jK}$	$\mathbf{kK}$	$i\mathbf{iK}$	$i\mathbf{jK}$	$i\mathbf{kK}$		

We can describe this as a double vector algebra or a double Clifford algebra of 3D space. Alternatively, we can take the algebraic product of the four quaternion units, 1, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$,and the four vector units i, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, to obtain + and − versions of:

$$
\begin{array}{llllllll}
\mathbf{i} & \mathbf{j} & \mathbf{k} & i\mathbf{i} & i\mathbf{j} & i\mathbf{k} & i & 1 \\
\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & i\boldsymbol{i} & i\boldsymbol{j} & i\boldsymbol{k} & & \\
\mathbf{i}\boldsymbol{i} & \mathbf{j}\boldsymbol{i} & \mathbf{k}\boldsymbol{i} & i\mathbf{i}\boldsymbol{i} & i\mathbf{j}\boldsymbol{i} & i\mathbf{k}\boldsymbol{i} & & \\
\mathbf{i}\boldsymbol{j} & \mathbf{j}\boldsymbol{j} & \mathbf{k}\boldsymbol{j} & i\mathbf{i}\boldsymbol{j} & i\mathbf{j}\boldsymbol{j} & i\mathbf{k}\boldsymbol{j} & & \\
\mathbf{i}\boldsymbol{k} & \mathbf{j}\boldsymbol{k} & \mathbf{k}\boldsymbol{k} & i\mathbf{i}\boldsymbol{k} & i\mathbf{j}\boldsymbol{k} & i\mathbf{k}\boldsymbol{k} & &
\end{array}
$$

This is exactly isomorphic to the previous algebra and can be described as a vector quaternion algebra. A third version of the same algebra could be obtained by complexifiying the algebraic product of two commutative sets of quaternion units $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, $\boldsymbol{I}$, $\boldsymbol{J}$, $\boldsymbol{K}$. This algebra has + and − versions of:

$$
\begin{array}{llllllll}
\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} & i\boldsymbol{i} & i\boldsymbol{j} & i\boldsymbol{k} & i & 1 \\
\boldsymbol{I} & \boldsymbol{J} & \boldsymbol{K} & i\boldsymbol{I} & i\boldsymbol{J} & i\boldsymbol{K} & & \\
\boldsymbol{iI} & \boldsymbol{jI} & \boldsymbol{kI} & i\boldsymbol{iI} & i\boldsymbol{jI} & i\boldsymbol{kI} & & \\
\boldsymbol{iJ} & \boldsymbol{jJ} & \boldsymbol{kJ} & i\boldsymbol{iJ} & i\boldsymbol{jJ} & i\boldsymbol{kJ} & & \\
\boldsymbol{iK} & \boldsymbol{jK} & \boldsymbol{kK} & i\boldsymbol{iK} & i\boldsymbol{jK} & i\boldsymbol{kK} & &
\end{array}
$$

This can be described as a complexified double quaternion algebra.

2 The gamma matrices and the H4 algebra

The three 64-part algebras are completely isomorphic. The units can be represented as a group of order 64, with a minimum of 5 generators. Their physical significance is that they are also isomorphic to the gamma algebra of the Dirac equation, based on 4 × 4 matrices. In fact all possible gamma matrices can be derived from the products of two commuting sets of Pauli matrices, say σ_1, σ_2, σ_3 and Σ_1, Σ_2, Σ_3. Relativistic quantum mechanics, it seems, requires a dual vector space. This is in addition to the 'doubling' produced by the complex nature of each vector space. 1, 2.

The 5 generators of the group can be matched to the 5 gamma matrices in a number of ways, for example:

$$\gamma_0 = i\boldsymbol{k}; \quad \gamma_1 = \boldsymbol{i}\mathbf{i}; \quad \gamma_2 = \boldsymbol{i}\mathbf{j}; \quad \gamma_3 = \boldsymbol{i}\mathbf{k}; \quad \gamma_5 = i\boldsymbol{j}.$$

There are many ways of doing this but the overall structure is always the same.
A particular subalgebra of the 64-part algebra is the H4 algebra. This can be obtained using coupled quaternions, with units 1, $\boldsymbol{iI}$, $\boldsymbol{jJ}$, $\boldsymbol{kK}$. The result is a cyclic but commutative algebra with multiplication rules

$$\boldsymbol{iI}\,\boldsymbol{iI} = \boldsymbol{jJ}\,\boldsymbol{jJ} = \boldsymbol{kK}\,\boldsymbol{kK} = 1 \tag{10}$$

$$\boldsymbol{iI}\,\boldsymbol{jJ} = \boldsymbol{jJ}\,\boldsymbol{iI} = \boldsymbol{kK} \tag{11}$$

$$\boldsymbol{jJ}\,\boldsymbol{kK} = \boldsymbol{kK}\,\boldsymbol{jJ} = \boldsymbol{iI} \tag{12}$$

$$\boldsymbol{kK}\,\boldsymbol{iI} = \boldsymbol{iI}\,\boldsymbol{kK} = \boldsymbol{jJ} \tag{13}$$

The same algebra can be achieved with the negative values of the paired vector units 1, $-\mathbf{iI}$, $-\mathbf{jJ}$, $-\mathbf{kK}$. (1 is equivalent here to $-ii$.) This time we have:

$$(-\mathbf{iI})(-\mathbf{iI}) = (-\mathbf{jJ})(-\mathbf{jJ}) = (-\mathbf{kK})(-\mathbf{kK}) = \mathbf{1} \tag{14}$$

$$(-\mathbf{iI})(-\mathbf{jJ}) = (-\mathbf{jJ})(-\mathbf{iI}) = (-\mathbf{kK}) \tag{15}$$

$$(-\mathbf{jJ})(-\mathbf{kK}) = (-\mathbf{kK})(-\mathbf{jJ}) = (-\mathbf{iI}) \tag{16}$$

$$(-\mathbf{kK})(-\mathbf{iI}) = (-\mathbf{iI})(-\mathbf{kK}) = (-\mathbf{jJ}) \tag{17}$$

If we use the symbols I $= \boldsymbol{iI} = -\mathbf{iI}$, J $= \boldsymbol{jJ} = -\mathbf{jJ}$, K $= \boldsymbol{kK} = -\mathbf{kK}$, 1, to represent this algebra, we can structure the relationships in a group table:

*		1	I	J	K
1		1	I	J	K
I		I	1	K	J
J		J	K	1	I
K		K	I	J	1

The group is a Klein-4 group, a noncyclic group of order 4.

3 Nilpotent quantum mechanics

One of the most significant aspects of the algebraic versions of the gamma algebra is that they allow us to create a very powerful and streamlined version of relativistic quantum mechanics. 1, 2 The simplest way to derive this is to begin with Einsteins energy-momentum conservation equation (with the usual convention that $c = 1$):

$$E^2 - p^2 - m^2 = 0 \tag{18}$$

We can now use our algebra to factorize this equation, Here we will use the combination of four quaternion units (1, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$) and four multivariate vector units (i, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$) though we could equally use the double vector or complex double quaternion algebras. The eight base units (1, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, i) have a similar structure to Penrose's twistors, 3 with four real or norm -1 components and four imaginary or norm 1 components. There is a significant difference, however, in that the connection between the units of space and time is a quantum rather than a classically relativistic one. Even in conventional relativistic quantum mechanics, the connection between space and time is not that of a true 4-vector, but rather one mediated by the gamma matrices, with different gammas applied to the space and time components. The algebra now allows us to factorize (18) in the form

$$(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{i}\mathbf{j}p_y + \boldsymbol{i}\mathbf{k}p_z + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{i}\mathbf{j}p_y + \boldsymbol{i}\mathbf{k}p_z + \boldsymbol{j}m) = \mathbf{0} \tag{19}$$

or

$$(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) = \mathbf{0}. \tag{20}$$

If we now apply a canonical quantization procedure to the first bracket in these squared expressions, to replace the terms i and $\mathbf{p}$ by the operators $E \to i\partial/\partial t$, $\mathbf{p} \to -i\nabla$ (this time equating $\hbar$ to 1), and assume that the operators act on the phase factor for a free fermion, $e^{-i(Et-\mathbf{p}.\mathbf{r})}$, we obtain the nilpotent Dirac equation for a free fermion:

$$\left(\mp\boldsymbol{k}\frac{\partial}{\partial t} \mp i\boldsymbol{i}\nabla + \boldsymbol{j}m\right)(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\mathbf{e}^{-i(Et-\mathbf{p}.\mathbf{r})} = \mathbf{0} \tag{21}$$

If we use a multivariate vector for the $\mathbf{p}$ or ∇ term it automatically includes spin (through the extra $\times$ term in the full product). 4 So, here, $\mathbf{p}$ is interchangeable with $\sigma.\mathbf{p}$ and ∇ with $\sigma.\nabla$. However, if we should revert to using ordinary vectors at any time, we would have to include an explicit spin or angular momentum term.

As usual, 4 simultaneous solutions are required for the wavefunction: 2 for fermion / antifermion $\times$ 2 for spin up / spin down. Rather than a 4×4 matrix differential operator and a column vector wavefunction, we use a row vector operator and a column vector wavefunction, each of which may be represented in abbreviated form by $(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$. In the nilpotent formalism, the four solutions can be represented as, say:

$(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	*fermion*	*spin*	*up*
$(i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	*fermion*	*spin*	*down*
$(-i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	*antifermion*	*spin*	*down*
$(-i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	*antifermion*	*spin*	*up*

The observed particle state is the first in the column, while the others are the accompanying vacuum states, or states into which the observed particle could transform by respective P, T and C transformations:

$$\begin{array}{lllll} P & \boldsymbol{i}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{i} & = & (i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \\ T & \boldsymbol{k}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{k} & = & (-i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \\ C & -\boldsymbol{j}(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{j} & = & (-i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \end{array}$$

Replacing the observed fermion state spin up with any of the others would simultaneously transform all four states by P, T or C. It is often convenient to specify just the first term, with the others assumed to be automatic consequences. The relation between the P, T, C transformations and vacuum can be shown in a relatively simple way. If we take $(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ and post-multiply it by the idempotent $\boldsymbol{k}(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ any number of times, the only effect is to introduce a scalar multiple, which can be normalized away.

$$(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{k}(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{k}(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \ldots \to (\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \tag{22}$$

Similarly with $(\boldsymbol{j}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m$ or $(\boldsymbol{i}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m$. All these idempotent quantities can be regarded as vacuum operators, and $\boldsymbol{k}$, $\boldsymbol{i}$ and $\boldsymbol{j}$, or, equivalently, $\mathbf{K}$, $\mathbf{I}$ and $\mathbf{J}$, as coefficients of a 'vacuum space'. Nilpotent quantum mechanics (NQM) produces all the standard results of conventional relativistic quantum mechanics, which can easily be obtained by replacing (21) with

$$-i\gamma_5 \left(\gamma_0 \frac{\partial}{\partial t} + \gamma_1 \frac{\partial}{\partial x} + \gamma_2 \frac{\partial}{\partial y} + \gamma_3 \frac{\partial}{\partial z} + im \right) = 0 \tag{23}$$

Standard classic results obtainable through NQM include spin $\frac{1}{2}$, one-handed helicity for weakly interacting states, and the *zitterbewegung* which emerges as an automatic switching process between the four states in the wavefunction, and which is interpreted as a mass-generating switching between the fermion and its antifermion vacuum partner, and the two helicity states, which are already mixed in real fermions. NQM also produces many new results. 1 Among the most important are the descriptions of three different boson-type states, which are combinations of the fermion state which any of the P, T or C transformed ones, the result being a scalar wavefunction.

$(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\mp i\boldsymbol{k}\mathbf{E} \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	spin 1 boson
$(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\mp i\boldsymbol{k}\mathbf{E} \mp \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	spin 0 boson
$(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\pm i\boldsymbol{k}\mathbf{E} \mp \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	fermion-fermion combination

One of the most significant aspects of this formalization is that a spin 1 boson can be massless, but a spin 0 boson cannot, as then $(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p})(\mp i\boldsymbol{k}\mathbf{E} \mp \boldsymbol{i}\mathbf{p})$ would immediately zero: hence Goldstone bosons must become Higgs bosons in the Higgs mechanism.

The key aspect of NQM, is the fact that an operator of the form $(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ automatically generates a phase term on which it operates to produce a nilpotent amplitude of the form $(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, that is, one that squares to zero. We don't really need an equation. The fermion needn't be free. We can incorporate field terms or covariant derivatives into the operator, with, for example, $E \rightarrow i\partial/\partial t + e\phi + ...$, and $\mathbf{p} \rightarrow -i\nabla + e\mathbf{A} + ...$. We can still represent the operator as $(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, but the phase term will no longer be $e^{-i(Et - \mathbf{p.r})}$. It will be whatever is needed to create an amplitude of the general form $(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, which squares to zero, with the eigenvalues E and $\mathbf{p}$ representing the more complicated expressions that will result from the presence of the field terms. In principle, this means that we can do relativistic quantum mechanics for a fermion in any state, subject to any number of interactions, simply by defining an operator of the form $(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$. This will then uniquely determine the phase factor which makes the amplitude nilpotent. There is no need to define any equation at all:

$$\text{operator acting on phase factor}^2 = \text{amplitude}^2 = 0. \tag{24}$$

In NQM the total structure of the universe is exactly zero. Pauli exclusion, a fundamentally nonlocal phenomenon, is an immediate consequence. Imagine creating a fermion wavefunction of the form $\psi_f = (i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ from absolutely nothing; then we must simultaneously create the dual term, 'vacuum', $\psi_f = -(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, which negates it both in superposition and combination:

$$\psi_f + \psi_v = (i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) - (i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) = \mathbf{0} \tag{25}$$

$$\psi_f \psi_v = -(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) = \mathbf{0} \tag{26}$$

Pauli exclusion then says that no two fermions share the same vacuum.

As an example of the power of NQM, we may show a calculation involving the Coulomb interaction. The $U(1)$ symmetry group for the Coulomb interaction comes from the characterization of a fermion as a point source with spherical symmetry. It is a purely scalar symmetry defined only by the magnitude of the charge, or source of the interaction. This is effectively equivalent to defining a coupling constant for the interaction, which maintains its value independent of the distance from the source. Here, we use a version of Dirac's standard prescription for converting the differential operator to polar coordinates,5 with the explicit inclusion of a fermionic spin / angular momentum term:

$$\left(\pm \boldsymbol{ik}E \pm \boldsymbol{i} \left(\frac{\partial}{\partial r} + \frac{1}{r} \pm i \left(\frac{j + \frac{1}{2}}{r} \right) \right) + \boldsymbol{j}m \right). \tag{27}$$

The fundamental condition necessary to assign this operator to a fermion state is that it maintains Pauli exclusion and leads to a nilpotent solution when applied to a phase factor. This leads to the local manifestation of the $U(1)$ symmetry. It can be seen, simply by inspection, that it will be impossible to obtain a nilpotent solution (i.e. a nilpotent amplitude) and Pauli exclusion with any phase factor unless the operator $\boldsymbol{ik}E$ includes a potential energy term varying with $1 \,/\, r$ to cancel out the effect of that in the $\boldsymbol{i}$ part of the operator. So, simply requiring spherical symmetry for a point particle, requires a term of the form $A \,/\, r$ to be added to E.

$$\left(\pm \boldsymbol{ik} \left(E + \frac{A}{r} \right) \pm \boldsymbol{i} \left(\frac{\partial}{\partial r} + \frac{1}{r} \pm i \left(\frac{j + \frac{1}{2}}{r} \right) \right) + \boldsymbol{j}m \right). \tag{28}$$

Deriving the solution for this case provides a model for all other cases. If all point particles are spherically symmetric sources, then the minimum nilpotent operator will be of the form (28). To establish that this is a nilpotent, we must now find the phase to which this must apply to create a nilpotent amplitude. This is a convenient example for showing how an operator fixes the phase factor and quite quickly produces the characteristic solution for the Coulomb force (hydrogen atom, etc.). The solution for (28) is relatively straightforward. The ease of calculation is due to the fact that the structure provides dual information about both fermion and vacuum. We apply the specified operator to the phase factor

$$e^{-ar} r^{\gamma} \sum_{\nu=0} a_{\nu} r^{\nu} \tag{29}$$

to find the amplitude (derived, as in the conventional solution, by inspired guesswork or trial and error), and equate the squared amplitude to zero.

$$\begin{aligned} 4\left(E + \frac{A}{r}\right)^2 \quad &= \quad -2\left(-a + \frac{\gamma}{r} + \frac{\nu}{r} + ... + i\left(\frac{j + \frac{1}{2}}{r}\right)\right) \\ &\quad -2\left(-a + \frac{\gamma}{r} + \frac{\nu}{r} + ... - i\left(\frac{j + \frac{1}{2}}{r}\right)\right) + 4m^2 \end{aligned} \tag{30}$$

Equating constant terms, we find:

$$a = \sqrt{m^2 - E^2} \tag{31}$$

Equating terms in $1/r^2$, with $\nu = 0$:

$$\gamma = -1 + \sqrt{\left(j + \frac{1}{2}\right)^2 - A^2} \tag{32}$$

Assuming the power series terminates at n', and equating coefficients of $1/r$ for $\nu = n'$:

$$2EA = -2\sqrt{m^2 - E^2}(\gamma + 1 + n') \tag{33}$$

and

$$\frac{E}{m} = \frac{1}{\sqrt{1 + \frac{A^2}{(\gamma+1+n')^2}}} = \frac{1}{\sqrt{1 + \frac{A^2}{\left(\sqrt{(j+\frac{1}{2})^2 - A^2} + n'\right)^2}}}. \tag{34}$$

When $A = Ze^2$ we have the 'hydrogen atom' solution in just 6 lines!

4 The fermion as a singularity

One way of looking at fermion structure is to say that it requires two 'spaces' to define a particle singularity. We can describe one of these as real space and the other as the 'vacuum space' which we have previously defined. This space is closely connected with charge and the weak, strong and electric interactions, as well as with T, P and C transformations. The generators of the combined 64-part algebra, significantly, require the symmetry of one space to be broken while the other is preserved:

$\mathbf{K}$	$i\mathbf{Ii}$ $i\mathbf{Ij}$ $i\mathbf{Ik}$	$i\mathbf{J}$
energy	*momentum*	*mass*
time	*space*	*proper time*

The space with the unbroken symmetry (lower case symbols) is real space, the space of observation. The space with the broken symmetry (upper case symbols) is 'vacuum space', the space of all unobservable quantities (time, mass, charge, etc). The creation of a singularity using these two spaces determines that they are precisely dual and that each contains the same information as the other, though in a different form as regards observation. The fermionic singularity produces an asymmetry or chirality in the space of observation because of its combination in the asymmetric nilpotent structure with the unobserved dual vacuum space.

The combination of fermion singularity and spatial duality has many manifestations: spin $\frac{1}{2}$ and nonzero rest mass occur because the fermion 'rotation' has to negotiate 2 spaces but with an observed asymmetry; *zitterbewegung* comes from the switching determined by the duality between the spaces; spin chirality of fermions emerges through exactly the same process as the chirality producing mass via *zitterbewegung*, because the spinor process produces an observed asymmetry between the spaces that are dual in their original formulation. And it becomes apparent that the *zitterbewegung* mass is exactly that produced by the chirality of vacuum space in the Higgs mechanism. Berry phase is an expression of the singularity of the fermion state and is equivalent to spin $\frac{1}{2}$ (topology with a singularity produces an extra twist, equivalent to $\frac{1}{2}$). The pole in the fermion propagator occurs at the boundary between observed space ($+E$) and vacuum space ($-E$), the combination which produces the singularity.

A possible analogy between the two spaces is, if we create a knot out of two pieces of string, say red and blue, but imagine that each doesn't know that the other exists (which is effectively the meaning of commutativity). We then imagine seeing the universe from the point of view of one of them, say, the blue string. From the blue perspective ('blue space' / lower case symbols), the blue string is straight, so we must devise some special contortion to create the state of the red string from the blue's perspective. The spatial 'double twist' becomes equivalent to a singularity, an additional structure within the space. (The paired quaternion / vector units, $\mathrm{I} = \boldsymbol{iI} = -\mathbf{iI}$, $\mathrm{J} = \boldsymbol{jJ} = -\mathbf{jJ}$, $\mathrm{K} = \boldsymbol{kK} = -\mathbf{kK}$, in fact, define a minimal degree of mathematical knottedness in that each operated on by one of the others produces the third, with no anticommutativity.)

Penrose has examined something similar from the point of view of twistor theory, which has a family resemblance to the algebra of the dual space in that it is constructed of four real units and four imaginary. Visually, the effect can be represented in the Robinson congruence. 3, 6 Penrose's theory, however, assumes a classical 4-dimensional relation between space and time or momentum and energy, while NQM requires a quantum connection to be made via 'vacuum space' ($\boldsymbol{k}$, $\boldsymbol{i}$, $\boldsymbol{j}$), or through the 'gamma matrices':

$i\boldsymbol{k}$	$\boldsymbol{i}\mathbf{i}$ $\boldsymbol{i}\mathbf{j}$ $\boldsymbol{i}\mathbf{k}$	$\boldsymbol{j}$
$\mathbf{K}$	$i\mathbf{Ii}$ $i\mathbf{Ij}$ $i\mathbf{Ik}$	$i\mathbf{J}$
energy	*momentum*	*mass*
time	*space*	*proper time*

In effect, Penrose has to eliminate the mass and take the scalar product of the space-time to preserve the 4-vector structure which he has privileged.

The twistors derive their dual 4-D vector space from the intrinsic duality of a 3-D vector space, in requiring vectors and pseudovectors. However, NQM really requires an additional duality – a dual dual space, which does not require an arbitrary extension to 4-D. The apparent '4-dimensionality' comes from a combination of 2 × 3-D. Mass emerges from this extra duality even if we assume that the intrinsic motion of the particles is at the speed of light. Defining a physical singularity in terms of two vector spaces produces mass, as well as spin $\frac{1}{2}$and chirality.

5 Defining a dual space spinor

In standard relativistic quantum mechanics, the wavefunction, say ψ, is multiplied by a 4-spinor, a summation of 4 terms which adds to 1. ψ is a solution of the Dirac equation, and so is ψ multiplied by any of the 4 terms in the spinor. Individual terms in the spinor are used as projection operators to project out individual states fermion / antifermion, spin up / down. The nilpotent formalism doesn't need spinors because the terms are already projected, but it is possible to set it up in such a way as spinors can be used. The most convenient way is to use both pre- and post-multiplication of ψ, as with the C, P, T operators. This dual multiplication emerges from the fact that the nilpotent wavefunction is already pre-multiplied by an algebraic operator, by comparison with the conventional one.

All the standard aspects of spin and helicity are easily recovered with NQM. This means that it is possible to find a spinor structure which will generate the NQM state vector. A set of primitive idempotents constructing a spinor can be defined in terms of the H4 algebra, constructed from the dual vector spaces:

$$
\begin{gathered}
(1-\mathbf{iI}-\mathbf{jJ}-\mathbf{kK})/4 \\
(1-\mathbf{iI}+\mathbf{jJ}+\mathbf{kK})/4 \\
(1+\mathbf{iI}-\mathbf{jJ}+\mathbf{kK})/4 \\
(1+\mathbf{iI}+\mathbf{jJ}-\mathbf{kK})/4
\end{gathered}
$$

As required the 4 terms add up to 1, and are orthogonal as well as idempotent, all products between them being 0. The same terms can be generated using coupled quaternions rather than vectors:

$$
\begin{gathered}
(1+\boldsymbol{iI}+\boldsymbol{jJ}+\boldsymbol{iI})/4 \\
(1+\boldsymbol{iI}-\boldsymbol{jJ}-\boldsymbol{iI})/4 \\
(1-\boldsymbol{iI}+\boldsymbol{jJ}-\boldsymbol{iI})/4 \\
(1-\boldsymbol{iI}-\boldsymbol{jJ}+\boldsymbol{iI})/4
\end{gathered}
$$

Complexified vector quaternions produce the same structures as the dual vectors:

$$
\begin{gathered}
(1-i\boldsymbol{i}\mathbf{i}-i\boldsymbol{j}\mathbf{j}-i\boldsymbol{k}\mathbf{k})/4 \\
(1-i\boldsymbol{i}\mathbf{i}+i\boldsymbol{j}\mathbf{j}+i\boldsymbol{k}\mathbf{k})/4 \\
(1+i\boldsymbol{i}\mathbf{i}-i\boldsymbol{j}\mathbf{j}+i\boldsymbol{k}\mathbf{k})/4 \\
(1+i\boldsymbol{i}\mathbf{i}+i\boldsymbol{j}\mathbf{j}-i\boldsymbol{k}\mathbf{k})/4
\end{gathered}
$$

These spinor structures were produced following discussions with J. B. Almeida, who has been working on an extensive theory of spinor structure.

The 'spaces' in the spinor structure are notably completely dual. The system, however, introduces chirality, for the signs cannot be completely reversed. We can only reverse two of them, e.g.

$$(1+\mathbf{iI}-\mathbf{jJ}+\mathbf{kK})/4$$
$$(1+\mathbf{iI}+\mathbf{jJ}-\mathbf{kK})/4$$
$$(1-\mathbf{iI}-\mathbf{jJ}-\mathbf{kK})/4$$
$$(1-\mathbf{iI}+\mathbf{jJ}+\mathbf{kK})/4$$

Pre- and post-multiplying a 'pre-spinor' form of the nilpotent by either the original set of double vector spinors, or the set with signs reversed, typically gives results such as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i\boldsymbol{k}\mathbf{k} & 0 & 0 \\ 0 & 0 & -i\boldsymbol{i}\mathbf{i} & 0 \\ 0 & 0 & 0 & -i\boldsymbol{j}\mathbf{j} \end{pmatrix} \begin{pmatrix} i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m \\ i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m \\ i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m \\ i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i\boldsymbol{k}\mathbf{k} & 0 & 0 \\ 0 & 0 & i\boldsymbol{i}\mathbf{i} & 0 \\ 0 & 0 & 0 & -i\boldsymbol{j}\mathbf{j} \end{pmatrix}$$

$$= \begin{pmatrix} (i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) & (i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) & (-i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) & (-i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \end{pmatrix} \tag{35}$$

which is the full 'spinor' form of the nilpotent wavefunction, with the chirality assigned to the mass term. (An alternative approach would be to assume that the columns in the first 4×4 matrix bear the coefficients 1, $\mathbf{k}$, $\mathbf{i}$, $\mathbf{j}$, and the rows 1, $i\boldsymbol{k}$, $i\boldsymbol{i}$, $i\boldsymbol{j}$, the position being reversed in the second 4×4 matrix; a version of this technique has been used previously to relate the nilpotent version of the Dirac equation to the conventional one based on matrices. 1 Clearly, any two nonidentical spinor matrices will produce a physically meaningful version of the 4-component wavefunction.

One of the remarkable things about the spinor structures generated is that they have the exact form of the components of the two forms of the Berwald-Moor metric:

$$(t-x-y-z)(t-x+y+z)(t+x-y+z)(t+x-y+z) \tag{36}$$

$$(t+x+y+z)(t+x-y-z)(t-x+y-z)(t-x-y+z) \tag{37}$$

If we multiply the 4 components in any order, we will always get zero. In a sense this is like defining a singularity in 'spinor space'. The zero product can thus be interpreted as a fermionic singularity arising from the distortion introduced into the vacuum (or spinor) space by the application of a nilpotent condition. The space becomes quartic because it is created out of two quadratic spaces. We can see this from the fact that the spinor structure ultimately emerges from 4×4 matrices which are created from two sets of 2×2 matrices, which are isomorphic to the units of the usual quadratic vector spaces.

As the two vector spaces are dual, it is possible to restructure physical equations so that their positions are reversed, and so the singularity in spinor (= vacuum) space implies that there must also be a singularity in the observed 'real' space. The quartic Berwald-Moor metric becomes an expression of the fundamentally rotationally quartic nature of the underlying algebra. While multiplication of the units of the algebra produces rotations in the spaces and identity after a complete cycle, multiplication of the spin metric shows that it describes a singularity.

In fact, the H4 algebra has many manifestations at a fundamental level in physics. A long-standing theory of my own is that the fundamental parameters mass(-energy), time, charge (electric, strong and weak) and space have a Klein-4 symmetry relating their properties to each other. $1, 7-9$ Klein-4, as we have shown, is essentially the group structure of H4. The same applies to identity and the T, C and P transformations, which are related to the respective properties of mass, time, charge and space. Also, their fundamental algebraic natures are respectively scalar, pseudoscalar, quaternion and vector, which, when expressed as the Clifford algebra equivalents scalar, trivector, bivector and vector (where these are taken in 1-D), also have a Klein-4 symmetry. If we take mass, time, charge and space as successive descriptions of the universe generated by a 'universal rewrite system' (as work done over the last decade suggests we should $1, 10-11$, then we have four commutative algebras existing as a simultaneous description. In effect, because the first two are scalar and complex, this reduces to a combination of scalar, complex coefficient and quaternion acting as though it were a vector space, and another vector space. The combination is not physical, and so is unobservable. This is what we have called 'vacuum space'. The breaking of the symmetry of this 'space' in creating the 5 generators of the algebra is the ultimate source of the breaking of symmetry between the physical interactions. 1

6 Using discrete differentiation

A discrete or anticommutative differentiation process, developed by Kauffman, 12 offers us a possible link between quantum and classical conditions. In this mathematics, the differentials are replaced by commutators. Defining

$$\frac{dF}{dt} = [F,H] = [F,E] \tag{38}$$

and

$$\frac{\partial F}{\partial X_i} = [F,P_i] \tag{39}$$

we can remove the phase factor from the amplitude and the mass term from the operator (and $\frac{\partial F}{\partial t}$ can replace $\frac{dF}{dt}$ where there is no explicit use of a velocity operator). In our physical application, we can define a nilpotent amplitude

$$\psi = i\boldsymbol{k}E + \boldsymbol{i}\mathbf{i}P_{\mathbf{1}} + \boldsymbol{i}\mathbf{j}P_{\mathbf{2}} + \boldsymbol{i}\mathbf{k}P_{\mathbf{3}} + \boldsymbol{j}m) \tag{40}$$

and an operator

$$D = i\boldsymbol{k}\frac{\partial}{\partial t} - \boldsymbol{i}\mathbf{i}\frac{\partial}{\partial X_{\mathbf{1}}} - \boldsymbol{i}\mathbf{j}\frac{\partial}{\partial X_{\mathbf{2}}} - \boldsymbol{i}\mathbf{k}\frac{\partial}{\partial X_{\mathbf{3}}} \tag{41}$$

with

$$\frac{\partial \psi}{\partial t} = [\psi,H] = [\psi,E] \tag{42}$$

and

$$\frac{\partial \psi}{\partial X_i} = [\psi,P_i]. \tag{43}$$

With some straightforward algebraic manipulation, we find that

$$\begin{aligned} D &= i\psi(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{i}P_1 + \boldsymbol{i}\mathbf{j}P_2 + \boldsymbol{i}\mathbf{k}P_3 + \boldsymbol{j}m) + i(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{i}P_1 + \boldsymbol{i}\mathbf{j}P_2 + \boldsymbol{i}\mathbf{k}P_3 + \boldsymbol{j}m)\psi \\ &\quad -2i(E - P_1^2 - P_2^2 - P_3^2 - m^2). \end{aligned} \tag{44}$$

When is ψ nilpotent, then

$$D\psi = \left(\boldsymbol{k}\frac{\partial}{\partial t} + i\boldsymbol{i}\nabla\right)\psi = 0. \tag{45}$$

This is a Dirac equation using discrete differentials. Generalising this to four states, with D and ψ represented as 4-spinors, then

$$D\psi = \left(\boldsymbol{k}\frac{\partial}{\partial t} \pm i\boldsymbol{i}\nabla\right)(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{i}P_1 \pm \boldsymbol{i}\mathbf{j}P_2 \pm \boldsymbol{i}\mathbf{k}P_3 + \boldsymbol{j}m) = \mathbf{0} \tag{46}$$

becomes the equivalent to the Dirac equation in this calculus. Significantly we did not use i or $i\hbar$ in defining the differentials, though this is usually required in canonical quantization. The equation is thus valid, where nilpotency is a fundamental condition, in discrete classical as well as in quantum contexts.

In a further development, we can also extend the definition of D, following Kauffman, to include covariant terms, such as A_i, so that D becomes $D - A_i$. The covariant terms A_i can be seen as representing either a field source or an expression of the distortion of the Euclidean space-time structure, for example, that produced by the presence of mass in general relativity. This means that, if we choose to use structures of this kind to replace the direct use of mass, then a massless covariant D operator provides us with a convenient route to achieving this.

In this context, we observe that Bogoslovsky 13 sees the field of a fermion-antifermion condensate as a source of space-time anisotropy, with a phase transition in which the particles acquire masses from the space-time, the mass shell taking the form of two hyperboloid inscribed cones. By introducing exponents into the expression for the metric function, Bogoslovsky finds a geometric phase transition, which could be interpreted as a mass-creating spontaneous-symmetry breaking in the fermion-antifermion consendate. According to Bogoslovsky, the generalised Lorentz transformations responsible for the process lead directly to the Berwald-Moor metric. In the discrete version of the double nilpotent representation of the bosonic state (or 'fermion-antifermion condensate'), no mass term appears in the operator, but the differentials may be replaced by covariant derivatives, and so the opportunity arises to represent the appearance of mass directly in terms of an anisotropic space-time structure. Of course, the dual space structure we have used is directly responsible for the creation of mass, as this emerges with spin $\frac{1}{2}$, chirality and *zitterbewegung* from the creation of the fermionic singularity.

7 Conclusion

An analysis of the true nature of the gamma algebra and its origins suggests that the most significant aspects of relativistic quantum mechanics and the fermionic state –

singularity, nilpotency, spin $\frac{1}{2}$, chirality, *zitterbewegung*, the origin of mass, and symmetry breaking – can be described through a spinor structure which is a manifestation of the ultimate spatial distortion – a singularity. The singularity is created through a combination of two quadratic spaces, made dual through a nilpotent connection. In fact, if we reverse the topological argument for explaining spin $\frac{1}{2}$ and Berry phase, this is probably the only true way of creating a physical singularity in nature. The Berwald-Moor metric, by appearing in the spinor space which defines this singularity, has a truly fundamental role to play in quantum physics.The nilpotent condition, however, can be applied beyond quantum physics, and a version of the nilpotent Dirac equation can be applied to systems that are classical but discrete, if we use a calculus based on commutators rather than differentials. It is possible that the Berwald-Moor metric may be significant also under these, as well as under quantum, conditions.

References

[1] Rowlands, P., *Zero to Infinity: The Foundations of Physics*, World Scientific, Singapore and Hackensack, NJ (2007).

[2] Rowlands, P., *Physical interpretations of nilpotent quantum mechanics*, arXiv: 1004.1523 (2010).

[3] Penrose R., *Twistor quantization and the curvature of spacetime*, Int J Theor Phys **1** (1968), 61-99.

[4] Hestenes, D., *Space-Time Algebras*, Gordon and Breach, New York, 1966.

[5] Dirac, P. A. M., *The Principles of Quantum Mechanics*, fourth edition, Clarendon Press, Oxford 1958.

[6] Penrose R., *The Road to Reality*, Jonathan Cape, London, 2004.

[7] Rowlands, P. *The fundamental parameters of physics*, Speculat Sci Tech, **6** (1983), 69-80.

[8] Rowlands, P., *A foundational approach to physics*, arXiv:physics/0106054 (2001).

[9] Rowlands, P., *Are there alternatives to our present theories of physical reality*, arXiv:0912.3433 (2009).

[10] Rowlands, P. and Diaz, B., *A universal alphabet and rewrite system*, arXiv:cs.OH/0209026 (2002).

[11] Diaz, B. and Rowlands, P., *A computational path to the nilpotent Dirac equation*, Int. J. Comp. Anticip. Syst. **16** (2005), 203-218.

[12] Kauffman, L. H., *Non-commutative worlds*, New J. Phys. **6** (2004), 2-46.

[13] Bogoslovsky, G. Yu., *Lorentz symmetry violation without violation of relativistic symmetry*, Phys. Lett. A **350** (2006), 5-10.