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MASSES IN WEYL GEOMETRIES

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Abstract

We shall present a non-standard framework to describe a matter fluid in Weyl geometries and investigate conservation laws. In particular we shall obtain a set of properties which allow to obtain in this generalized setting the standard relation between conservation of the energy-momentum tensor and number of particles. Coupling with matter is also discussed in relation with conformal transformations.

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1 Introduction

Weyl geometries provide a more general framework than the one based on Riemannian geometry used in General Relativity. Weyl setting also naturally appears in axiomatic approaches to General Relativity (see [4]). It allows a clear separation of geometry and gravity which is worth investigating, not only from a geometrical prespective but also from a physical one, since at the very least it allows to decide on the geometry of spacetime on an experimental basis rather than assuming a Riemannian basis. Moreover, a wide class of gravitational theories (see [5], [6]) are naturally within the scheme of Weyl geometries (and in fact in a subclass that does not encounter the typical holonomy problems of Weyl geometries).

This new framework allows both to have new insights in old problems (such as representing matter in Special Relativity, providing a framework for effective action principles from quantum gravity and supergravity, just to quote some) as well as it produces a new set of problems to be investigated (e.g. how one can define observational protocols in Weyl geometry?).

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All physical experiments use electromagnetic field in quite a peculiar way. Although this issue was usually overlooked (and practically irrelevant) at the time of mechanical experiments, it was finally stressed during 19th and 20th centuries when Physics recognized to be working in terms of fields.

With special relativity (SR) Einstein realized that the physical properties of the electromagnetic field are fundamentally entangled with our notion of space and time (or, more correctly, *spacetime*) on which our observational protocols rely, for example through synchronization. SR is in fact a good model of spacetime and electromagnetic field in which observational protocols are made explicit. As such, and in view of later studies of general relativity (GR), SR is fundamentally a theory of empty space, with electromagnetic field but no other matter.

As soon as massive particles are considered they generate a gravitational field that cannot be described in SR. Of course one can in some case neglect gravitational effects and consider massive particles as test particles; however, from a fundamental viewpoint SR is not compatible with masses.

When Einstein was led to consider the gravitational field he was forced to take the only direction that his contemporary geometric technologies allowed to use. Gravitation was identified with the curvature of a spacetime metric, leading to standard GR. At that time, general connections were still on the way of being invented and the only curvature known was metric curvature.

When general connections were finally invented (maybe feeling something to be improved in GR in the relation among electromagnetic field, gravitational field and matter) Einstein tried the way of a unifying theory of electromagnetic and gravitational fields in which both fields were the output of a connection. Einstein could not reach a satisfactory solution to this problem (later solved in [1]).

In 1972 another fundamental contribution was given by Ehlers, Pirani and Schild (EPS). They started by assuming that one can observe and trace lightrays and particle worldlines in spacetime and they listed a set of axioms about these congruences of trajectories together with their compatibility, resorting to physically reasonable properties. EPS axioms allow to derive the geometric structure of spacetime starting from potentially observable data about lightrays and particles. And the final conclusion of EPS is that lightrays determine a conformal structure on spacetime (i.e. a sheaf of lightcones or a class $[g]$ of conformally equivalent Lorentzian metrics) while free fall of massive particles do determine a *projective structure*, i.e. a class of connections $[\Gamma]$ which share the same geodesic worldlines.

Massive particles move along geodesic worldlines of a connection Γ which is neither physically nor mathematically forced to be the Levi-Civita connection of the metric determined by lightrays. These two structures have just to be *EPS-compatible* in the sense that lightlike geodesics with respect to $\{g\}$ have to belong to the family of auto parallel curves of Γ. It can be shown that that happens iff the connection Γ is chosen so that there exists a covector A such that:

$$\Gamma^{\alpha}_{\beta\mu} = \{g\}^{\alpha}_{\beta\mu} + (g^{\alpha\epsilon} g_{\beta\mu} - 2\delta^{\alpha}_{(\beta}\delta^{\epsilon}_{\mu)})A_{\epsilon} \tag{1}$$

In other words, EPS showed that lightrays and mass particles uniquely determine a so-called *Weyl geometry* $(M, [g], \Gamma)$. The connection Γ is obtained by a gauge fixing of the projective structure (which is in fact a gauge symmetry of the action: see [8]) by the gauge condition $\nabla^{(\Gamma)}_\lambda g_{\mu\nu} = -2A_\lambda g_{\mu\nu}$. In this setting the conformal structure $[g]$ is directly related to lightrays and lightcones while the connection Γ is directly related to the free fall of massive particles.

EPS framework clearly shows that it is unlogical to require or impose at a purely kinematical level that the connection is the Levi-Civita connection of the metric g. There is no reason to assume such a strict relation between the metric and affine structures. Of course, being unlogical it does not imply being physically false. Standard GR, in which these two structures do in fact coincide by assumption, might be the *true* choice, though that should be decided by experiments and not assumed *a priori*.

EPS framework clearly points towards a formalism *à la Palatini* (which by the way was another finding of Einstein; see [2]), in which the metric and the connection are *a priori* unrelated and field equations then decide dynamically the relation between the two structures. It is well-known that in vacuum (or with electromagnetism and scalar matter fields) in standard Palatini formalism the connection is forced to be the Levi-Civita connection of the metric, which would in fact explain why GR works so well in simple situations such as the experiments in the Solar system.

By the way, there is another issue related to EPS formalism which is against the assumptions of standard GR. In EPS lightrays determine a conformal structure, not a metric. Choosing a representative of the conformal structure, i.e. a Lorentzian metric g, to represent the conformal structure is in fact a sort of fixing of the conformal gauge invariance which the dynamics of the gravitational field has to preserve. Hence one has conformal transformations acting on metrics and on connections and the Lagrangian has to be covariant with respect to such transformations. However, conformal transformations are related to lightrays, not to massive particles, and the connection Γ is left unchanged by such transformations. Accordingly, a conformal transformation has a particular physical meaning and it acts by rescaling the metric but leaving the connection unchanged. If so, imposing kinematically a strict relation between the connection and the metric, namely $\Gamma = \{g\}$, would be incompatible with such a group symmetry.

In any event, physically speaking one should resign to determine the metric structure by just observing lightrays (or more generally the electromagnetic field which in dimension $m = 4$ is known to be conformally invariant). The metric struture is then historically used to gauge observational protocols and experiments. Then one should observe massive particle free fall and determine the connection among the EPS-compatible structures. In other words, electromagnetic and mass phenomenologies are distinct both from the experimental point of view and on the role they play in the definition of the geometry of spacetime.

Now, we want to suggest that this new perspective opens the possibility to solve in a different more general way the problem faced by Einstein of finding a model of gravitational, electromagnetic and matter fields by assuming a much larger class of theories, in which standard GR is just one possibility, in which one should set up experiments to determine which is the theory that is physically realized by the gravity we know. In particular we shall show how one could consider a matter fluid on Minkowski spacetime, generating a gravitational field which is encoded in a (curved) Weyl connection which is compatible with the flat Minkowski metric structure. This spacetime is curved, in the sense that the connection Γ *is curved*, even though the metric structure is the standard flat Minkowski which is in fact unaffected by the presence of the matter fluid, being it determined by the Physics of lightrays.

Hereafter, in Section 2 we shall briefly review how a fluid on a conformal structure $(M, [g])$ essentially determines a (unique) Weyl geometry $(M, [g], \Gamma)$ in which in fact the fluid flows along geodesics of Γ; see [3].

In Section 3 we shall investigate how the system behaves from the viewpoint of its stress tensors and its relation to conservation laws and geodesics of Γ. In standard GR this issue is very well known. However, in Weyl geometries one has two connections (namely, $\{g\}$ and Γ) and whenever a connection is used one has to declare which one of these two has to be used. This cannot be done without constraints; the geodesic equation of Γ should be made to work together with conservation of the stress tensor (and Bianchi identities) in order to produce the expected conservation laws of the fluid.

In Section 4 the relation between the conservation of the number of particles is related to the conservation of the energy-momentum tensor. We shall show that this can be done by selecting a preferred conformal frame and a parametrization of worldlines.

2 Weyl Geometries Compatible with a Fluid

Hereafter we shall briefly review the results of [3]. Let us consider, on a spacetime M of dimension m, a conformal structure $(M, \mathfrak{P})$ and any $\mathfrak{P}$-timelike vector field u. For any gauge fixing of the conformal structure $g \in \mathfrak{P} \equiv [g]$ one can normalize u to be a g-unit vector n.

Let $\gamma : \mathbb{R} \to M$ be an integral curve of the vector field n. We can arbitrarily reparametrize the curve γ to obtain a different representative $\gamma \circ \phi$ of the same trajectory. If the original curve γ was a Γ-geodesic motion (for a connection Γ) then $\gamma \circ \phi$ is a Γ-geodesic trajectory. Accordingly, one has

$$n^\mu \nabla^{(\Gamma)}_\mu n^\alpha = \varphi \cdot n^\alpha \tag{2}$$

for some scalar field $\varphi(x)$.

In GR one has no much choice for the fluid flow lines generated by n; the connection is freezed to be the Levi-Civita connection of g and the vector field n has to be selected in the small class of geodesic fields. In a Weyl setting one has a wider freedom in chosing the connection in the class of EPS-compatible connections given by (1). One can rely on this freedom to show that for any timelike vector field n there exists an EPS-compatible connection Γ for which n is Γ-geodesic, i.e. (2) holds true. One can easily check that A has to fixed as

$$A_\nu = n^\mu \nabla^{(g)}_\mu n_\nu + \varphi n_\nu \tag{3}$$

Notice how, once u is given and a parametrization of curves is fixed by choosing the scalar field φ, one can uniquely determine the covector A and thence the connection Γ. If one started from a different conformal representative $\tilde{g} = \Phi^2 \cdot g$ this would amount to redefine accordingly the covector $\tilde{A} = A + d\ln\Phi$ to obtain the same Γ, the unit vector $\tilde{n}^\lambda = \Phi^{-1} n^\lambda$ and the scalar field $\tilde{\varphi} = \Phi^{-1}(\varphi - n^\mu \nabla^{(*)}_\mu \ln\Phi)$, where $\nabla^{(*)}_\mu$ denotes the covariant derivative in the case it is independent of the connection. We refer to [3] for details and proofs.

3 Fluid Conservation Laws

The fluid is described, besides by its flow lines which are generated by u, by two scalar fields $\rho(x)$ and $p(x)$ describing particle density and pressure. The (symmetric) energy-momentum tensor of the fluid is in the form

$$T_{\mu\nu} = p g_{\mu\nu} + (p+\rho) n_\mu n_\nu \tag{4}$$

In standard GR, one has a strict relation between Bianchi identities (associated to field equations), conservation of energy-monentum tensor $\nabla_\nu T^{\mu\nu} = 0$, matter field equations (which together with fluid equation of state determine the evolution of the fluid) and conservation laws which are associated to conservation of the number of particles (and the fluid energy).

In Weyl setting one should describe the same sort of relations, by suitably specifying when covariant derivatives are induced by g and when they are induced by Γ.

The conservation of energy-momentum tensor is

$$\nabla^{(\Gamma)}_\nu T^{\mu\nu} = 0 \qquad \Rightarrow \tag{5}$$

$$\begin{aligned} (p+\rho)\nabla^{(\Gamma)}_\nu n^\nu &= (p-\rho)\varphi - n^\nu \nabla^{(*)}_\nu \rho \\ (g^{\mu\nu} + n^\mu n^\nu)\nabla^{(*)}_\nu p &= 2pn^\nu \nabla^{(g)}_\nu n^\mu \end{aligned}$$

where we used (1) and (3).

By starting from a different conformal gauge fixing $\tilde{g} = \Phi^2 \cdot g$ one has different pressure and density, since physical rods and therefore measures depend on the choice of the conformal factor Φ. In principle one sets $\tilde{p} = \Phi^n p$ and $\tilde{\rho} = \Phi^n \rho$ (with the power n to be determined later in view of conservation of the number of particles) and the energy momentum tensor is

$$\tilde{T}_{\mu\nu} := \tilde{p}\tilde{g}_{\mu\nu} + (\tilde{p}+\tilde{\rho})\tilde{n}_\mu \tilde{n}_\nu = \Phi^{n+2}(p g_{\mu\nu} + (p+\rho) n_\mu n_\nu) = \Phi^{n+2} T_{\mu\nu} \tag{6}$$

Of course, unless $n = -2$ (as we shall see below this happens in dimension $m = 3$), the energy-momentum tensor is not conformally invariant and, more importantly, its conservation is not preserved by conformal transformations. If $T_{\mu\nu}$ is conserved then in general $\tilde{T}_{\mu\nu}$ is not. Accordingly, one has to specify in which gauge the conservation of energy-momentum tensor has to be imposed. We shall discuss this issue below in greater detail.

Also conservation of number of particles can be discussed at kinematical level. If ρ is related to the density of particles of the fluid, then one would like to define a quantity J^μ which can be integrated on a spatial region Σ to determine the number N_Σ of fluid particles hitting Σ. The number N_Σ must be constant along the flow of n.

An object to be integrated along an hypersurface Σ is a $(m-1)$-form. There is a natural choice:

$$J = \sqrt{g} T^{\mu\nu} n_\nu ds_\mu =: J^\mu ds_\mu \qquad N_\Sigma := \int_\Sigma J^\mu ds_\mu \tag{7}$$

The number of particles N_Σ is conserved along the flow of n iff one has $dJ = 0$. The current J is sensible to conformal transformations since measures are. It can be redefined out of any conformal framework and one has

$$\tilde{J} = \sqrt{\tilde{g}} \tilde{T}^{\mu\nu} \tilde{n}_\nu ds_\mu = \Phi^m \sqrt{g} \Phi^{n-2} T^{\mu\nu} \Phi n_\nu ds_\mu = \Phi^{m+n-1} J \tag{8}$$

The integral N_Σ counts how many fluid particles hit the region Σ and as such Physics imposes that it has to be independent of the conformal framework. This forces n to be fixed as $n = 1 - m$ so that J is conformally invariant and its integral is accordingly invariant.

This thence prescribes the following conformal transformations

$$\tilde{p} = \Phi^{1-m} p \qquad \tilde{\rho} = \Phi^{1-m} \rho \tag{9}$$

$$\tilde{p} = \Phi^{1-m} p \qquad \tilde{\rho} = \Phi^{1-m} \rho \tag{10}$$

This could be expected since ρ and p denotes the *spatial densities* (of particles and pressure).

4 Conservation of $T_{\mu\nu}$

In a general Weyl context the conservation of energy-momentum tensor and its relation with conservation of the current J needs to be deeply reviewed. One has in fact from conservation of number of particles:

$$\partial_\mu J^\mu = \nabla^{(\Gamma)}_\mu (\sqrt{g} T^{\mu\nu} n_\nu) = \sqrt{g} (\nabla^{(\Gamma)}_\mu T^{\mu\nu} n_\nu + T^{\mu\nu} \nabla^{(\Gamma)}_\mu n_\nu) + \nabla^{(\Gamma)}_\mu \sqrt{g}\, T^{\mu\nu} n_\nu = 0 \tag{11}$$

In standard GR one has $\Gamma = \{g\}$ and hence $\nabla_\mu \sqrt{g} = 0$; if n is a Killing vector also $T^{\mu\nu} \nabla_\mu n_\nu = 0$ and conservation of particles, $dJ = 0$, is implied by conservation of the energy-momentum stress tensor (which, usually, is eventually implied by Bianchi identities). However, we have to observe that conservation of J does not on the contrary imply conservation of energy-momentum tensor. Only the projection of conservation laws along n is in fact used. Moreover, being n a Killing vector is a sufficient but by no means necessary condition.

In a general Weyl context one has in fact

$$\partial_\mu J^\mu = \sqrt{g} (\nabla^{(\Gamma)}_\mu T^{\mu\nu} n_\nu + T^{\mu\nu} \nabla^{(\Gamma)}_\mu n_\nu) + m \sqrt{g} A_\mu\, T^{\mu\nu} n_\nu = 0 \tag{12}$$

Thence one should somehow impose that in general one has

$$T^{\mu\nu}\nabla^{(\Gamma)}_\mu n_\nu = 0 \qquad T^{\mu\nu}A_\mu n_\nu = 0 \tag{13}$$

without being too demanding on the vector n.

Here the situation is easier for dust ($p = 0$). Accordingly, let us first consider this case.

$$\begin{aligned} T^{\mu\nu}\nabla^{(\Gamma)}_\mu n_\nu = & \ \rho n^\mu n^\nu \nabla^{(\Gamma)}_\mu (n^\alpha g_{\alpha\nu}) = \\ = & \ \rho(n_\alpha n^\mu \nabla^{(\Gamma)}_\mu n^\alpha - 2n^\mu A_\mu) = \rho(-\varphi + 2\varphi) = \varphi\rho \end{aligned} \tag{14}$$

Now we have to stress that the scalar φ can be chosen at will. Here of course we would like to fix $\varphi = 0$. This corresponds to require that the integral curves of n are not only geodesics trajectories of Γ, but in fact they are geodesics motions. We here started to be general enough but we are forced back to geodesic motions.

As far as the second condition is concerned one has

$$T^{\mu\nu}A_\mu n_\nu = -\rho A_\mu n^\mu = \rho\varphi = 0 \tag{15}$$

which is also satisfied for $\varphi = 0$.

The situation with pressure is more complicated.

One can easily show that

$$T^{\mu\nu}\nabla^{(\Gamma)}_\mu n_\nu = p\nabla^{(g)}_\mu n^\mu + (2\rho - mp)\varphi \qquad T^{\mu\nu}A_\mu n_\nu = \rho\varphi \tag{16}$$

$$\begin{aligned} T^{\mu\nu}\nabla^{(\Gamma)}_\mu n_\nu = & \ p g^{\mu\nu}\nabla^{(\Gamma)}_\mu n_\nu + (p+\rho) n^\nu n^\mu \nabla^{(\Gamma)}_\mu n_\nu = \\ = & \ p\nabla^{(\Gamma)}_\mu n^\mu + 2pn^\mu A_\mu + (p+\rho) n_\alpha n^\mu \nabla^{(\Gamma)}_\mu n^\alpha - 2(p+\rho) n^\mu A_\mu = \\ = & \ p\nabla^{(\Gamma)}_\mu n^\mu - 2p\varphi - (p+\rho)\varphi + 2(p+\rho)\varphi = p\nabla^{(\Gamma)}_\mu n^\mu + 2\rho\varphi = \\ = & \ p\nabla^{(g)}_\mu n^\mu + pn^\lambda (g^{\mu\epsilon} g_{\lambda\mu} - \delta^\mu_\lambda \delta^\epsilon_\mu - \delta^\mu_\mu \delta^\epsilon_\lambda) A_\epsilon + 2\rho\varphi = \\ = & \ p\nabla^{(g)}_\mu n^\mu + pn^\lambda (\delta^\epsilon_\lambda - \delta^\epsilon_\lambda - m\delta^\epsilon_\lambda) A_\epsilon + 2\rho\varphi = p\nabla^{(g)}_\mu n^\mu + (2\rho - mp)\varphi \end{aligned} \tag{17}$$

and one has to face the incompressibility condition $\nabla^{(g)}_\mu n^\mu = 0$. This can be solved by using the freedom in the conformal gauge fixing. One can show that there is always a conformal representative for which

$$\nabla^{(\tilde g)}_\mu \tilde n^\mu = 0 \tag{18}$$

and then in this conformal frame we fix $\tilde\varphi = 0$.

One has

$$\begin{aligned} \nabla^{(\tilde g)}_\mu \tilde n^\mu = & \ \nabla^{(g)}_\mu \tilde n^\mu - \tilde n^\lambda (g^{\mu\epsilon} g_{\lambda\mu} - \delta^\mu_\lambda \delta^\epsilon_\mu - \delta^\mu_\mu \delta^\epsilon_\lambda) \partial_\epsilon \ln\Phi = \\ = & \ \nabla^{(g)}_\mu \tilde n^\mu - \tilde n^\lambda (\delta^\epsilon_\lambda - \delta^\epsilon_\lambda - m\delta^\epsilon_\lambda) \partial_\epsilon \ln\Phi = \nabla^{(g)}_\mu \tilde n^\mu + m\tilde n^\epsilon \partial_\epsilon \ln\Phi = \\ = & \ \tfrac{1}{1}(\nabla^{(g)}_\mu n^\mu + (m-1) n^\epsilon \partial_\epsilon \ln\Phi) \end{aligned} \tag{19}$$

Now, the fact is that whaterver $\nabla^{(g)}_\mu n^\mu$ one can always find a conformal factor Φ such that

$$n^\epsilon \partial_\epsilon \ln\Phi = -\frac{1}{m-1}\nabla^{(g)}_\mu n^\mu \tag{20}$$

(fix coordinates in which $n = \partial_0$). Using such a conformal factor to change conformal representative, in the new conformal frame one has $\nabla^{(\tilde{g})}_\mu \tilde{n}^\mu = 0$.

For the second condition to hold one has

$$T^{\mu\nu} A_\mu n_\nu = \quad -(\rho + p) A_\mu n^\mu + p A_\mu n^\mu = -\rho A_\mu n^\mu = \rho\varphi \tag{21}$$

which also vanishes under the same condition.

Thus in general one needs not to require that n is Killing. In fact the fluid (with pressure) selects a preferred conformal frame $\tilde{g}$. In that preferred frame one has a preferred Weyl connection with

$$\tilde{A}_\nu = \tilde{n}^\mu \nabla^{(\tilde{g})}_\mu \tilde{n}_\nu \tag{22}$$

for which $\tilde{n}^\mu \nabla^{(\Gamma)}_\mu \tilde{n}^\nu = 0$. With these choices, not only the fluid velocities are a geodesic field, but the conservation law $dJ = 0$ is equivalent to the conservation of fluid energy-momentun tensor, $\nabla^{(\Gamma)}_\mu T^{\mu\nu} = 0$.

5 Conformal transformations

If we have to take seriously Weyl geometry and EPS framework, once matter is added then it must be free fall along geodesics of Γ, not along geodesics of $\{g\}$ as in standard GR.

Now it is well known that massive particles move along the eikonal approximation of the corresponding field (e.g. Klein-Gordon field). Thus in turn the EPS request is in fact a prescription to fix matter-gravity coupling so that it finally result in massive particles to move along geodesics of Γ.

This has longly been discussed in literature (see [9]), whether minimal coupling should be imposed in the Einstein frame or in the Jordan frame, i.e. whether matter coupled naturally to the metric g or to the connection Γ.

Let us here consider the special case of Klein-Gordon matter field. We shall show that Klein-Gordon equations are *almost* covariant with respect to conformal transformation (which in $f(R)$ models connect g to the connection Γ which is in fact the Levi-Civita connection to a conformal metric g').

In fact, a conformal transformation maps a Klein-Gordon equation into a similar equation with a point-dependent mass (actually a function of the conformal factor). Thus in most situations (i.e. in regions where the conformal factor is almost constant as in virtually all cases regarding tests of GR) the conformal transformations finally amounts to a rescaling of effective masses. When the conformal factor is near 1 (as in standard GR or in the Solar system) then the conformal transformation leave Klein-Gordon equation approximately invariant and the rescaling can be hidden by observation errors.

Still when gravity-matter interactions are important (as in non-vacuum solutions, e.g. Friedman-Robertson-Walker cosmological solutions) or when the conformal factor is non-trivial (e.g. in strong coupling regime) then the mass rescaling can be important.

The transformation rules of the Klein-Gordon equation with respect to conformal rescaling of the metric and the scalar field can be easily obtained using a simple fact concerning the conformal Laplacian operator $\boxdot = \Box + \frac{1}{6} R$.

One can easily see that in view of the transformation $g \mapsto g' = pg$ one has:

$$\boxdot\varphi + m^2\varphi = p^{\frac{3}{2}}\tilde{\boxdot}\tilde{\varphi} + m^2 p^{\frac{1}{2}}\tilde{\varphi} = p^{\frac{3}{2}}\left(\tilde{\boxdot}\tilde{\varphi} + \frac{m^2}{p}\tilde{\varphi}\right) = 0 \tag{23}$$

If one starts from a Minkowski spacetime, then $R = 0$ and $\boxdot = \Box$. In this case, a conformal spacetime $\tilde{g}_{\mu\nu} = p \cdot \eta_{\mu\nu}$ is conformally flat and one has the scalar field to obey the equation

$$\tilde{\Box}\tilde{\varphi} + \frac{1}{6}\tilde{R}\tilde{\varphi} + \frac{m^2}{p}\tilde{\varphi} = \tilde{\Box}\tilde{\varphi} + \left(\frac{m^2}{p} + \frac{1}{6}\tilde{R}\right)\tilde{\varphi} = 0$$

This is again a Klein-Gordon-like equation with an effective mass

$$\tilde{m}^2 = \frac{m^2}{p} + \frac{1}{6}\tilde{R}$$

Let us remark that even starting from a massless Klein-Gordon field ($m = 0$) the final effective mass receives a contribution from the curvature of the conformal metric. Second, the curvature itself can be expressed in terms of the conformal factor (since the original metric is flat) and one has

$$\tilde{m}^2 = \frac{m^2}{p} + \frac{1}{6p}\left(-3\Box \ln p - \frac{3}{2p}\tilde{\nabla}_\mu p \tilde{\nabla}^\mu_{\cdot} p\right) = \frac{m^2}{p} - \frac{1}{2p^2}\left(\Box p - \frac{1}{2}\tilde{\nabla}_\mu p \tilde{\nabla}^\mu_{\cdot} p\right)$$

As a side effect, a solution of Klein-Gordon coupled with a metric g is a solution of a Klein-Gordon (with an effective mass depending on the conformal mass) for any conformal metric $\tilde{g}$. This is a first evidence that one can eventually have matter fields coupled with the Einstein metric and hence in eikonal approximation, test particles free falling along geodesics of $\tilde{g}$. All these effects are completely trivial in vacuum models when $p = \mathit{const}$, though they are switched on within matter.

6 Conclusions and Perspectives

We showed that for any conformal structure $[g]$ on a spacetime M and for any timelike vector field u one can always determine an EPS-compatible connection Γ for which the vector field u is geodesic. Then one can determine the conformal frame and parametrizations along worldlines so that one has the standard relation among the different conservations associated to the fluid.

Of course we are not here suggesting that the model we presented here is physically sound. One should specify a dynamics and then investigate the relation with Bianchi identities. We chose this model to show that had Einstein known it, he could have possibly tried this way to model matter and gravity and make them compatible with SR.

This research is part of a larger project aiming to model a generic self-gravitating fluid in EPS formalism. One can specify the conformal structure to be the Minkowskian one by setting $g = \eta$ and still be free to model any congruence of (timelike) worldlines as the flow of a fluid. In this framework the gravitational field is encoded into the covector A which in turn determines the Weyl connection Γ.

More investigations are required in this direction to show that in fact one can set the coupling with matter so that the dynamics in fact realized what prescribed in EPS (i.e. free falling determined by Γ). We here first sketched this at the level of field equations while it should be worked out in detail at action level.

Moreover, all observational protocols commonly used in GR relying on uniqueness of parallel transport. When conformal transformations are consider when automatically one has a whole family of parallel transports. What impact has these on observational protocols (and then on GR tests) are to be discussed in details as well.

Let us stress that in the EPS setting there is no freedom in choosing the connection associated to free fall. In EPS framework the free fall of particles is *by construction* described by the connection Γ while the Levi-Civita connection of g plays just the kinematical role of *reference frame* in the affine space of connections (which is moreover conformally covariant since one can start from any representative of the conformal structure).

The extra degrees of freedom to determine in Γ are thence encoded into the covector A which is kinematically free to be generic. The dynamics of the theory determines then the connection Γ fixing the covector A in terms of matter fields and g.

Of course Weyl geometries are affected by physical interpretation problems mainly related to the (possibly non-trivial) holonomy of the connection Γ; see [4]. However, these problems arise only if Γ is not metric while metric connections do not generate any physical problem of this kind and have just to be interpreted correctly. We stress that in all $f(R)$ models with non-exotic matter (or in vacuum) dynamics forces a posteriori the connection Γ to be automatically metric (and in fact to be the Levi-Civita connection of a metric conformal to the g originally entering the Lagrangian; see [5] and [6]), unless one introduces a matter Lagrangian in which matter couples directly with the connection Γ (a case in which the connection can be also not metric; see [7]).

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