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FINITELY SEPARABLE SOLVABLE MOUFANG LOOPS

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Abstract

We study finitely separable solvable Moufang loops and solved the membership problem for nilpotent Moufang loops.

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1 Indroduction

The set L of elements with operations of multiplication of the right and left divisions $\cdot,,\ \backslash,\ /$ is called loop, if there exists an element e in L, is for cloth $e \cdot x = x \cdot e = x$, and if the $x = xy/y = y\backslash(yx) = x/y \cdot y = y \cdot (y\backslash x)$ for all elements x, y in L. Similarly for groups, we say that that subloop H of a loop L is finitely separated from the element $x \notin H$ if there is homomorphism φ of L into a finite loop for which $x\varphi \notin H\varphi$. Loop with all non-trivial finite separable subloops we call finite separable. The condition of finite separability is stronger than the condition of finite approximability (loop L is called finite approximated (residually) finite if for $x \neq e$ the is homomorphism of L into a finite loop that $a\varphi \neq e$. In this paper we study in detail the structure of finite separable soluble Moufang loops. In particular, it appears that solvable Moufang loops with maximum condition for subloops finite separable, which amplifies results from the theory of residual finite theory ([1], [2]). It turned out that in a Hausdorff topology all subloops of finite separable solvable Moufang loop are closed sets. At the end prove the existence of an algorithm to solve the membership problem for nilpotent Moufang loops (commutative Moufang loops or nilpotent A-loops).

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2 Notations, observations and comments

Let H be a subset of elements of the loop L. The set of elements of L which may be obtained applying a finite number of basic operations to elements of H is said to be

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a *subloop* generated by the elements of H. If this set coincides with H, i.e. for each pair of elements x and y of H, the elements $x \cdot y, y/x$ and $x\backslash y$ also belong to H, then H is a subloop of the loop L.

The substitutions $T_x, R_{x,y}, L_{x,y}$, where x and y are elements of the loop L, are defined by means of the following relations:

$$T_x = R_x L_x^{-1}, R_{x,y} = R_x R_y R_{xy}^{-1}, L_{x,y} = L_x L_y L_{yx}^{-1}, \text{where } yL_x = xR_y = x \cdot y.$$

All these substitutions of the loop L generate the group of *inner substitutions* $J(L)$ ([3], [4]).

A subloop H of the loop L is named *normal* in L if

$$xH = Hx,\ Hx \cdot y = H \cdot xy,\ x \cdot yH = xy \cdot H \tag{1}$$

for any $x, y \in L$.

It is clear that conditions (1) are equivalent to the conditions:

$$HT_x = H,\ HR_{x,y} = H,\ HL_{x,y} = H$$

for any $x, y \in L$. That is why we may say that the subloop H of the loop L is normal in L if H is invariant with respect to the inner substitution of the group $J(L)$ [3].

For elements x, y and z of the loop L, the *associator* $[x, y, z]$ and the *commutator* $[x, y]$ are defined, respectively, by means of the following equations:

$$[x, y, z] = (x \cdot yz)\backslash(xy \cdot z), \quad [x, y] = (yx)\backslash(xy).$$

The subloop of L generated by all the associators and commutators of L is called *associant-commutant* of the loop L and will be denoted by L'. The *center* of the loop L is said to be the subset

$$Z(L) = \{x \in L | [x, y, z] = [y, z, x] = [x, y] = e \ \forall\, y, z \in L\}.$$

It is easy to see that the associant-commutant L' and the center $Z(L)$ are normal subloops in L. We mention also: *each subloop of the loop L which belongs to the center $Z(L)$ or contains the associant-commutant L' is normal in L.*

Let L be a loop and n a natural number. A sequence of subloops

$$\{e\} = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_{n-1} \subseteq H_n = L \tag{2}$$

is said to be *normal* if for each $i \in \{0, 1, \ldots, n\}$ the subloop H_i is normal in L. The number n is called the *length* of the sequence, the subloops H_i, $i \in \{0, 1, \ldots, n\}$ are named terms of the series, while the factor-loops $H_i/H_{i-1} \in \{1, 2, \ldots n\}$ are called the *factors of the sequence*. A natural sequence (2) is said to be central if all its factors are *central*, i.e.

$$H_i/H_{i-1} \subseteq Z(L/H_{i-1}) \quad \forall\, i \in \{1, 2, \ldots n\}$$

or equivalently

$$[H_i, L] \subseteq H_{i-1} \quad \forall\, i \in \{1, 2, \ldots, n\}$$

where $[A, B]$ means the normal subloop generated in L by all the associators and commutators of the form $[x, y, z]$,$[y, z, x]$ and $[x, y](x \in A; y, z, \in B)$. A loop L is called (central-)*nilpotent* if it has a (finite) central sequence, while the least length of all its central sequences is named the *nilpotence class* of the loop L. A loop having a normal sequence with Abelian factors is said to be *solvable*, while the least length of such sequence is called *class of solvability*.

For a loop L we define subloops

$$L_1 = L,\ L_{i+1} = [L_i, L],\ \text{for } i \geq 1, \tag{3}$$

$$Z_0(L) = \{e\},\ Z_{j+1}(L)/Z_j(L) = Z(L/Z_j(L)),\ \text{for } j \geq 0. \tag{4}$$

For simplicity, let us denote $Z_j(L) = Z_j$. It is clear that if a certain subloop Z_j coincides with the whole loop L or a certain sobloop L_j coincides with the unity subloop $\{e\}$, then the loop L is nilpotent. Conversely, let L be a nilpotent loop and let (2) be an arbitrary central sequence of the loop L. The definitions from above and hypothesis imply the following inclusions: $H_0 \subseteq Z_0$, $H_1 \subseteq Z_1, \ldots$; $L_1 \subseteq H_n$, $L_2 \subseteq H_{n-1}, \ldots$ It is evident that (3) and (4) represent normal sequences, i.e. each of them contains the unitary subloop $\{e\}$ and the loop L and its number of terms is equal to the class of nilpotence. As for the group, these sequences are said to be *central descending* and *central ascending*.

If the loop L is truly one of the following identities

$$x(y \cdot xz) = (xy \cdot x)z,\ (xy \cdot z)y = x(y \cdot zy),\ xy \cdot zx = x(yz \cdot x),\ xy \cdot zx = (x \cdot yz)x,$$

it is called a *Moufang loop* (see their study [3], [4], [5]).

3 Some properties of Moufang loops

Here we formulate the properties in the form of lemmas. In the beginning we present some known properties of Moufang loops, which are needed in the future.

Lemma 1. *([3], [4]). In Moufang loop any three elements linked by associative law generates a group. In particular, a Moufang loop is di-associative, i.e. each pair of its elements generates an associative subloop.*

Lemma 2. *([3]). In Moufang loop are true identities:*

$$xT_y = x[x, y], \tag{5}$$

$$xL_{z,y} = x[x, y, z]^{-1} \tag{6}$$

Now we prove the following lemmas.

Lemma 3. *If L is a nilpotent (Moufang) loop and H is a subloop normal in L and different from the unity, then $H \cap Z \neq 1$.*

Proof. Let k be the least natural number which verifies the condition $H \cap Z_k \neq 1$. Then

$$[H \cap Z_k, L] \subseteq H \cap Z_{k-1}.$$

But $H \cap Z_{k-1} = 1$, hence $H \cap Z_k \subseteq Z$ and $k = 1$. □

Lemma 4. *In any nilpotent Moufang loop L the set P of all periodic elements is a normal subloop.*

Proof. Indeed, let $a, b \in P$. According to Lemma 1 the subloop generated by the elements a, b is associative and, since it is nilpotent, it is finite. Hence, a^{-1} and ab have finite orders, so they belong to P. Now, if m is the order of the element a and α is an arbitrary inner substitution of the group $J(L)$, which according to [3], [4] is a semi-automorphism, then $(a^\alpha)^m = (a^m)^\alpha = 1$, so $a^\alpha \in P$. □

From Lemmas 3 and 4 it follows that *if a nilpotent Moufang loop contains periodic elements different from the unity, then also the center of this loop contains periodic elements.*

Lemma 5. *If a nilpotent Moufang loop L is torsion-free (i.e. without elements of finite order), then the factor loop L/Z is torsion-free too.*

Proof. If the conclusion of the lemma is not true, then, according to the corollary of Lemmas 3 and 4, the center of the factor loop L/Z contains periodic elements, so there exists an element a such that $a \in Z_2$, $a \notin Z$ and $a^n \in Z$ for a certain integer $n > 0$. Then, for arbitrary elements $b, c \in L$, the associator $[a, b, c]$ belongs to the center Z and it follows that (according (5), (6) to Lemma 2)

$$a^n = a^n[a^n, b, c]^{-1} = a^n L_{c,b} = (aL_{c,b})^n = (a[a, b, c]^{-1})^n = a^n[a, b, c]^{-n}$$

and

$$[a, b, c]^n = 1.$$

Similarly we obtain

$$a^n = a^n[a^n, b] = a^n T_b = (aT_b)^n = (a[a, b])^n = a^n[a, b]^n$$

and

$$[a, b]^n = 1.$$

But, since $[a, b, c]$ and $[a, b]$ belong to the center Z and Z is torsion-free, then

$$[a, b, c] = 1, [a, b] = 1.$$

i.e. $a \in Z$. By this contradiction we finished the proof of the lemma. □

4 Limited Moufang loops

Next we shall show that the following assertion is trues: *the subloops, the factor loops and the direct product of a finite number of finitely separable loops are also finitely separable.*

It is clear that all the subloops of a finitely separable loop L will be finitely separable. Let us consider now a normal subloop H of L. Let us denote $\bar{L} = L/H$, let $\bar{K}$ be a certain subloop of $\bar{L}$ and $\bar{a} \in \bar{L}$, $\bar{a} \notin \bar{K}$. Let K be the complete pre-image of $\bar{K}$ in L by the morphism $\phi : L \to L/H$, and a the pre-image of the element $\bar{a}$ in L by the same morphism ϕ. It is easy to see that $a \notin K$ by hypothesis; in L we shall find a normal subloop N such that L/N be finite and $aN \notin K/N$. Let ψ be the loop morphism of the loop L with the kernel HN. Then L^{ψ} is finite and at the same time representing a contradiction. Since $Ker\phi \subseteq Ker\psi$, there exists a loop morphism λ of the loop $\bar{L}$ on L^{ϕ} such that $a^{\psi} = a^{\phi}$, $\forall\, a \in L$. Then $a^{\psi} \notin K^{\psi}$ implies $a^{\phi\lambda} \notin K^{\phi\lambda}$, i.e. $\bar{a} \notin \bar{K}^{\lambda}$.

We prove the theorem. For this it is enough to investigate the direct product of two loops $L = H \times K$. Let A be a certain subloop of L, B and C be the corresponding projections of H and K. We denote $M = A \cap H$ and let $a \in L$, $a \notin A$, $a = bc$, $b \in H$, $c \in K$. If $a \notin BC$, then $b \notin B$ or $c \notin C$ and in order to prove the theorem it is enough to consider the factor loop L/K or L/H. In order to do it suppose that $a \in BC$. Let $b_0c \in A$, $b_0 \in B$. Then $a/b_0c = b/b_0 \notin A$, so t$b/b_0 \notin M$. According to the hypothesis, the element b/b_0 and the subloop $M \subseteq H$, which does not contain this element, are separable in H. This means that in H there is a normal loop H_0 such that the factor loop H/H_0 is finite and $b/b_0 \notin H_0M$. Let us show that $a \notin H_0A$. Assume the contrary, $a = h \cdot b_1c_1 = hb_1 \cdot c_1$, $h \in H_0$, $b_1c_1 \in A$, hence $b = hb_1, c = c_1$. Since $b_1c \in A$ and $b_0c \in A$, then $b_1/b_0 \in A \cap H = M$. Therefore, $b/b_0 = hb_1/b_0 \in H_0M$ contradicts the selection of H_0. Thus, by the canonical mapping of $L = H \times K$ onto $H/H_0 \times K$, the image of the element a does not belong to the subloop A. Applying once more the argument indicated for the product $H/H_0 \times K$, we obtain that the image of the element a does not belong to the image of the subloop A in the finite loop $H/H_0 \times K/K_0$; but this is what it was to be proved.

We investigate now an example where we indicate an finitely approximable non-separable loop. In the variety of commutative nilpotent of class 2 Moufang loops we consider the free loop F of strictly countable rank and the loop L given by the generators $x, x_i, i = 1, 2, \ldots$, and the relations

$$x_i^3 = 1, [x_i, x_{2i}, x_{3i}] = x, i = 1, 2, \ldots,$$

$$[x_j, x_k, x_l] = 1, (j, k, l) \notin \{(i, 2i, 3i) | i = 1, 2, \ldots\}.$$

This loop L has the associant-commutant $H = \{1, x, x^2\}$, which coincides with its center. But the factor loop of L by its center is the direct sum of an infinite number of cyclic groups of order 3. Hence, L/H and H are Abelian groups with finitely separable subgroups. We show now that L is not finitely separable. Indeed, let N be a certain normal subloop of finite index of L. Then, for certain $i, j, i \neq j, x_i = x_j \, mod\, N$ or $x_i x_j^{-1} \in N$. From here $x = [x_i, x_{2i}, x_{3i}] = [x_i x_j^{-1}, x_{2i}, x_{3i}] \in N$, i.e. all the normal subloops contain the element

x and then L is not finitely separable. At the same time, an approximable finite loop F, which is mapped homomorphically onto L, can not be finitely separable.

According to [6], Abelian groups, the orders of all elements which are powers fixed prime number is called primary. Abelian group G is called limited if all its primary periodical part of P are finite, and the factor group G/P is finitely separable Abelian group.

Definition 1. *Loop L is called* limited *if it has at least one finite series whose factors are limited Abelian groups.*

From the proof of the last assertion and Schreier theorem about the thickening of the normal sequence for loops (see [3]) it follows that the factors of any finite normal sequence of a limited solvable loop are limited Abelian groups. Hence, it follows that : *the subloops, the factor subloops and the direct product of a finite number of limited solvable loops are limited solvable loops.*

A.I. Mal'cev in [7] studied limited solvable groups, and proved that a limited solvable group is finitely separable. In the following we will show that this result can be extended to limited solvable Moufang loops.

Lemma 6. *Let L be a Moufang loop and L^n be the subloop generated by n-th powers of all the elements of L. If the loop L is solvable and limited, then the factor loops $L/L^n, n = 1, 2, \ldots,$ are finite.*

Proof. Indeed, according to [6], L^n is normal in L, then L/L^n is a solvable limited loop, the orders of its elements dividing n. Hence it follows that the factors of the normal sequence of L/L^n have only a finite number of primary components and then are finite, and together with them is finite the loop L/L^n itself. □

Lemma 7. *If the associant-commutant L' of a Moufang loop L is finite, then in L the following identities hold*

$$\left[x^{m!}, y, z\right] = 1, \left[x^{m!}, y\right] = 1,$$

where $m = |L'|$.

Proof. Indeed, let a, b be arbitrary elements of L. Then in the sequences of elements

$$[a, b, c], [a^2, b, c], \ldots, [a^{m+1}, b, c]$$

and

$$[a, b], [a^2, b], \ldots, [a^{m+1}, b]$$

not all the pairs of elements are different. Hence, for certain $i, j, k, l \leq m + 1, i \neq j, k \neq l$

$$[a^i, b, c] = [a^j, b, c], \quad [a^k, b] = [a^l, b].$$

From these relations and (5), (6) we obtain

$$a^{m!}L_{c,b} = (a^{j-i}L_{c,b})^{m!/j-i} = \left[a^j L_{c,b}(a^i L_{c,b})^{-1}\right]^{m!/j-i} =$$

$$(a^j a^{-i})^{m!/j-i} = a^{m!},$$
$$a^{m!}T_b = (a^{l-k}T_b)^{m!/l-k} = \left[a^l T_b (a^k T_b)^{-1}\right]^{m!/l-k} =$$
$$(a^l a^{-k})^{m!/l-k} = a^{m!},$$

hence
$$[a^{m!}, b, c] = 1, \quad [a^{m!}, b] = 1$$
and the lemma is proved. □

Lemma 8. *A solvable limitated Moufang loop is finitely approximable.*

Proof. For Abelian groups the lemma is true. We shall apply the induction method, assuming that the given Moufang loop L is solvable of class n and that for Moufang loops with solvability class less than n the lemma is true.

Let b be an arbitrary element of L. If a does not belong to L', then the image $\bar{a}$ of the element a in the factor loop L/L' will be different from the unity. Since L/L' is an Abelian group, there exists a normal subgroup N/L' with finite index which does not contain the element $\bar{a}$, and we obtained what we untented. Let $a \in L'$. By hypothesis, in L' there is a normal subloop M of a certain finite index s which does not contain the element a. According to Lemma 6, the subloop L'^s is normal in L' and has a certain finite index s'. The subloop L'^s is normal in L and $L'^s \subseteq M$. Let $H = L/L'^s$, K and b the corresponding images of L' and a in H. We have $b \neq 1, b \notin K, K$ is a normal finite subloop in H and H/K is an Abelian group. Then the power of the associant-commutant of the loop H is equal to a finite number s'. Now, if $b \notin H^{s'!}$ on account of Lemma 6, the complete preimage in L of $H^{s'!}$ will be the subloop we are looking for. Let $b \in H^{s'!}$. According to Lemma 7, the generating elements of the loop $H^{s'!}$ belong to the center of the loop H. Therefore, all the subloops of $H^{s'!}$ are normal in H. Since $H^{s'!}$ is a limited Abelian group, it follows that it is finitely approximated. Hence, taking into account that in H this group has a finite index, it follows that H is also finitely approximated. □

From Lemma 8, in particular we obtain the following

Corollary 1. *Each finite subloop of a limited solvable Moufang loop L is finitely separable.*

Proof. Indeed, let $a \notin H, H = \{h_1, \ldots, h_m\}$. According to Lemma 8, in L there is a normal subloop N_i of finite index such that $h_i^{-1}a \notin N_i, i = 1, \ldots, m$. Then $N = \bigcap_{i=1}^{n} N_i$, will be a normal subloop of finite index in L for which $a \notin HN$, we looked for. □

Theorem 1. *All the subloops of a limited solvable Moufang loop are finitely separable.*

Proof. For Abelian groups the theorem is true, so we shall apply the induction, assuming that the given Moufang loop is solvable of class $n > 1$ and that for solvable Moufang loops of class smaller than n the theorem is proved.

Let B be a subloop of a Moufang loop $L, a \in L$ and $a \notin B$. If $a \notin BL'$, than in the Abelian group L/BL' we shall find a normal subloop N/BL' of finite index which does not contain the element a. Then the normal subloop N may be considered as the loop we look for and which separates a from B.

Suppose that $a \in BL'$. Then $a = bh$, where $b \in B, h \in L', T = B \cap L'$. Since $h \notin T$ and the Moufang loop L' is solvable of class $n-1$, we shall find in L' a normal subloop of finite index M for which $h \notin TM$. Let us denote by s the index of M in L'. Then the factor loop $L'/{L'}^s$ is finite according to Lemma 6, ${L'}^s$ is normal in L and $h \notin T{L'}^s$. If it happens that $a \in B{L'}^s$, then $a = b_0h_0$, where $b_0 \in B, h_0 \in {L'}^s$, and we shall have

$$bh = b_0h_0, \quad h = b^{-1} \cdot b_0h_0 = (b^{-1}b_0 \cdot h_0)\left[b^{-1}, b_0, h_0\right],$$

$$b^{-1}b_0 = h\left[b^{-1}, b_0, h_0\right]^{-1} \cdot h_0^{-1} \in B \cap L' = T, \quad h \in T{L'}^s,$$

which contradicts the hypothesis. Thus $a \notin B{L'}^s$.

Let us denote $L/{L'}^s = H$ and let the considered images of the loops L', B and of the element a in H be, respectively, K, C, b. Then we have: K is a normal subloop in H, the factor H/K is an Abelian group, $b \notin C$, KC is a normal subloop in H. Let us denote $m = |K|$ and $N = (KC)^{m!}$. The subloop N is normal in H. We show that $N \subseteq C$. Let $K = f_1, \ldots, f_m$ and f, c be arbitrary elements, respectively, of K, C. Then in the sequence $fc, (fc)^2, \ldots, (fc)^{m+1}$ of elements of $FC = \bigcup_{i=1}^{m} f_iC$ there are two elements $(fc)^i, (fc)^{m+1}, i \neq j, i \leq m+1, j \leq m+1$, which belong to the same set f_kC for a certain $k \leq m$. Thus

$$(fc)^i = f_kc', (fc)' = f_kc",$$

where $c', c" \in C$. Then

$$(fc)^i c'^{-i} = (fc)^j c"^{-1},$$
$$(fc)^{i-j} = c"^{-1}c' \in C,$$

hence

$$(fc)^{m!} \in C.$$

Therefore, all the elements generating the subloop $(KS)^{m!}$ belong to C, so $(KS)^m$ is contained in C. Let us denote $N = (KS)^{m!}$ and consider the factor loop $H/N = H_1$. Let C_1 and b_1 be the images of C and b in H_1. Since $b_1 \notin C_1$, and C_1 is finite, according to Corollary 1, there exists in H_1 a normal subloop N_1 such that $b_1 \notin C_1N_1$. In this way, we constructed the chain of loop morphisms $L \to H \to H_1 \to H_1/N_1$. The result of the morphism of this chain is the loop morphism of L onto the finite loop H_1/N_1, for which the image of the element a is not contained in the image of the subloop B, as we wanted. □

Corollary 2. *Limited solvable Moufang loops is not only sufficient but also necessary for the finite separability subloops in the case of Moufang loops without torsion.*

Proof. In fact, let L be a solvable Moufang loop without torsion. finitely separable. Separability of loops L involves the separability of subloops and factor loops L and, consequently, the separability factors in the normal soluble series $L \supset L' \supset L'' \supset \ldots \supset L^{(n)} = \{e\}$ where $L' = [L, L], L'' = [L', L'], \ldots, L^{(n)} = [L^{(n-1)}, L^{(n-1)}]$. Since by Lemma 5, all the factors L/L', L'/L'', …, $L^{(n-1)}/L^{(n)}$ are Abelian groups without torsion, sou are limited, Therefore, considered Moufang loop L is limited. □

From Theorem 1, in particular, resulted finite separation of Moufang loops with maximality condition for subloops (a condition in which every increasing number of its subloops terminates). Of these loops are, for example, finitely generated nilpotent Moufang loops, finitely generated commutative Moufang loops and finitely generated nilpotent A-loops. Maximality condition for these loops are proved in [8], [9], [10]. We can assume that similar results are Lemmas 3, 4, 5, 6, 7, 8 and Theorem 1 can also be proved for the A-loops.

5 Applications

a) Following [1], we build on a finite approximate loop a topology. Indeed, let L is a finitely residual loop and $S = \{H_i \mid i \in I\}$ – a system composed from all of its normal subloops. Under the definition of opened sets we can understand adjacent classes of normal subloops from S, also any reunion of adjacent classes. We ca notice that the intersection of ani two sub-loops from S belong also to the S sets and intersection of all sub-loops from S according to the finitely residuality of L, is the unity subloop. From these proprieties results that the open sets define the L as a Hausdorff topology where the basic operations of the loop are continuous. By this way the loop L transforms itself in a topologic loop. Let's suppose that the L loop contains a H subloop and $a \in L$, $a \notin H$. Let's suppose that for a specific homomorphism φ of L loop in a finite loop $a\varphi \notin H\varphi$ and let it be $N = \ker\varphi = \{x \in L \mid x\varphi = e\}$ the nucleus of this homomorphism. Then the adjacent class $[a]$ that corresponds to the normal subloop N does not intersect with the H subloop. Taking in consideration that that the set $[a]$ is an opened one than the H is finitely separable from the element a. Particularly, we obtain the following: the finitely separability of H in relation to the all elements of L that do not belong to H is equal to the fact that H should be closed in finitely topology. In conclusion the loops with finitely separable subloops are those loops whose subloops in finite topology are finite sets. Now according to Theorem 1, we can affirm the following statement.

Theorem 2. *All subloops of finite separability solvable Moufang loops are closed sets.*

b) Let the loop L be given by a finite system of generators $a_1, \dots, a_m$, and a finite system R of relations. Suppose that besides this, we give a finite number of elements of L, which are represented as words $u, u_1, \dots, u_n$ from the generators $a_1, \dots, a_m$ of this loop L. The problem about the membership of an element u to the subloop B, generated by the elements $u_1, \dots, u_n$, consists in indicating an algorithm in order to establish if the element u belongs or does not to B.

Theorem 3. *If the nilpotent Moufang loop (commutative Moufang loop or nilpotent A-loop) L is defined by a finite number of generators and by a finite number of relations, then for the loop L the membership problem of a subloop element is algorithmic solvable.*

In order to prove this is it sufficient to use a solvable process whose step $2s-1$ consists in what follows. From the beginning, according to McKinsey known method, we distribute all finite and nilpotent Moufang loops L_i in a sequence $L_1, L_2, \dots$. Now in each loop L_s we choose, in all possible ways, as much elements $a_i^s \in L_s, i = 1, \ldots, m$, as are given generators $a_1, \ldots, a_m$ in the loop L. Assuming that the words $u, u_1, \ldots, u_n$ of $a_1, \ldots, a_m$ are fixed, we distribute in a sequence all the words $v_1, v_2, \ldots$ of $u_1, \ldots, u_n$. We consider now the loop L_s and see if $a_1^s, \ldots, a_m^s$ are the generators of L_s and if the relations of L are verified for them. If not, then we pass to the next step. If yes, then we determine if the element $u(a_1^s, \ldots, a_m^s)$ belongs to the subloop generated in L_s by $u_1(a_1^s, \ldots, a_n^s), \ldots, u_n(a_1^s, \ldots, a_m^s)$. If not, then the process is interrupted and in the loop L the element u does not belong to the subloop generated by the elements $u_1, \ldots, u_n$. If yes, we pass to the step of the algorithm with the number $2s$, which consists in determining if in L holds the equality $u = v_s$. Since Moufang loop is nilpotent, according to [8]([9], [10]) there exists the solvability algorithm of equality problem of two words. If $u \neq v_s$, then we pass to the next step.

Since a finitely generated and nilpotent Moufang loop is limited, Theorem 1 assures that the process, after a finite number of steps is interrupted and the problem of the membership of u to the subloop generated in L by the elements $u_1, \ldots, u_n$ is solvable.

Results Theorems 1, 2, and 3 can be transferred to the CH-quasigroups and quasigroup distribution using structural links with the commutative Moufang loops, which are described in [12], [13], [14], [15]. Mention, notion finite separable quasigroups is defined analogous to the universal algebras.

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