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TH-WAVES PROPAGATION IN CRYSTALS SUBJECT TO INITIAL FIELDS

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Abstract

In this paper we are dealing with the study of the coupling conditions for propagation of planar guided waves in a piezoelectric semi-infinite plane subject to initial electro-mechanical fields. We analyze the propagation of non-piezoelectric TH waves in anisotropic crystals, subject to an initial mechanical field, for two cases: parallel-sided plate, resp. layer on a substrate. Last case is related to the well known Love type wave described in seismology problems. We obtain and analyze the dispersion relations for various classes of anisotropy.

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1 Introduction

The problems related to electroelastic materials subject to incremental fields superposed on initial mechanical and electric fields have attracted considerable attention last period, due their complexity and to multiple applications. Last decade we dealt with various problems in the field, such as progressive waves and attenuated waves propagation in piezoelectric crystals subject to an electromechanical bias, and the propagation of waveguides in monoclinic crystals subject to initial fields (see papers [9]-[23], or the chapter [8] for an overview of our results).

In this paper we present new results related to guided waves propagation in anisotropic crystals subject to initial fields. We analyze here the propagation of non-piezoelectric TH waves subject to initial mechanical field for two cases: parallel-sided plate, resp. layer on a substrate. Our results generalize in the second case, for initial mechanical fields, the classical results from seismology concerning Love waves propagation (see [6] and [7]). Using the general results obtained in chapter [8], we obtain and analyze the dispersion relations into a parallel-sided plate, resp. into a layer on a substrate, for various classes of anisotropy.

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2 Fundamental equations. Geometric hypotheses

We assume the material to be an elastic dielectric, which is nonmagnetizable and conducts neither heat, nor electricity. Consequently, we shall use the quasi-electrostatic approximation of the equations of balance. Furthermore, we assume that the elastic dielectric is homogeneous, and that we apply on initial large homogeneous deformations and an initial large homogeneous electric field.

To describe this situation we use three different configurations : the *reference configuration* B_R in which at time $t = 0$ the body is undeformed and free of all fields; the *initial configuration* $\overset{\circ}{B}$ in which the body is deformed statically and carries the large initial fields; the *present (current)configuration* B_t obtained from $\overset{\circ}{B}$ by applying time dependent incremental deformations and fields. In what follows, all the fields related to the initial configuration $\overset{\circ}{B}$ will be denoted by a superposed "$\circ$".

In this case the *homogeneous field equations* take the following form:

$$\overset{\circ}{\rho}\, \ddot{\boldsymbol{u}} = \text{div}\, \boldsymbol{\Sigma},\ \text{div}\, \boldsymbol{\Delta} = 0$$
$$\text{rot}\, \boldsymbol{e} = 0 \ \Leftrightarrow\ \boldsymbol{e} = -\text{grad}\, \varphi \tag{1}$$

where $\overset{\circ}{\rho}$ is the mass density, $\boldsymbol{u}$ is the incremental displacement from $\overset{\circ}{B}$ to B_t, $\boldsymbol{\Sigma}$ is the incremental electromechanical nominal stress tensor, $\boldsymbol{\Delta}$ is the incremental electric displacement vector, $\boldsymbol{e}$ is the incremental electric field and φ is the incremental electric potential. All incremental fields involved into the above equations depend on the spatial variable $\boldsymbol{x}$ and on time t.

We suppose the following *incremental constitutive equations*:

$$\Sigma_{kl} = \overset{\circ}{\Omega}_{klmn}\, u_{m,n} + \overset{\circ}{\Lambda}_{mkl}\, \varphi_{,m}$$
$$\Delta_k = \overset{\circ}{\Lambda}_{kmn}\, u_{n,m} + \overset{\circ}{\epsilon}_{kl}\, e_l = \overset{\circ}{\Lambda}_{kmn}\, u_{n,m} - \overset{\circ}{\epsilon}_{kl}\, \varphi_{,l}. \tag{2}$$

In these equations $\overset{\circ}{\Omega}_{klmn}$ are the components of the instantaneous elasticity tensor, $\overset{\circ}{\Lambda}_{kmn}$ are the components of the instantaneous coupling tensor and $\overset{\circ}{\epsilon}_{kl}$ are the components of the instantaneous dielectric tensor. The instantaneous coefficients can be expressed in terms of the classical moduli of the material and on the initial applied fields as follows:

$$\overset{\circ}{\Omega}_{klmn} = \overset{\circ}{\Omega}_{nmlk} = c_{klmn} + \overset{\circ}{S}_{kn}\, \delta_{lm} - e_{kmn}\, \overset{\circ}{E}_l - e_{nkl}\, \overset{\circ}{E}_m - \eta_{kn}\, \overset{\circ}{E}_l \overset{\circ}{E}_m,$$
$$\overset{\circ}{\Lambda}_{mkl} = e_{mkl} + \eta_{mk}\, \overset{\circ}{E}_l,\ \ \overset{\circ}{\epsilon}_{kl} = \overset{\circ}{\epsilon}_{lk} = \delta_{kl} + \eta_{kl}, \tag{3}$$

where c_{klmn} are the components of the constant elasticity tensor, e_{kmn} are the components of the constant piezoelectric tensor, ϵ_{kl} are the components of the constant dielectric tensor,

$\overset{\circ}{E}_i$ are the components of the initial applied electric field and $\overset{\circ}{S}_{kn}$ are the components of the initial applied symmetric (Cauchy) stress tensor.

From the previous field and constitutive equations we obtain the following *fundamental system of equations*:

$$\overset{\circ}{\rho}\,\ddot{u}_l = \overset{\circ}{\Omega}_{klmn}\,u_{m,nk} + \overset{\circ}{\Lambda}_{mkl}\,\varphi_{,mk},\ \overset{\circ}{\Lambda}_{kmn}\,u_{n,mk} - \overset{\circ}{\epsilon}_{kn}\,\varphi_{,nk} = 0,\ l = \overline{1,3}. \tag{4}$$

In what follows we shall describe the *geometric hypotheses* for our problem. The crystal is assumed to be semi-infinite, occupying the region $x_2 > 0$, and the waves are supposed to propagate along x_1 axis. The plane x_1x_2 containing the surface normal and the propagation direction is called *sagittal plane.* Furthermore, we suppose that the guide of waves has the properties invariant with time t and with x_1 variable. In these conditions, if the material behaves linearly and without attenuation, the *normal modes* will have the form:

$$u_j(\boldsymbol{x},t) = u_j^0(x_2,x_3)\exp[\mathrm{i}(\omega t - kx_1)],\quad j = \overline{1,4}. \tag{5}$$

Here u_1, u_2, u_3 are the mechanical displacements, while u_4 stands for the electric potential φ. In the previous relations k represents the *wave number*, ω defines the *frequency* of the wave and $\mathrm{i}^2 = -1$. Using these hypotheses, the equations (4) become:

$$\overset{\circ}{\Omega}_{klmn}\,u_{m,nk} + \overset{\circ}{\Lambda}_{mkl}\,\varphi_{,mk} = -\overset{\circ}{\rho}\,\omega^2 u_l,\ \overset{\circ}{\Lambda}_{kmn}\,u_{n,mk} = \overset{\circ}{\epsilon}_{kn}\,\varphi_{,nk},\ l = \overline{1,3}. \tag{6}$$

We define the non-dimensional variable $X_2 = kx_2$ and we neglect the effects of diffraction in x_3 direction, so that $\partial/\partial x_3 = 0$. From the other hypotheses it yields the derivation rules $\partial/\partial x_1 = -\mathrm{i}k$ and $\partial/\partial x_2 = k\partial/\partial X_2$. Finally, we introduce the *phase velocity* of the guided wave as $V = \omega/k$.

3 Coupling conditions for waveguide propagation in crystals

To analyze the coupling of plane waveguide, using the previous hypotheses, we introduce the *differential operators* with complex coefficients, as follows:

$$\begin{aligned}
\overset{\circ}{\Gamma}_{il} &= \overset{\circ}{\Omega}_{1il1} - \overset{\circ}{\Omega}_{2il2}\,\frac{\partial^2}{\partial X_2^2} + \mathrm{i}(\overset{\circ}{\Omega}_{1il2} + \overset{\circ}{\Omega}_{1li2})\frac{\partial}{\partial X_2},\\
\overset{\circ}{\gamma}_{l} &= \overset{\circ}{\Lambda}_{11l} - \overset{\circ}{\Lambda}_{22l}\,\frac{\partial^2}{\partial X_2^2} + \mathrm{i}(\overset{\circ}{\Lambda}_{12l} + \overset{\circ}{\Lambda}_{21l})\frac{\partial}{\partial X_2},\\
\overset{\circ}{\epsilon} &= \overset{\circ}{\epsilon}_{11} - \overset{\circ}{\epsilon}_{22}\,\frac{\partial^2}{\partial X_2^2} + 2\mathrm{i}\,\overset{\circ}{\epsilon}_{12}\,\frac{\partial}{\partial X_2}.
\end{aligned} \tag{7}$$

In these conditions, after a lengthy, but elementary calculus, we obtain that the differential system (6) has the following form:

$$\begin{pmatrix} \overset{\circ}{\Gamma}_{11} - \overset{\circ}{\rho}\, V^2 & \overset{\circ}{\Gamma}_{12} & \overset{\circ}{\Gamma}_{13} & \overset{\circ}{\gamma}_1 \\ \overset{\circ}{\Gamma}_{12} & \overset{\circ}{\Gamma}_{22} - \overset{\circ}{\rho}\, V^2 & \overset{\circ}{\Gamma}_{23} & \overset{\circ}{\gamma}_2 \\ \overset{\circ}{\Gamma}_{13} & \overset{\circ}{\Gamma}_{23} & \overset{\circ}{\Gamma}_{33} - \overset{\circ}{\rho}\, V^2 & \overset{\circ}{\gamma}_3 \\ \overset{\circ}{\gamma}_1 & \overset{\circ}{\gamma}_2 & \overset{\circ}{\gamma}_3 & -\overset{\circ}{\epsilon} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = 0. \tag{8}$$

Here the coefficients are defined by relations (7). The system (8) is a homogeneous differential system of four equations with four unknowns, i.e. the components of the mechanical displacement and the electric potential, having as coefficients complex differential operators in non-dimensional variable X_2. It generalizes the similar system from the case without initial fields, derived in [7].

In what follows we shall analyze the coupling conditions of the guided plane wave propagation in two particular cases.

3.1 Sagittal plane normal to a direct axis of order two

In this case, we suppose that *the sagittal plane x_1x_2 is normal to a dyad axis* (x_3 in our case). Then, the elastic constants with one index equal to 3 are zero (see [7] for details). After a short inspection of the coefficients of the system (8), using Voigt convention, we find:

$$\begin{aligned} \overset{\circ}{\Gamma}_{13} &= -[e_{15} + \mathrm{i}(e_{14} + e_{25})\frac{\partial}{\partial X_2} - e_{24}\frac{\partial^2}{\partial X_2^2}]\, \overset{\circ}{E}_1 - [\eta_{11} + 2\mathrm{i}\eta_{12}\frac{\partial}{\partial X_2} - \eta_{22}\frac{\partial^2}{\partial X_2^2}]\, \overset{\circ}{E}_1 \overset{\circ}{E}_3, \\ \overset{\circ}{\Gamma}_{23} &= -[e_{15} + \mathrm{i}(e_{14} + e_{25})\frac{\partial}{\partial X_2} - e_{24}\frac{\partial^2}{\partial X_2^2}]\, \overset{\circ}{E}_2 - [\eta_{11} + 2\mathrm{i}\eta_{12}\frac{\partial}{\partial X_2} - \eta_{22}\frac{\partial^2}{\partial X_2^2}]\, \overset{\circ}{E}_2 \overset{\circ}{E}_3\ . \end{aligned} \tag{9}$$

We can easily observe that $\overset{\circ}{\Gamma}_{13}$ and $\overset{\circ}{\Gamma}_{23}$ does not depend on the initial stress field components, but on the initial electric field components, only. Thus, $\overset{\circ}{\Gamma}_{13} = \overset{\circ}{\Gamma}_{23} = 0$ if $\overset{\circ}{E}_1 = \overset{\circ}{E}_2 = 0$.

Moreover, if we suppose that *the dyad axis is direct* (this means that the sagittal plane is normal to a direct axis of order two), it follows that the crystal belongs to the class 2 of the *monoclinic system* ($A_2 \parallel x_3$). In this particular case the piezoelectric constants with no index equal to 3 are zero (as described in [7]). Therefore, we obtain:

$$\overset{\circ}{\gamma}_1 = (\eta_{11} + 2\mathrm{i}\eta_{12}\frac{\partial}{\partial X_2} - \eta_{22}\frac{\partial^2}{\partial X_2^2})\, \overset{\circ}{E}_1,\ \overset{\circ}{\gamma}_2 = (\eta_{11} + 2\mathrm{i}\eta_{12}\frac{\partial}{\partial X_2} - \eta_{22}\frac{\partial^2}{\partial X_2^2})\, \overset{\circ}{E}_2\ . \tag{10}$$

So, we obtain that $\overset{\circ}{\gamma}_1 = \overset{\circ}{\gamma}_2 = 0$ if $\overset{\circ}{E}_1 = \overset{\circ}{E}_2 = 0$.

In conclusion, we derive the following result concerning the decomposition of the fundamental system (8):

If the axis x_3 is a direct dyad axis and if $\overset{\circ}{E}_1 = \overset{\circ}{E}_2 = 0$, the system (8) reduces to two independent subsystems, as follows:

a) The first subsystem:

$$\begin{pmatrix} \overset{\circ}{\Gamma}_{11} - \overset{\circ}{\rho}\, V^2 & \overset{\circ}{\Gamma}_{12} \\ \overset{\circ}{\Gamma}_{12} & \overset{\circ}{\Gamma}_{22} - \overset{\circ}{\rho}\, V^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0. \tag{11}$$

defines a non-piezoelectric guided wave, polarized in the sagittal plane x_1x_2, which depends on the initial stress field, only. We shall denote it by $\overset{\circ}{P}_2$. These characteristics are due to the form of the involved coefficients:

$$\begin{aligned} \overset{\circ}{\Gamma}_{11} &= c_{11} + \overset{\circ}{S}_{11} + 2\mathrm{i}(c_{16} + \overset{\circ}{S}_{12})\frac{\partial}{\partial X_2} - (c_{66} + \overset{\circ}{S}_{22})\frac{\partial^2}{\partial X_2^2}, \\ \overset{\circ}{\Gamma}_{12} &= c_{16} + \mathrm{i}(c_{12} + c_{66})\frac{\partial}{\partial X_2} - c_{26}\frac{\partial^2}{\partial X_2^2}, \\ \overset{\circ}{\Gamma}_{22} &= c_{66} + \overset{\circ}{S}_{11} + 2\mathrm{i}(c_{26} + \overset{\circ}{S}_{12})\frac{\partial}{\partial X_2} - (c_{22} + \overset{\circ}{S}_{22})\frac{\partial^2}{\partial X_2^2}. \end{aligned} \tag{12}$$

b) The second subsystem:

$$\begin{pmatrix} \overset{\circ}{\Gamma}_{33} - \overset{\circ}{\rho}\, V^2 & \overset{\circ}{\gamma}_3 \\ \overset{\circ}{\gamma}_3 & -\overset{\circ}{\epsilon} \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = 0. \tag{13}$$

has as solution a transverse-horizontal wave, with polarization after the axis x_3, which is piezoelectric and electrostrictive active, and depends on the initial mechanical and electrical fields. It is denoted by $\overline{\overset{\circ}{TH}}$ and generalizes the famous Bleustein-Gulyaev wave (see [7], to compare). The components involved into this equation have the form:

$$\begin{aligned} \overset{\circ}{\Gamma}_{33} &= c_{55} + \overset{\circ}{S}_{11} + 2\mathrm{i}(c_{45} + \overset{\circ}{S}_{12})\frac{\partial}{\partial X_2} - (c_{44} + \overset{\circ}{S}_{22})\frac{\partial^2}{\partial X_2^2} \\ &\quad -2[e_{15} + \mathrm{i}(e_{14} + e_{25})\frac{\partial}{\partial X_2} - e_{24}\frac{\partial^2}{\partial X_2^2}]\, \overset{\circ}{E}_3 - [\eta_{11} + 2\mathrm{i}\eta_{12}\frac{\partial}{\partial X_2} - \eta_{22}\frac{\partial^2}{\partial X_2^2}]\, \overset{\circ}{E}{}_3^2, \\ \overset{\circ}{\gamma}_3 &= e_{15} + \mathrm{i}(e_{14} + e_{25})\frac{\partial}{\partial X_2} - e_{24}\frac{\partial^2}{\partial X_2^2} + [\eta_{11} + 2\mathrm{i}\eta_{12}\frac{\partial}{\partial X_2} - \eta_{22}\frac{\partial^2}{\partial X_2^2}]\, \overset{\circ}{E}_3, \\ \overset{\circ}{\epsilon} &= \overset{\circ}{\epsilon}_{11} + 2\mathrm{i}\, \overset{\circ}{\epsilon}_{12}\, \frac{\partial}{\partial X_2} - \overset{\circ}{\epsilon}_{22}\, \frac{\partial^2}{\partial X_2^2} = 1 + \eta_{11} + 2\mathrm{i}\eta_{12}\frac{\partial}{\partial X_2} - (1 + \eta_{22})\frac{\partial^2}{\partial X_2^2}. \end{aligned} \tag{14}$$

3.2 Sagittal plane parallel to a mirror plane

We suppose now that *the sagittal plane x_1x_2 is normal to an inverse dyad axis* (x_3 in our case) or, equivalently, if *the sagittal plane is parallel to a mirror plane M*. It follows that the crystal belongs to the class m of the *monoclinic system* ($M \perp x_3$). In this particular case the elastic constants with one index equal to 3 are zero, as well as the piezoelectric constants with one index equal to 3, which vanish (see [7] for details).

Analyzing the coefficients of the system (8) in this case, we find:

$$\overset{\circ}{\Gamma}_{13}= -[e_{11} + \mathrm{i}(e_{21} + e_{16})\frac{\partial}{\partial X_2} - e_{26}\frac{\partial^2}{\partial X_2^2}]\ \overset{\circ}{E}_3 - [\eta_{11} + 2\mathrm{i}\eta_{12}\frac{\partial}{\partial X_2} - \eta_{22}\frac{\partial^2}{\partial X_2^2}]\ \overset{\circ}{E}_1\overset{\circ}{E}_3,$$

$$\overset{\circ}{\Gamma}_{23}= -[e_{16} + \mathrm{i}(e_{26} + e_{12})\frac{\partial}{\partial X_2} - e_{22}\frac{\partial^2}{\partial X_2^2}]\ \overset{\circ}{E}_3 - [\eta_{11} + 2\mathrm{i}\eta_{12}\frac{\partial}{\partial X_2} - \eta_{22}\frac{\partial^2}{\partial X_2^2}]\ \overset{\circ}{E}_2\overset{\circ}{E}_3,$$

$$\overset{\circ}{\gamma}_3= (\eta_{11} + 2\mathrm{i}\eta_{12}\frac{\partial}{\partial X_2} - \eta_{22}\frac{\partial^2}{\partial X_2^2})\ \overset{\circ}{E}_3\ . \tag{15}$$

It yields that $\overset{\circ}{\Gamma}_{13}=\overset{\circ}{\Gamma}_{23}= 0$ and $\overset{\circ}{\gamma}_3= 0$ if $\overset{\circ}{E}_3= 0$.

Thus, *if the axis x_3 is an inverse dyad axis and if $\overset{\circ}{E}_3= 0$, the fundamental system (8) splits into two parts*, as follows.

a) The first subsystem has the form:

$$\begin{pmatrix} \overset{\circ}{\Gamma}_{11} - \overset{\circ}{\rho}\, V^2 & \overset{\circ}{\Gamma}_{12} & \overset{\circ}{\gamma}_1 \\ \overset{\circ}{\Gamma}_{12} & \overset{\circ}{\Gamma}_{22} - \overset{\circ}{\rho}\, V^2 & \overset{\circ}{\gamma}_2 \\ \overset{\circ}{\gamma}_1 & \overset{\circ}{\gamma}_2 & -\overset{\circ}{\epsilon} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_4 \end{pmatrix} = 0. \tag{16}$$

It has as solution a guided wave with sagittal plane polarization, associated with the electric field (*via* the electric potential $u_4 = \varphi$), providing piezoelectric and electrostrictive effects, and depending on the initial stress and electric fields. It is denoted by $\overset{\circ}{\overline{P}}_2$. The electric field, associated with this wave, is contained in the sagittal plane, since $E_3 = \partial\varphi/\partial x_3 = 0$. This fact is consistent with the hypothesis $\overset{\circ}{E}_3= 0$. These features of $\overset{\circ}{\overline{P}}_2$ wave are obtained from the analysis of the corresponding coefficients:

$$\overset{\circ}{\Gamma}_{11}= c_{11}+ \overset{\circ}{S}_{11} - 2e_{11}\ \overset{\circ}{E}_1 - \eta_{11}\ \overset{\circ}{E}{}_1^2 + 2\mathrm{i}[c_{16}+ \overset{\circ}{S}_{12} - (e_{16} + e_{21})\ \overset{\circ}{E}_1 - \eta_{12}\ \overset{\circ}{E}{}_1^2]\frac{\partial}{\partial X_2}$$
$$- (c_{66}+ \overset{\circ}{S}_{22} - 2e_{26}\ \overset{\circ}{E}_1 - \eta_{22}\ \overset{\circ}{E}{}_1^2)\frac{\partial^2}{\partial X_2^2},$$

$$\overset{\circ}{\Gamma}_{12}= c_{16} - e_{16}\ \overset{\circ}{E}_1 - e_{11}\ \overset{\circ}{E}_2 - \eta_{11}\ \overset{\circ}{E}_1\overset{\circ}{E}_2 + \mathrm{i}[c_{12} + c_{66} - (e_{12} + e_{26})\ \overset{\circ}{E}_1$$
$$- (e_{21} + e_{16})\ \overset{\circ}{E}_2 - 2\eta_{12}\ \overset{\circ}{E}_1\overset{\circ}{E}_2]\frac{\partial}{\partial X_2} - (c_{26} - e_{22}\ \overset{\circ}{E}_1 - e_{26}\ \overset{\circ}{E}_2 - \eta_{22}\ \overset{\circ}{E}_1\overset{\circ}{E}_2)\frac{\partial^2}{\partial X_2^2}, \tag{17}$$

$$\overset{\circ}{\Gamma}_{22}= c_{66}+ \overset{\circ}{S}_{11} - 2e_{16}\ \overset{\circ}{E}_2 - \eta_{11}\ \overset{\circ}{E}{}_2^2 + 2\mathrm{i}[c_{26}+ \overset{\circ}{S}_{12} - (e_{26} + e_{12})\ \overset{\circ}{E}_2 - \eta_{12}\ \overset{\circ}{E}{}_2^2]\frac{\partial}{\partial X_2}$$
$$- (c_{22}+ \overset{\circ}{S}_{22} - 2e_{22}\ \overset{\circ}{E}_2 - \eta_{22}\ \overset{\circ}{E}{}_2^2)\frac{\partial^2}{\partial X_2^2},$$

respectively:

$$\overset{\circ}{\gamma}_1 = e_{11} + \eta_{11}\overset{\circ}{E}_1 + \mathrm{i}(e_{16} + e_{21} + 2\eta_{12}\overset{\circ}{E}_1)\frac{\partial}{\partial X_2} - (e_{26} + \eta_{22}\overset{\circ}{E}_1)\frac{\partial^2}{\partial X_2^2},$$

$$\overset{\circ}{\gamma}_2 = e_{16} + \eta_{11}\overset{\circ}{E}_2 + \mathrm{i}(e_{12} + e_{26} + 2\eta_{12}\overset{\circ}{E}_2)\frac{\partial}{\partial X_2} - (e_{22} + \eta_{22}\overset{\circ}{E}_2)\frac{\partial^2}{\partial X_2^2}, \tag{18}$$

$$\overset{\circ}{\epsilon} = \overset{\circ}{\epsilon}_{11} + 2\mathrm{i}\,\overset{\circ}{\epsilon}_{12}\frac{\partial}{\partial X_2} - \overset{\circ}{\epsilon}_{22}\frac{\partial^2}{\partial X_2^2} = 1 + \eta_{11} + 2\mathrm{i}\eta_{12}\frac{\partial}{\partial X_2} - (1 + \eta_{22})\frac{\partial^2}{\partial X_2^2}.$$

b) The second subsystem reduces to a single equation, as follows:

$$(\overset{\circ}{\Gamma}_{33} - \overset{\circ}{\rho}\,V^2)u_3 = 0. \tag{19}$$

Its root corresponds to a transverse-horizontal wave, non-piezoelectric, and influenced by the initial stress field, only. It is called $\overset{\circ}{TH}$ wave. In this equation:

$$\overset{\circ}{\Gamma}_{33} = c_{55} + \overset{\circ}{S}_{11} + 2\mathrm{i}(c_{45} + \overset{\circ}{S}_{12})\frac{\partial}{\partial X_2} - (c_{44} + \overset{\circ}{S}_{22})\frac{\partial^2}{\partial X_2^2}. \tag{20}$$

4 The decoupling of mechanical and electric boundary conditions

In this section we analyze the decomposition of the mechanical, resp. electric boundary conditions at the surface $x_2 = 0$.

4.1 Mechanical boundary conditions

On the boundary surface $x_2 = 0$ the mechanical conditions are supposed to concern the surface stresses Σ_{2i} with $i = \overline{1,3}$. Following the incremental constitutive equations $(2)_1$, we find in this case that:

$$\begin{aligned}
\Sigma_{21} = {} & \overset{\circ}{\Omega}_{2111} u_{1,1} + \overset{\circ}{\Omega}_{2121} u_{2,1} + \overset{\circ}{\Omega}_{2131} u_{3,1} + \overset{\circ}{\Omega}_{2112} u_{1,2} + \overset{\circ}{\Omega}_{2122} u_{2,2} + \overset{\circ}{\Omega}_{2132} u_{3,2} \\
& + \overset{\circ}{\Lambda}_{121} u_{4,1} + \overset{\circ}{\Lambda}_{221} u_{4,2}, \\
\Sigma_{22} = {} & \overset{\circ}{\Omega}_{2211} u_{1,1} + \overset{\circ}{\Omega}_{2221} u_{2,1} + \overset{\circ}{\Omega}_{2231} u_{3,1} + \overset{\circ}{\Omega}_{2212} u_{1,2} + \overset{\circ}{\Omega}_{2222} u_{2,2} + \overset{\circ}{\Omega}_{2232} u_{3,2} \\
& + \overset{\circ}{\Lambda}_{122} u_{4,1} + \overset{\circ}{\Lambda}_{222} u_{4,2}, \\
\Sigma_{23} = {} & \overset{\circ}{\Omega}_{2311} u_{1,1} + \overset{\circ}{\Omega}_{2321} u_{2,1} + \overset{\circ}{\Omega}_{2331} u_{3,1} + \overset{\circ}{\Omega}_{2312} u_{1,2} + \overset{\circ}{\Omega}_{2322} u_{2,2} + \overset{\circ}{\Omega}_{2332} u_{3,2} \\
& + \overset{\circ}{\Lambda}_{123} u_{4,1} + \overset{\circ}{\Lambda}_{223} u_{4,2}.
\end{aligned} \tag{21}$$

a) For a *sagittal plane normal to a direct axis of order two* and if $\overset{\circ}{E}_1 = \overset{\circ}{E}_2 = 0$ we obtain:

$$\begin{aligned}
\Sigma_{21} &= -k\mathrm{i}[(c_{16} + \overset{\circ}{S}_{12})u_1 + c_{66}u_2] + k\frac{\partial}{\partial X_2}[(c_{66} + \overset{\circ}{S}_{22})u_1 + c_{26}u_2],\\
\Sigma_{22} &= -k\mathrm{i}[c_{12}u_1 + (c_{26} + \overset{\circ}{S}_{12})u_2] + k\frac{\partial}{\partial X_2}[c_{26}u_1 + (c_{22} + \overset{\circ}{S}_{22})u_2],\\
\Sigma_{23} &= -k\mathrm{i}[(\overset{\circ}{S}_{12} + c_{45} - e_{25}\overset{\circ}{E}_3 - e_{14}\overset{\circ}{E}_3 - \eta_{12}\overset{\circ}{E}{}_3^2)u_3 + (e_{14} + \eta_{12}\overset{\circ}{E}_3)u_4]\\
&\quad + k\frac{\partial}{\partial X_2}[(c_{44} + \overset{\circ}{S}_{22} - 2e_{24}\overset{\circ}{E}_3 - \eta_{22}\overset{\circ}{E}{}_3^2)u_3 + (e_{24} + \eta_{22}\overset{\circ}{E}_3)u_4].
\end{aligned} \tag{22}$$

Consequently, the mechanical boundary conditions on the plane $x_2 = 0$, under the previous conditions, reduce to the equalities (22), for given stresses Σ_{2i} with $i = \overline{1,3}$.

As regards the boundary conditions associated with the waves previously derived, for $\overset{\circ}{P}_2$ wave we have relations (22) with $u_3 = u_4 = 0$ (it yields that $\Sigma_{23} = 0$ for this wave), while for $\overline{\overset{\circ}{TH}}$ we have the same relations with $u_1 = u_2 = 0$ (it results that $\Sigma_{21} = \Sigma_{22} = 0$ in this case).

b) For a *sagittal plane parallel to a mirror plane* and if $\overset{\circ}{E}_3 = 0$ we derive:

$$\begin{aligned}
\Sigma_{21} &= k[(-\mathrm{i})(c_{16} + \overset{\circ}{S}_{12} - e_{16}\overset{\circ}{E}_1 - e_{21}\overset{\circ}{E}_1 - \eta_{12}\overset{\circ}{E}{}_1^2) + (c_{66} + \overset{\circ}{S}_{22} - 2e_{26}\overset{\circ}{E}_1\\
&\quad - \eta_{22}\overset{\circ}{E}{}_1^2)\frac{\partial}{\partial X_2}]u_1 + k[(-\mathrm{i})(c_{66} - e_{26}\overset{\circ}{E}_1 - e_{16}\overset{\circ}{E}_2 - \eta_{12}\overset{\circ}{E}_1\overset{\circ}{E}_2) + (c_{26} - e_{22}\overset{\circ}{E}_1 - e_{26}\overset{\circ}{E}_2\\
&\quad - \eta_{22}\overset{\circ}{E}_1\overset{\circ}{E}_2)\frac{\partial}{\partial X_2}]u_2 + k[(-\mathrm{i})(e_{16} + \eta_{12}\overset{\circ}{E}_1) + (e_{26} + \eta_{22}\overset{\circ}{E}_1)\frac{\partial}{\partial X_2}]u_4,\\
\Sigma_{22} &= k[(-\mathrm{i})(c_{12} - e_{12}\overset{\circ}{E}_1 - e_{21}\overset{\circ}{E}_2 - \eta_{12}\overset{\circ}{E}_1\overset{\circ}{E}_2) + (c_{26} - e_{22}\overset{\circ}{E}_1 - e_{26}\overset{\circ}{E}_2\\
&\quad - \eta_{22}\overset{\circ}{E}_1\overset{\circ}{E}_2)\frac{\partial}{\partial X_2}]u_1 + k[(-\mathrm{i})(c_{26} + \overset{\circ}{S}_{12} - e_{12}\overset{\circ}{E}_2 - e_{26}\overset{\circ}{E}_2 - \eta_{12}\overset{\circ}{E}{}_2^2) + (c_{22} + \overset{\circ}{S}_{22} - 2e_{22}\overset{\circ}{E}_2\\
&\quad - \eta_{22}\overset{\circ}{E}{}_2^2)\frac{\partial}{\partial X_2}]u_2 + k[(-\mathrm{i})(e_{12} + \eta_{12}\overset{\circ}{E}_2) + (e_{22} + \eta_{22}\overset{\circ}{E}_2)\frac{\partial}{\partial X_2}]u_4,\\
\Sigma_{23} &= k[(-\mathrm{i})(c_{45} + \overset{\circ}{S}_{12}) + (c_{44} + \overset{\circ}{S}_{22})\frac{\partial}{\partial X_2}]u_3.
\end{aligned} \tag{23}$$

Consequently, the mechanical boundary conditions on the plane $x_2 = 0$, under the previous conditions, reduce the the equalities (23), for given stresses Σ_{2i} with $i = \overline{1,3}$.

For $\overline{\overset{\circ}{P}}_2$ wave we have relations (23) with $u_3 = 0$ (it yields $\Sigma_{23} = 0$ for this wave), while for $\overset{\circ}{TH}$ wave we obtain the boundary conditions from (23) with $u_1 = u_2 = u_4 = 0$ (it results $\Sigma_{21} = \Sigma_{22} = 0$ in this case).

We conclude that *the stresses on the horizontal surface* $x_2 = 0$*, associated with the guided waves polarized in the sagittal plane (i.e.* $\overset{\circ}{P}_2$ *and* $\overset{\circ}{\overline{P}}_2$*), become decoupled from those associated with transverse horizontal waves (i.e.* $\overset{\circ}{\overline{TH}}$ *and* $\overset{\circ}{TH}$*), when* x_3 *is a dyad axis normal to the sagittal plane* x_1x_2. Our results generalizes the classical boundary conditions for piezoelectric guided waves without initial fields, as described in [7].

4.2 Electric boundary conditions

On the boundary surface of the domain we suppose the electric boundary condition of the type:

$$\Delta_n = \Delta_i n_i = -\overline{w} \text{ , with } i = \overline{1,3}, \tag{24}$$

where the normal component of the electrical displacement Δ_n is related to the surface density of electric charge $\overline{w}$.

In our case, as the boundary of the domain $x_2 > 0$ is the plane $x_2 = 0$, the previous boundary condition becomes:

$$\Delta_2 = \overline{w} \text{ on } x_2 = 0. \tag{25}$$

Using the constitutive equation $(2)_2$ and the derivation rules, we find that:

$$\begin{aligned}\Delta_2 =& \overset{\circ}{\Lambda}_{2nm} u_{m,n} - \overset{\circ}{\epsilon}_{2l} \varphi_{,l} = k(-\mathrm{i}\, \overset{\circ}{\Lambda}_{211} + \overset{\circ}{\Lambda}_{221} \frac{\partial}{\partial X_2})u_1 + k(-\mathrm{i}\, \overset{\circ}{\Lambda}_{212} + \overset{\circ}{\Lambda}_{222} \frac{\partial}{\partial X_2})u_2 \\ &+k(-\mathrm{i}\, \overset{\circ}{\Lambda}_{213} + \overset{\circ}{\Lambda}_{223} \frac{\partial}{\partial X_2})u_3 + k(\mathrm{i}\, \overset{\circ}{\epsilon}_{12} - \overset{\circ}{\epsilon}_{22} \frac{\partial}{\partial X_2})u_4.\end{aligned} \tag{26}$$

a) For a *sagittal plane normal to a direct axis of order two* and if $\overset{\circ}{E}_1 = \overset{\circ}{E}_2 = 0$ we obtain the following electrical boundary condition:

$$k[(-\mathrm{i}\, e_{25} + e_{24}) + \overset{\circ}{E}_3 (-\mathrm{i}\, \eta_{12} + \eta_{22})\frac{\partial}{\partial X_2}]u_3 + k(\mathrm{i}\, \overset{\circ}{\epsilon}_{12} - \overset{\circ}{\epsilon}_{22} \frac{\partial}{\partial X_2})u_4 = \overline{w} \text{ on } x_2 = 0. \tag{27}$$

It is obvious that this type of boundary condition suits to the $\overset{\circ}{\overline{TH}}$ wave, only.

b) For a *sagittal plane parallel to a mirror plane* and if $\overset{\circ}{E}_3 = 0$ we derive the following electrical boundary condition:

$$\begin{aligned}&k[(-\mathrm{i}\, e_{21} + e_{26}) + \overset{\circ}{E}_1 (-\mathrm{i}\, \eta_{12} + \eta_{22}\frac{\partial}{\partial X_2})]u_1 + k[(-\mathrm{i}\, e_{26} + e_{22}) \\ &+ \overset{\circ}{E}_2 (-\mathrm{i}\, \eta_{12} + \eta_{22}\frac{\partial}{\partial X_2})]u_2 + k(\mathrm{i}\, \overset{\circ}{\epsilon}_{12} - \overset{\circ}{\epsilon}_{22} \frac{\partial}{\partial X_2})u_4 = \overline{w} \text{ on } x_2 = 0.\end{aligned} \tag{28}$$

It is evident that this kind of boundary condition is specific to the wave $\overset{\circ}{\overline{P}}_2$, only.

5 $\overset{\circ}{TH}$ wave propagation

We remind the case of monoclinic crystals with the sagittal plane parallel to a mirror plane and $\overset{\circ}{E}_3 = 0$. There, the fundamental system (8) splits into two subsystems, the second one reducing to a single equation:

$$(\overset{\circ}{\Gamma}_{33} - \overset{\circ}{\rho} V^2)u_3 = 0. \tag{29}$$

Its root corresponds to a transverse-horizontal wave, non-piezoelectric, and influenced by the initial stress field, only. It is called $\overset{\circ}{TH}$ wave. In this equation:

$$\overset{\circ}{\Gamma}_{33} = c_{55} + \overset{\circ}{S}_{11} + 2\mathrm{i}(c_{45} + \overset{\circ}{S}_{12})\frac{\partial}{\partial X_2} - (c_{44} + \overset{\circ}{S}_{22})\frac{\partial^2}{\partial X_2^2}. \tag{30}$$

Substituting $X_2 = kx_2$, $V = \omega/k$ and denoting $u_3 = u(x_2)$, the previous equation has the form:

$$(c_{44} + \overset{\circ}{S}_{22})\frac{\mathrm{d}^2 u}{\mathrm{d}x_2^2} - 2k\mathrm{i}(c_{45} + \overset{\circ}{S}_{12})\frac{\mathrm{d}u}{\mathrm{d}x_2} + [\overset{\circ}{\rho}\,\omega^2 - k^2(c_{55} + \overset{\circ}{S}_{11})]u = 0. \tag{31}$$

In order to simplify the resolution of equation (31), we suppose that the medium is *isotropic in relation with TH waves*. This is the case when x_3 is along a crystal tetrad or hexad axis, or when the whole medium is isotropic. So that, $c_{45} = 0$ and $c_{55} = c_{44}$.

In this case the equation (31) becomes:

$$(1 + \frac{\overset{\circ}{S}_{22}}{c_{44}})u'' - 2k\mathrm{i}\frac{\overset{\circ}{S}_{12}}{c_{44}}u' + [\frac{\omega^2}{v_T^2} - k^2(1 + \frac{\overset{\circ}{S}_{11}}{c_{44}})]u = 0, \tag{32}$$

where $v_T = \sqrt{\frac{c_{44}}{\overset{\circ}{\rho}}}$ is the TH wave velocity in the case without initial fields. This equation is related to the boundary condition $(23)_3$, which has here the form:

$$(-k\mathrm{i})\,\overset{\circ}{S}_{12}\,u + (c_{44} + \overset{\circ}{S}_{22})u' = \Sigma_{23},\ \text{on } x_2 = 0. \tag{33}$$

The equation (32), with the boundary condition (33), will be solved for the following particular problems.

5.1 Parallel-sided plate

The plate is located between the surfaces $x_2 = 0$ and $x_2 = -h$. We suppose that we have homogeneous boundary condition of type (33), i.e. $\Sigma_{23} = 0$, on $x_2 = 0$ and $x_2 = -h$.

We suppose that the solution is sinusoidal, with the form:

$$u(x_2) = u_0 \cos[\frac{n\pi}{h}(x_2 + h)],\ n = 0, 1, 2, ... \tag{34}$$

In this way, the homogeneous boundary conditions are satisfied if the initial shear stress $\overset{\circ}{S}_{12}=0$. Moreover, for n even (odd) the displacement is symmetric (antisymmetric).

Substituting relation (34) into the equation (32) we obtain the *dispersion relation*:

$$(n\pi/h)^2 = \frac{(\omega/v_T)^2 - k^2(1+\overset{\circ}{S}_{11}/c_{44})}{1+\overset{\circ}{S}_{22}/c_{44}},\ n = 0, 1, 2, ... \tag{35}$$

which has, in normalized variables, the form:

$$(\frac{\omega h}{\pi v_T})^2 = (\frac{kh}{\pi})^2(1 + \frac{\overset{\circ}{S}_{11}}{c_{44}}) + n^2(1 + \frac{\overset{\circ}{S}_{22}}{c_{44}}). \tag{36}$$

An alternate form of the dispersion relation may be obtained using $f = \omega/(2\pi)$ as *wave frequency* and $\lambda = 2\pi/k$ as *wave length*:

$$\frac{2fh}{v_T} = \sqrt{(\frac{2h}{\lambda})^2(1 + \frac{\overset{\circ}{S}_{11}}{c_{44}}) + n^2(1 + \frac{\overset{\circ}{S}_{22}}{c_{44}})}. \tag{37}$$

These forms of the dispersion relation for $\overset{\circ}{TH}$ wave propagation in a parallel-sided plate generalize classical results (see [7]).

The velocity oh the guided $\overset{\circ}{TH}$ wave:

$$V = \omega/k = v_T\sqrt{1 + \frac{\overset{\circ}{S}_{11}}{c_{44}} + (\frac{n\pi}{kh})^2(1 + \frac{\overset{\circ}{S}_{22}}{c_{44}})} \tag{38}$$

depends on the frequency, except the mode of order zero, which has the velocity $\overset{\circ}{v}_T = v_T\sqrt{1 + \frac{\overset{\circ}{S}_{11}}{c_{44}}}$.

5.2 Layer on a substrate. Love waves

We suppose an elastic layer $(-h < x_2 < 0)$ bonded to an elastic substrate $(x_2 > 0)$, which are isotropic in relation with TH waves. Moreover, we suppose that $\overset{\circ}{S}_{12}=0$.

In the substrate the displacement must vanish for $x_2 \to \infty$. So, we seek the solution of the equation (32) in the form:

$$u(x_2) = u_0 \exp(-k\chi x_2), \quad \mathrm{Re}[k\chi] > 0, \quad x_2 > 0. \tag{39}$$

The corresponding characteristic equation is:

$$k^2\chi^2(1 + \frac{\overset{\circ}{S}_{22}}{c_{44}}) + \frac{\omega^2}{v_T^2} - k^2(1 + \frac{\overset{\circ}{S}_{11}}{c_{44}}) = 0, \tag{40}$$

which implies

$$V = \frac{\omega}{k} < v_T \sqrt{1 + \frac{\overset{\circ}{S}_{11}}{c_{44}}} \tag{41}$$

under the realistic hypotheses $|\overset{\circ}{S}_{11}|/c_{44} < 1$ and $|\overset{\circ}{S}_{22}|/c_{44} < 1$.

Moreover, we obtain:

$$\chi = \sqrt{\frac{1 + \dfrac{\overset{\circ}{S}_{11}}{c_{44}} - \dfrac{V^2}{v_T^2}}{1 + \dfrac{\overset{\circ}{S}_{22}}{c_{44}}}}. \tag{42}$$

For the layer, we distinguish the variables using a hat. We seek the solution of the equation (32) in a sinusoidal form:

$$u(x_2) = \hat{u}_0 \cos k\hat{\chi}(x_2 + h), \quad -h < x_2 < 0. \tag{43}$$

The mechanical stress $\hat{\Sigma}_{32} = (\hat{c}_{44} + \overset{\circ}{S}_{22}) u'(x_2) = 0$, at the free surface $x_2 = -h$, where the displacement is maximal.

From the characteristic equation we obtain:

$$(k\hat{\chi})^2 = \frac{\dfrac{\omega^2}{\hat{v}_T^2} - k^2 (1 + \dfrac{\overset{\circ}{S}_{11}}{\hat{c}_{44}})}{1 + \dfrac{\overset{\circ}{S}_{22}}{\hat{c}_{44}}} > 0. \tag{44}$$

This implies that

$$V = \frac{\omega}{k} > \hat{v}_T \sqrt{1 + \frac{\overset{\circ}{S}_{11}}{\hat{c}_{44}}}, \tag{45}$$

where $\hat{v}_T = \sqrt{\dfrac{\hat{c}_{44}}{\overset{\circ}{\rho}}}$.

The inequalities (41) and (45) shows us that the velocity of the TH wave in the substrate is greater that of the TH wave in the layer: $v_T > \hat{v}_T$.

Consequently, the Love wave velocity V satifies the following fundamental inequalities:

$$\hat{v}_T \sqrt{1 + \frac{\overset{\circ}{S}_{11}}{\hat{c}_{44}}} < V < v_T \sqrt{1 + \frac{\overset{\circ}{S}_{11}}{c_{44}}}. \tag{46}$$

Moreover, we obtain:

$$\hat{\chi} = \sqrt{\frac{\dfrac{V^2}{\hat{v}_T^2} - (1 + \dfrac{\overset{\circ}{S}_{11}}{\hat{c}_{44}})}{1 + \dfrac{\overset{\circ}{S}_{22}}{\hat{c}_{44}}}}. \tag{47}$$

At the interface $x_2 = 0$ between layer and substrate we suppose the continuity of displacement, so we obtain

$$\hat{u}_0 = \frac{u_0}{\cos(k\hat{\chi}h)},$$

and the continuity of stress, which yields the relation

$$(c_{44} + \overset{\circ}{S}_{22}) u_0 \, \chi = (\hat{c}_{44} + \overset{\circ}{S}_{22}) \hat{u}_0 \, \hat{\chi} \sin(k\hat{\chi}h).$$

Hence, we obtain the *dispersion relation* in the form:

$$\tan(k\hat{\chi}h) = \frac{(c_{44} + \overset{\circ}{S}_{22})\chi}{(\hat{c}_{44} + \overset{\circ}{S}_{22})\hat{\chi}}, \tag{48}$$

which is influenced by the initial mechanical fields (see the expressions (42) and (47) of χ and $\hat{\chi}$). This dispersion equation has an infinite number of solutions given by:

$$(kh)_n = \frac{1}{\hat{\chi}} \tan^{-1} [\frac{(c_{44} + \overset{\circ}{S}_{22})\chi}{(\hat{c}_{44} + \overset{\circ}{S}_{22})\hat{\chi}}] + n\frac{\pi}{\hat{\chi}}, \quad n = 0, 1, 2, ... \tag{49}$$

A detailed analysis of the obtained solution, as well as the study of the energy balance on the Love wave will be given in a forthcoming paper.

6 Conclusions

In this work we dealt with the study of the coupling conditions for propagation of planar guided waves in a piezoelectric semi-infinite plane subject to initial electro-mechanical fields. These results were obtained together with Dr. Iulian Ana, and are published in papers [21, 22].

If the sagittal plane is normal to a direct, resp. inverse dyad axis, we derived that the fundamental equations' system decomposes for particular choices of the initial electric field. In this way we obtained mechanical and piezoelectric waves generalizing the classical guided waves from the case without initial fields. Furthermore, we obtained a similar decomposition of mechanical and electrical boundary conditions, which enable us to characterize the obtained guided waves.

Here we reported new results concerning TH-waves propagation in prestressed layered materials. Our results generalize, for initial mechanical fields, classical results from seismology concerning Love waves propagation (see [6] and [7]). Using the general results obtained in chapter [8], we obtain and analyze the dispersion relations into a parallel-sided plate, resp. into a layer on a substrate, for various classes of anisotropy.

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