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COINCIDENCE PRODUCING OPERATORS ON A LARGE FIXED POINT STRUCTURE

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Abstract

In this paper we investigate the following problem (see I.A. Rus, *Fixed Point Structure Theory*, Cluj Univ. Press, Cluj-Napoca, 2006, pp. 193-194): Let X be a set with structure and $(X, S(X), M)$ be a large fixed point structure on X. Let $U \in S(X)$. An operator $p : U \to U$ is by definition coincidence producing operator on $(X, S(X), M)$ if for each $f \in M(U)$ there exists $u \in U$ such that $f(u) = p(u)$. The problem is to study the existence and the properties of these class of operators.

For the case of topological space with fixed point property with respect to continuous operators, the starting papers are given by W. Holsztynski (*Une généralization du théorème de Brouwer sur les points invariants*, Bull. Acad. Pol. Sc., 12(1964), 603-606) and by H. Schirmer (*Coincidence producing maps onto trees*, Canad. Math. Bull., 10(1967), 417-423) and for the case of metric spaces with fixed point property with respect to nonexpansive operators, by W.A. Kirk (*Universal nonexpansive maps*, 95-101, in Proc. 8^{th} ICFPTA (2007), Yokohama Publ., 2008).

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1 Introduction

Let $(X, S(X), M)$ be a large fixed point structure ($l.f.p.s.$). By definition $(X, S(X), M)$ is:

(1) a coincidence producing $l.f.p.s.$ if

$$U \in S(X),\ f, g \in M(U) \Rightarrow C(f, g) := \{u \in U \mid f(u) = g(u)\} \neq \emptyset;$$

(2) a $l.f.p.s.$ with the coincidence property if

$$U \in S(X),\ f, g \in M(U),\ f \circ g = g \circ f \Rightarrow C(f, g) \neq \emptyset.$$

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An operator $p : U \to U, U \in S(X)$ is coincidence producing with respect to $(X, S(X), M)$ if for each $f \in M(U)$ we have that, $C(f, p) \neq \emptyset$.

The basic problems of the coincidence theory (in terms of the above notions) are the following (see [26], [27]):

Problem 1 (The general problem). *Let $U \in S(X)$ and $f, g \in M(U)$. In which conditions we have that, $C(f, g) \neq \emptyset$?*

Problem 2. *Which l.f.p.s. are coincidence producing fixed point structures ?*

Problem 3. *Which l.f.p.s. are with the coincidence property ?*

Problem 4. *To investigate the existence and the properties of the coincidence producing operators with respect to a given l.f.p.s.*

The aim of this paper is to study the Problem 4. To do this, we begin our considerations with some examples of *l.f.p.s.*

2 Fixed point structures

Let $\mathscr{C}$ be a class of structured sets (ordered sets, topological spaces, metric spaces, Banach spaces, Hilbert spaces, ...).

Let Set^* be the class of nonempty set and if X is a nonempty set, then

$$P(X) := \{Y \subset X \mid Y \neq \emptyset\}.$$

Now we consider the following multivalued operators:

$P : \mathscr{C} \multimap Set^*,\ X \mapsto P(X);$
$S : \mathscr{C} \multimap Set^*,\ X \mapsto S(X) \subset P(X);$
$M : D_M \subset P(\mathscr{C}) \times P(\mathscr{C}) \multimap \mathbb{M}(P(\mathscr{C}), P(\mathscr{C})),\ (U, V) \mapsto M(U, V) \subset \mathbb{M}(U, V).$

Here, $P(\mathscr{C}) := \{U \in P(X) \mid X \in \mathscr{C}\}$ and

$$\mathbb{M}(U, V) := \{f : U \to V \mid f \text{ an operator}\},$$
$$\mathbb{M}(U) := \mathbb{M}(U, U) \text{ and } M(U) := M(U, U).$$

Definition 1. *(see [24], [27]). By a large fixed point structure (l.f.p.s.) on $X \in \mathscr{C}$ we understand a triple $(X, S(X), M)$ with the following properties:*

(i) $U \in S(X) \Rightarrow (U, U) \in D_M$*;*

(ii) $U \in S(X),\ f \in M(U) \Rightarrow F_f := \{u \in U \mid f(u) = u\} \neq \emptyset.$

Now we consider some examples of *l.f.p.* structures.

Example 1 (The trivial *l.f.p.s.*). *In this case, $\mathscr{C} := Set^*$, $X \in Set^*$, $S(X) := \big\{\{x\} \mid x \in X\big\}$ and $M(U) := \mathbb{M}(U)$.*

Example 2 (The *l.f.p.s.* **of Zorn).** *$\mathscr{C}$ is the class of partially ordered set. For $(X, \leq)$ a partially ordered set,*

$$S(X) := \{U \in P(X) \mid (U, \leq) \text{ has at least a maximal element}\}$$

and $M(U) := \{f : U \to U \mid u \leq f(u),\ \forall\ u \in U\}$. The Zorn's fixed point theorem implies that $(X, S(X), M)$ is a l.f.p.s.

Example 3 (The *l.f.p.s.* **of Tarski).** *$\mathscr{C}$ is the class of the complete lattices. For $(X, \leq)$ a complete lattice, $S(X) := \{U \in P(X) \mid (U, \leq)$ is a complete lattice$\}$ and $M(U) := \{f : U \to U \mid f$ is increasing$\}$. From Tarski's fixed point theorem we have that $(X, S(X), M)$ is a l.f.p.s.*

Example 4 (The *l.f.p.s.* **of contractions).** *$\mathscr{C}$ is the class of the complete metric spaces. For (X, d) a complete metric space, $S(X) := P_{cl}(X) := \{U \in P(X) \mid U$ is a closed subset$\}$ and $M(U) := \{f : U \to U \mid f$ is a contraction$\}$. From the contraction principle it follows that $(X, S(X), M)$ is a l.f.p.s.*

Example 5 (The *l.f.p.s.* **of α-contractions).** *$\mathscr{C}$ and S are as in Example 4. For $0 < \alpha < 1$, $M(U) := \{f : U \to U \mid f$ is an α-contraction$\}$.*

From the various variants of the Schauder's fixed point theorem we have the following examples.

Example 6 (The first *l.f.p.s.* **of Schauder).** *$\mathscr{C}$ is the class of Banach spaces. For X a Banach space,*

$$S(X) := P_{cp,cv}(X) := \{U \in P(X) \mid U \text{ is compact and convex}\}$$

and $M(U) := C(U, U) := \{f : U \to U \mid f$ is continuous$\}$.

Example 7 (The second *l.f.p.s.* **of Schauder).** *$\mathscr{C}$ is the class of Banach spaces. For X a Banach space,*

$$S(X) := P_{b,cl,cv}(X) := \{U \in P(X) \mid U \text{ is bounded, closed and convex}\}$$

and $M(U) := \{f : U \to U \mid f$ is complete continuous$\}$.

Example 8 (The third *l.f.p.s.* **of Schauder).** *$\mathscr{C}$ is the class of Banach spaces. For X a Banach space, $S(X) := \{X\}$ and*

$$M(X) := \{f : X \to X \mid f \text{ is continuous and } \overline{f(X)} \text{ is compact}\}.$$

Example 9 (The *l.f.p.s.* **of Schauder-Granas).** *$\mathscr{C}$ is the class of Banach spaces. For X a Banach space, $S(X) := \{X\}$ and*

$$M(X) := \{f : X \to X \mid f \text{ is complete continuous and is quasibounded with the quasinorm, } |f| < 1\}.$$

Example 10 (The *l.f.p.s.* of Browder-Ghöde-Kirk). *$\mathscr{C}$ is the class of the uniformly convex Banach spaces, $S(X) := P_{b,cl,cv}(X)$ and $M(U) := \{f : U \to U \| \; f$ is nonexpansive$\}$. From the fixed point theorem of Browder-Ghöde-Kirk, it follows that $(X, S(X), M)$ is a l.f.p.s.*

Remark 1. *It is clear that for any fixed point theorem we have at least an example of l.f.p.s.*

Remark 2. *For the notions and for the fixed point theorems which appear in the above examples, see [10], [17], [23], [28], ...*

Remark 3. *For the notions of fixed point structure and l.f.p.s., and for other examples and counterexamples, see [27], [26] and [24]. See also [28].*

Remark 4. *For the fixed point theory in the category theory see [27] and the references therein.*

3 Historical roots of the Problem 4

For the case of topological space with fixed point property with respect to continuous operators, the starting paper was given by W. Holsztynski ([11]). Let X and Y be two topological spaces. Holsztynski, called an operator $p \in C(X, Y)$ universal, if it satisfies the following implication:

$$f \in C(X, Y) \Rightarrow C(f, p) \neq \emptyset.$$

Holsztynski established that if a pair (X, Y) of topological spaces has a universal operator, then Y is a topological space with fixed point property with respect to continuous operators, i.e.,

$$f \in C(Y, Y) \Rightarrow F_f \neq \emptyset.$$

In [30], H. Schirmer renamed *universal operator* as *coincidence producing* one. After that, some authors used "universal" and others "producing" ([1]-[4], [12], [13], [16], [18], [19], [29], [30], [31],...).

For the case of metric space with fixed point property with respect to nonexpansive operators, W.A. Kirk established that if a pair (X, Y) of metric spaces has a nonexpansive universal operator, then Y is with fixed point property with respect to nonexpansive operators.

There are notions in which appear the coincidence producing concept as a term of definitions. For example, M. Furi, M. Martelli and A. Vignoli ([7]) present the following definition:

Let X be a Banach space. A continuous operator $p : X \to X$ is called a strong surjection if the equation, $p(x) = f(x)$, has a solution for any continuous operator $f : X \to X$ with $\overline{f(X)}$ compact.

In our terminology, this definition takes the following form: *A continuous operator $p : X \to X$ is a strong surjection iff p is coincidence producing operator with respect to the l.f.p.s. of Schauder.*

On the other hand, some coincidence point theorems are in fact examples of coincidence producing operators with respect to suitable *l.f.p.* structures. Here are some examples:

Example 11. *Let us consider the Tychonoff's l.f.p.s., i.e., $\mathscr{C}$ is the class of locally convex Hausdorff spaces, $S(X) := P_{cp,cv}(X)$ and $M(U) := C(U,U)$.*

In [5] the authors give the following theorem:

Let X be a locally convex space, $Y \in P_{cp,cv}(X)$ and $f, p \in C(Y,Y)$. If p is Vietoris operator (i.e., $p(Y) = Y$ and $p^{-1}(y)$ is acyclic with respect to Čech homology with the coefficients in $\mathbb{Q}$, for all $y \in Y$), then there exists $y^* \in Y$ such that $f(y^*) = p(y^*)$.

In our terminology this theorem takes the following form:

A Vietoris operator is coincidence producing with respect to the l.f.p.s. of Tychonoff.

Example 12. *One of the Krasnoselskii's fixed point theorem can be presented as the following coincidence point theorem:*

Let X be a Banach space, $Y \in P_{b,cl,cv}(X)$ and $f, g : Y \to Y$ be two operators. We suppose that:

- (i) *f is complete continuous;*
- (ii) *g is a contraction;*
- (iii) *$f(x) + g(y) \in Y$, for all $x, y \in Y$.*

Then, $C(f, 1_Y - g) \neq \emptyset$.

In our terminology, this theorem is as follows:

If $g : Y \to Y$ is a contraction, then $p := 1_Y - g$ is a coincidence producing with respect to the second Schauder's l.f.p.s.

Remark 5. *For the above Krasnoselskii's fixed point theorem, see [21], [28] and the references therein.*

Remark 6. *For the coincidence point theory see [1], [2], [6], [8], [9], [10], [14], [15], [17], [22], [25], [26], [28], …*

4 General remarks on coincidence producing operators on a given $l.f.p.s.$

Let $(X, S(X), M)$, $X \in \mathscr{C}$, be a *l.f.p.s.* We have

Lemma 1. *Let $U \in S(X)$ and $p : U \to U$ be a coincidence producing operator on $(X, S(X), M)$. If all constant operators from u to U are in $M(U)$, then p is a surjective operator.*

Proof. Let $u_0 \in U$. We consider the constant operator $\tilde{u}_0 : U \to U$, $u \mapsto u_0$. Since $\tilde{u}_0 \in M(U)$ and p is a coincidence producing operator, there exists $u^* \in U$ such that $p(u^*) = \tilde{u}_0(u^*)$, i.e., $p(u^*) = u_0$. So, p is surjective. □

This lemma suggests

Problem 5. *Which are the l.f.p. structures with the property that M selects all constant operators?*

Problem 6. *Which surjective operators are coincidence producing on a suitable l.f.p.s. ?*

Lemma 2. *Let $U \in S(X)$ and $p : U \to U$ be an operator. We suppose that:*

(i) *p has a right-inverse, p_r^{-1};*

(ii) *$h \circ p_r^{-1} \in M(U)$, for all $h \in M(U)$.*

Then p is coincidence producing on $(X, S(X), M)$.

Proof. From (ii) it follows that there exists $u^* \in U$ such that $h(p_r^{-1}(u^*)) = u^*$.
Let $v^* := p_r^{-1}(u^*)$. Then $v^* \in C(h,p)$. □

Problem 7. *If a surjective operator p has a property, in which conditions p_r^{-1} has the same property ?*

Problem 8. *Let U be a structured set and $p : U \to U$ be a surjection. In which conditions some p_r^{-1} is a morphism ?*

Lemma 3. *Let $U \in S(X)$ and $p : U \to U$ be a bijection. If $p^{-1} \circ h \in M(U)$ for all $h \in M(U)$, then p is coincidence producing on $(X, S(X), M)$.*

Lemma 4. *Let $(X, S(X), M)$ be a maximal l.f.p.s. (see [27], pp. 32-34). Let $(U,U) \in D_M$. Then:*

$$U \in S(X) \Leftrightarrow 1_U \text{ is coincidence producing on } (X, S(X), M).$$

Proof. By definition $(X, S(X), M)$ is maximal iff

$$S(X) = \{U \in P(X) \mid (U,U) \in D_M,\ f \in M(U) \Rightarrow F_f \neq \emptyset\}.$$

The proof follows from this definition. □

5 Examples

Example 13. *Let $(X, S(X), M)$ with $X \in \mathscr{C}$, be the Zorn's l.f.p.s. Let $U \in S(X)$ and $p : U \to U$. If p has a progressive right-inverse, then p is coincidence producing on $(X, S(X), M)$.*

Indeed, we have $u \leq p_r^{-1}(u)$. So, $h \circ p_r^{-1} \in M(U)$ for all $h \in M(U)$. Now we apply Lemma 2.

Example 14. *Let $(X, S(X), M)$ with $X \in \mathscr{C}$ be the Tarski's l.f.p.s. Let $U \in S(X)$ and $p : U \to U$. Then:*

(i) *If p has an increasing right-inverse, then p is coincidence producing on $(X, S(X), M)$;*

(*ii*) *If p is an ordered set isomorphism, then p is coincidence producing on $(X, S(X), M)$.*

Indeed, (i) follows from Lemma 2 and (ii) follows from Lemma 3.

Example 15. *Let $(X, S(X), M_\alpha)$ with $X \in \mathscr{C}$, be the l.f.p.s. of α-contractions. Let $U \in P_{cl}(X)$ and $p : U \to U$. We suppose that:*

(1) *p is surjective;*

(2) *there exists $\beta > \alpha$ such that*

$$d(p(x), p(y)) \geq \beta d(x, y),\ \forall\ x, y \in U.$$

Then, p is coincidence producing on $(X, P_{cl}(X), M_\alpha)$.

Indeed, let p_r^{-1} be a right-inverse of p. Then from (2) it follows

$$d(p_r^{-1}(x), p_r^{-1}(y)) \leq \frac{1}{\beta} d(x, y).$$

Let $h \in M_\alpha(X)$. Then

$$d(h \circ p_r^{-1}(x), h \circ p_r^{-1}(y)) \leq \frac{\alpha}{\beta} d(x, y),\ \forall\ x, y \in U.$$

Now the proof follows from Lemma 2.

Example 16. *The surjective isometries are coincidence producing on the l.f.p.s. of contractions.*

Example 17 (Generic example). *We present this example by the following question:*

Problem 5.1. *Let (X, τ) be a topological Hausdorff space. Let $M(X) \subset C(X, X)$ be a nonempty subset of the set of continuous operators from X to X. We suppose that X and M are such that $(X, \{X\}, M)$ is a l.f.p.s. We also suppose that there exists $p \in M(X)$, a coincidence producing operator on this l.f.p.s. For $f \in M(X)$, we consider the induced operator $\widehat{f} : P_{cp}(X) \to P_{cp}(X)$ defined by $\widehat{f}(A) := f(A) := \{f(a) \mid a \in A\}$, for $A \in P_{cp}(X)$. Let $\widehat{M}(P_{cp}(X)) := \{\widehat{f} \mid f \in M(X)\}$. We suppose that $(P_{cp}(X), \{P_{cp}(X)\}, \widehat{M})$ is a l.f.p.s. The problem is in which conditions on X and M, the induced operator $\widehat{p}$ is coincidence producing on $(P_{cp}(X), \{P_{cp}(X)\}, \widehat{M})$?*

For some particular cases of this problem, see [12], [20] and [3].

Here are other examples. Let (X, d) be a complete metric space, $M(X) := \{f : X \to X \mid f$ is a contraction $\}$. By a theorem of Nadler (see [10], [17], [28]), $(P_{cp}(X), \{P_{cp}(X)\}, \widehat{M})$ is a *l.f.p.s.* on the complete metric space $(P_{cp}(X), H_d)$, where H_d is the Pompeiu-Hausdorff metric. This *l.f.p.s.* is the induced *l.f.p.s.* from the *l.f.p.s.* of contractions, $(X, \{X\}, M)$.

Let $p : X \to X$ be a surjective isometry. The operator p is coincidence producing on $(X, \{X\}, M)$ (see Example 16). On the other hand we remark that $\widehat{p}$ is a surjective isometry on $(P_{cp}(X), H_d)$. So, $\widehat{p}$ is coincidence producing on $(P_{cp}(X), \{P_{cp}(X)\}, H_d)$.

Remark 7. *Let X and Y be two topological space, $M(X, Y) \subset C(X, Y)$ be a nonempty family of continuous operators from X to Y. For some examples of coincidence producing operator with respect to $M(X, Y)$ see [11], [12], [13], [29], [30], [31], [1]-[4], [18], [20], [32], ...*

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