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ON REPRODUCTIVE SOLUTIONS OF POSTIAN AND STONEAN EQUATIONS

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Abstract

A theorem due to Löwenheim provides the general reproductive solution of a Boolean equation whenever a particular solution is known. This theorem has been generalized either by passing from Boolean algebras to Post algebras and Stone algebras, or by using a parametric solution instead of a particular solution, or by combining the two directions. In the present paper we survey this research and generalize everything to Stone algebras.

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1 Introduction

The concept of a reproductive solution, introduced by Löwenheim [10], [11] in the case of Boolean equations, can be defined in a very general framework, for any kind of equations.

Consider a set U ("universe of discourse") and two entities t ("true") and f ("false"). An *equation over* U is a function $e : U \longrightarrow \{t, f\}$; its *solutions* are the elements $x \in U$ such that $e(x) = t$. By a *general solution* of the equation is meant a parametric representation of the *solution set* $S = \{x \in U \mid e(x) = t\}$, that is, a function $\varphi : U \longrightarrow U$ such that $\varphi(U) = S$. This means that $\varphi(x) \in S$ for all $x \in U$ and conversely, for every $x \in S$ there is $p \in U$ such that $x = \varphi(p)$. If the latter condition is fulfilled in the stronger form $x \in S \Longrightarrow x = \varphi(x)$, then φ is called a *general reproductive solution*, or simply a *reproductive solution*.

We also use expressions of the form "$x = \varphi(p)$ is a general/reproductive solution of the equation $e(x)$".

Note that in contrast with general solutions and reproductive solutions, we refer to an element of S as a *particular solution* of the equation.

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The theory of Boolean equations, which begins with Boole himself, studies (systems of) *Boolean equations* over an arbitrary Boolean algebra (not necessarily the two-element one). By Boolean equations we mean equations expressed by *Boolean functions*. The latter term designates the algebraic functions of the Boolean algebra, i.e., functions obtained from variables and constants by superpositions of the basic operations $\vee$ (disjunction), $\cdot$ (conjunction) and $'$ (negation) of the Boolean algebra. For the theory of Boolean equations and their applications we refer the reader e.g. to the monographs [13] and [14].

Let $(B, \vee, \cdot, ', 0, 1)$ be an arbitrary Boolean algebra; the conjunction $\cdot$ is also denoted by juxtaposition. Every system of Boolean equations over B is equivalent to a single Boolean equation of the form $f(x_1, \ldots, x_n) = 0$ (and also to an equation $g(x_1, \ldots, x_n) = 1$). Löwenheim proved [10], [11] that if $(\xi_1, \ldots, \xi_n) \in B^n$ is a particular solution of the equation $f(x_1, \ldots, x_n) = 0$, then

$$x_i = t_i f'(t_1, \ldots, t_n) \vee \xi_i f(t_1, \ldots, t_n) \ \ (i = 1, \ldots, n) \tag{1}$$

is a reproductive solution of the equation; here $'$ is the operation defined by $f'(x_1, \ldots, x_n) = (f(x_1, \ldots, x_n))'$.

It is convenient to use a compact vectorial notation like $X = (x_1, \ldots, x_n)$ and to equip the vector set B^n with the operations $\vee, \cdot$ defined componentwise and with the "scalar multiplication" $aX = (ax_1, \ldots, ax_n)$.

Using this notation, the Löwenheim theorem above says that if $\Xi \in B^n$ is a particular solution of the Boolean equation $f(X) = 0$, then formula

$$X = Tf'(T) \vee \Xi f(T) \tag{2}$$

is a reproductive solution of the equation.

The Löwenheim reproductive solution was generalized (in chronological order) to Post algebras by Carvallo [7], [8], Bordat [6] and Serfati [15], and to Stone algebras by Beazer [5].

Of course, if a general solution $G(X)$ of equation $f(X) = 0$ is known, we can choose a (convenient) particular solution Ξ in order to apply formula (2). Yet Banković [1] proved that if $P : T^n \longrightarrow T^n$ is an arbitrary transformation, then formula

$$X = Tf'(T) \vee G(P(T))f(T) \tag{3}$$

is also a reproductive solution of equation $f(X) = 0$. Later on, Banković [2] generalized this theorem to Post algebras and proved also the converse: every reproductive solution of equation $f(X) = 0$ is of the form (3) for a suitably chosen P.

Now recall a concept from universal algebra. A function $f : A^n \longrightarrow A$, where A is an abstract algebra, is said to have the *congruence substitution property* (CSP for short) if for every congruence $\sim$ of A, if $X, Y \in A^n$ satisfy $X \sim Y$ (meaning $x_i \sim y_i$ for $i = 1, \ldots, n$), then $f(X) \sim f(Y)$. It is well known that every algebraic function satisfies CSP, but the converse does not hold in general. However Grätzer proved [9] that in every Boolean algebra the algebraic functions (known as Boolean functions) coincide with the

CSP *functions*, i.e., the functions satisfying CSP. More generally, this also happens in Post algebras, as was proved by Beazer [4].

Recall also that a lattice $(L, \vee, \cdot, 0)$ with zero is said to be *pseudocomplemented* if for every element x there is a (necessarily unique) element x^*, called the *pseudocomplement* of X, such that for every y we have $xy = 0 \Longleftrightarrow y \leq x^*$. A pseudocomplemented distributive lattice satisfying the identity $x^* \vee x^{**} = 1$ is called a *Stone algebra*. In particular every Post algebra is a Stone algebra; see e.g. [14]. Proposition 5.1.12.

Beazer [5] studied CSP functions and equations expressed by CSP functions in Stone algebras. In particular he generalized Löwenheim's reproductive solution (2) to this framework.

In the sequel we work with Stone algebras and equations $f(X) = 0$ expressed by CSP functions f. We prove a theorem from which all the results mentioned above are obtained within this more general framework, along with the simpler reproductive solution

$$X = Tf^*(T) \vee G(T)f(T)\ . \tag{4}$$

Finally we prove a theorem which joins two other theorems, due to Banković [2] and Marinković [12], respectively, and we state an open problem.

2 The new results

Recall that if L is a Stone algebra, then $x^* \vee x^{**} = 1$ and the closure operator $x \mapsto x^{**}$ is a homomorphism of L onto the Boolean algebra $B(L) = \{x \in L \mid x^{**} = x\} = \{x^* \mid x \in L\}$ of *Boolean* (i.e., *complemented*) *elements* of L. Thus $(x \vee y)^{**} = x^{**} \vee y^{**}$, $(xy)^{**} = x^{**}y^{**}$, $x^{***} = x^*$, and $0, 1 \in B(L)$. Other important properties of the pseudocomplement x^* are $(x \vee y)^* = x^*y^*$, $(xy)^* = x^* \vee y^*$, $0^* = 1$ and $1^* = 0$. Consequently $x = 0 \Longleftrightarrow x^{**} = 0$. More generally:

Remark 1. *$xy = 0 \Longleftrightarrow x^{**}y = 0$, because*

$$xy = 0 \Longrightarrow x^{**}y^{**} = (xy)^{**} = 0 \Longrightarrow x^{**}y = 0 \Longrightarrow xy = 0\ .$$

We will also use the following property: if $\beta : B^n \longrightarrow B$ is a Boolean function, then $\beta(ax \vee bx') = \beta(a)x \vee \beta(b)x'$.

Notification *In the sequel $f(X) = 0$ is an equation in a Stone algebra L and $f : L^n \longrightarrow L$ is a* CSP *function. We use the notation $f^*(X) = (f(X))^*$.*

As was done in [12], we use the following result:

Lemma 1. (Beazer [5]) *If $f : L^n \longrightarrow L$ is a* CSP *function in a Stone algebra L, then $\overline{f} : B(L)^n \longrightarrow B(L)$ defined by $\overline{f}(X^{**}) = f^{**}(X)$ is a Boolean function and $f(X) = 0 \Longleftrightarrow \overline{f}(X^{**}) = 0\, (\Longleftrightarrow f^{**}(X) = 0)$.*

We are now in a position to prove:

Theorem 1. *Let $f : L^n \longrightarrow L$ be a* CSP *function and $\Phi, \Psi : L^n \longrightarrow L^n$. Then*

$$H(X) = \Phi(X) f^*(X) \vee \Psi(X) f(x) \tag{5}$$

is a general solution of equation $f(X) = 0$ if and only if

$$f(\Phi(X)) f^*(X) = 0 \ , \tag{6}$$

$$f(\Psi(X)) f(X) = 0 \ . \tag{7}$$

Proof. We have

$$H^{**}(X) = \Phi^{**}(X) f^*(X) \vee \Psi^{**}(X) f^{**}(X) \ ,$$

$$f(H(X)) = 0 \Longleftrightarrow \overline{f}(H^{**}(X)) = 0$$

$$\Longleftrightarrow \overline{f}(\Phi^{**}(X)) f^*(X) \vee \overline{f}(\Psi^{**}(X)) f^{**}(X) = 0$$

$$\Longleftrightarrow f^{**}(\Phi(X)) f^{***}(X) = 0 \ \& \ f^{**}(\Psi(X)) f^{**}(X) = 0$$

$$\Longleftrightarrow (f(\Phi(X)) f^*(X))^{**} = 0 \ \& \ (f(\Psi(X)) f(X))^{**} = 0$$

$$\Longleftrightarrow f(\Phi(X)) f^*(X) = 0 \ \& \ f(\Psi(X)) f(X) = 0 \ .$$

□

Remark 2. *In view of Remark 1, condition (7) is equivalent to*

$$f(\Psi(X)) f^{**}(X) = 0 \ . \tag{7'}$$

Here are a few consequences of Theorem 1.

Corollary 1. *The transformation*

$$H(X) = X f^*(X) \vee \Psi(X) f(X) \tag{8}$$

is a general solution of equation $f(X) = 0$ if and only if the identity (7) *holds, in which case $H(X)$ is a reproductive solution.*

Proof. The first claim follows from Theorem 1 with $\Phi(X) := X$. If $f(X) = 0$ then (8) implies $H(X) = X$, therefore the solution is reproductive. □

Corollary 2. *If $\Psi(X)$ is a general solution of equation $f(X) = 0$, then the transformation* (8) *is a reproductive solution of equation $f(X) = 0$.*

Proof. By Corollary 1, since $f(\Psi(X)) = 0$. □

Corollary 2 expresses the basic idea of Banković for passing beyond the Löwenheim theorem : replace the particular solution Ξ by a general solution $\Psi(X)$. However in [1], [2] and [12], formula (8) appears in a slightly more complicated form, with $\Psi(P(X))$ instead of $\Psi(X)$; we will come to this point in the next section of the paper.

Corollary 3. *The transformation*

$$H(X) = Xf^*(X) \vee X^* f(X) \tag{9}$$

is a general solution of equation $f(X) = 0$ if and only if $f(X^*)f(X) = 0$, in which case it is a reproductive solution.

Proof. By Corollary 1 with $\Psi(X) := X^*$. □

The next corollary generalizes Theorem 6 in [3].

Corollary 4. *The transformation*

$$H(X) = Xf^*(X) \vee X^* f^{**}(X) \tag{10}$$

is a general solution of equation $f(X) = 0$ if and only if $f(X^*)f^{**}(X) = 0$, in which case it is a reproductive solution.

Proof. By Corollary 3 with $f := f^{**}$, taking into account that $f^{***}(X) = f^*(X)$, $f(X) = 0 \Longleftrightarrow f^{**}(X) = 0$, and finally Remark 1. □

3 An open problem

Two more theorems, due to Banković and Marinković, can be joined in the following form:

Theorem 2. *Let L be a Stone algebra, $f : L^n \longrightarrow L$ a* CSP *function, $\Psi(X)$ a general solution of equation $f(X) = 0$, and $P : L^n \longrightarrow L^n$. Set*

$$H(X) = Xf^*(X) \vee \Psi(P(X))f^{**}(X)\ . \tag{11}$$

1) *If P satisfies*

$$\Psi(P(X)) = H(X)\ , \tag{12}$$

then $H(X)$ is a reproductive solution of equation $f(X) = 0$.

2) *If L is a Post algebra and $H(X)$ is a reproductive solution of equation $f(X) = 0$, then $H(X)$ is of the form* (11), *where P satisfies* (12).

Proof. 1) We have $f(X) = 0 \Longrightarrow H(X) = X$ by (11) and $f(H(X)) = 0$ by (12).

2) We have

$$H(X) = Xf^*(X) \vee H(X)f^{**}(X) \tag{13}$$

by Lemma 2 in [2]. Now (11) follows from (13) and (12). □

Theorem 2 reduced to Post algebras is essentially the main Theorem 6 in [2]. Simillary, Part 1) of Theorem 2 is essentially the main Theorem 4 in [12]. As a matter of fact, the proofs in [2] and [12] specify how to obtain (12): for each X such that $f(X) = 0$ choose T such that $X = \Psi(T)$ and put $P(X) = T$, otherwise $P(X)$ is arbitrary. Indeed, we get $\Psi(P(X)) = \Psi(T) = X = H(X)$.

The **open problem** mentioned in the Introduction is the generalization of Part 2) of Theorem 2 to Stone algebras.

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