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IMMISCIBLE HELE-SHAW DISPLACEMENT WITH CONSTANT VISCOSITY FLUIDS

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Abstract

We study the interfaces stability of immisicble displacement in a Hele-Shaw cell or porous medium, with applications to secondary oil recovery. A middle-region of constant length L exists between the displacing and displaced fluids (water and oil), filled by a third fluid with an unknown constant viscosity μ. The linear stability analysis is performed and the growth constant σ of perturbations is estimated in terms of L, μ and water and oil viscosities. The new element is a lower-upper estimate of σ in terms of L.

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1 Indroduction

A Hele-Shaw cell is the "gap" between two parallel plates, at a small distance δ. The Poiseuille solution of the Stokes equation is averaged across the gap and the obtained equation is similar with the Darcy law for the flow in porous media. The "permeability" is given in terms of the fluid viscosity and δ. This model is useful for the study of oil recovery from a horizontal porous medium. The flow can be visualized if the cell plates are transparent - see Jacob Bear (1972). Saffman and Taylor (1958) proved that if the displacing fluid is less viscous, the interface with the displaced fluids is unstable and the "fingering" phenomenon appears. If the oil is displaced from a porous medium by water, the above instability is giving "fingers" of water penetrating in oil, which remains trapped in the medium. The "fingering" phenomenon can be minimized by using an intermediate fluid (between water and oil), whose viscosity is an "a priori" unknown parameter.

In this paper we study the simpler case when the viscosity of the intermediate fluid is constant. We consider a Hele-Shaw cell in the fixed horizontal plane x_1Oy, filled by three fluids: water (with constant viscosity μ_1), an intermediate fluid with constant viscosity μ, and oil (with constant viscosity μ_2). The $0z$ axis is orthogonal on the plates and the gravity effects are neglected. The flow is due to the water velocity U far upstream, from the left part, in the positive direction Ox_1. In our model, the middle region between water an oil region (denoted by MR) is $Ut - L < x_1 < Ut$, where t is the time. We use also

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the assumption $\mu_1 < \mu < \mu_2$. In the moving reference $x = x_1 - Ut$, MR is the segment $(-L, 0)$. Two sharp interfaces exists between the above three immisicble fluids: water-MR and MR-oil. We study the linear stability of the flow, therefore the linear stability of the interfaces and get some estimates of the growth constant (in time) σ of the perturbations. Some upper bounds of σ were obtained, but not depending on the middle-region length L - see Pasa (2002), Daripa and Pasa (2004) and Daripa (2008). The new element of this paper is a lower-upper estimate of σ, depending on the middle region length L.

2 The linear stability system

The flow in the Hele-Shaw cell is governed by the equations

$$u_x + v_y = 0, \quad p_x = -\mu u, \quad p_y = -\mu v,$$

where the indices x, y denote the partial derivatives, (u, v) is the velocity, p is the pressure, μ is the viscosity (in all three regions). The boundary conditions are given by the Laplace's law: the pressure jump on the interfaces is balanced by the surface tension times the curvature.

In the moving reference system $x = x_1 - Ut$, we consider the following basic solution :

$$u = U,\ v = 0,\ P_x = -\mu U,\ P_y = 0, \tag{1}$$

with the interfaces

$$x = -L, \quad x = 0. \tag{2}$$

We insert the perturbations u', v', p' in the flow equations and get

$$(P + p')_x = -\mu(U + u'), \quad (P + p')_y = -\mu(v'), \tag{3}$$

$$(u + u')_x + (v + v')_y = 0, \tag{4}$$

therefore the perturbations equations are

$$u'_x + v'_y = 0; \quad p'_x = -\mu u'; \quad p'_y = -\mu v'. \tag{5}$$

We start our analysis with the perturbations of the velocity

$$u'(x, y, t) = f(x) \exp(iky + \sigma t).$$

From $(5)_1$ and $(5)_3$ we get

$$v' = -[f_x/ik] \exp(iky + \sigma t); \quad p' = -\mu[f_x/k^2] \exp(iky + \sigma t). \tag{6}$$

Cross derivation of the pressure equations $(5)_2, (5)_3$ is giving

$$(\mu u')_y = (\mu v')_x \Rightarrow -\mu f_{xx} + k^2 \mu f = 0. \tag{7}$$

We derive next the boundary conditions for (7). Out of the middle region we need far decay solutions:

$$f(x) = \exp[(x+L)k],\ x < -L, \qquad f(x) = \exp(-kx),\ x > 0,$$

therefore the limit values of the derivatives f_x at the *exterior* ends of MR are

$$f_x^-(-L) = kf(-L), \quad f_x^+(0) = -kf(0), \tag{8}$$

where $^-$ and $^+$ denote the "left" and "right" limits.

The perturbed (material) interface near an arbitrary point x_0 is denoted by

$$x = g(x_0, y, t),\ \text{with}\ g_t = u', \tag{9}$$

therefore

$$g(x_0, y, t) = [f(x_0)/\sigma]\exp(iky + \sigma t). \tag{10}$$

We estimate the pressure jump by using $(6)_2$ (recall P is the basic pressure) and the viscosities jumps on the interfaces:

$$p^+(x_0) = P(x_0, y, t) + P_x^+(x_0, y, t)\cdot g(x_0, y, t) + p'^+(x_0, y, t) \Rightarrow$$

$$p^+(x_0) = P(x_0) - \mu^+(x_0)\left[\frac{Uf(x_0)}{\sigma} + \frac{f_x^+(x_0)}{k^2}\right]\exp(iky + \sigma t), \tag{11}$$

$$p^-(x_0) = P(x_0) - \mu^-(x_0)\left[\frac{Uf(x_0)}{\sigma} + \frac{f_x^-(x_0)}{k^2}\right]\exp(iky + \sigma t) \tag{12}$$

We obtain the corresponding relations for $x_0 = 0,\ x_0 = -L$, where $\mu^+(0) = \mu_2,\ \mu^-(0) = \mu = \mu^+(-L),\ \mu^-(-L) = \mu_1$.

On the interfaces $x = 0, x = -L$ we consider the surface tensions T, S. The Laplace's law gives us

$$p^+(0) - p^-(0) = Tg_{yy}(0, y, t), \tag{13}$$
$$p^+(-L) - p^-(-L) = Sg_{yy}(0, y, t),$$

therefore from the equations (11) - (13) we obtain

$$\mu^-(0)\left[\frac{Uf(0)}{\sigma} + \frac{f_x^-(0)}{k^2}\right] - \mu^+(0)\left[\frac{Uf(0)}{\sigma} + \frac{f_x^+(0)}{k^2}\right] = -T\frac{f(0)}{\sigma}k^2 \tag{14}$$

$$\mu^-(-L)\left[\frac{Uf(-L)}{\sigma} + \frac{f_x^-(-L)}{k^2}\right] - \mu^+(-L)\left[\frac{Uf(-L)}{\sigma} + \frac{f_x^+(-L)}{k^2}\right] = \tag{15}$$
$$-S\frac{f(-L)}{\sigma}k^2$$

The above relations are giving us the limit values of f_x at the *interior* ends of the MR:

$$f_x^-(0) = \frac{1}{\mu}\left[\frac{k^2U(\mu_2 - \mu) - Tk^4}{\sigma}f(0) - \mu_2 kf(0)\right] := (\frac{e}{\sigma} + q)f(0), \tag{16}$$

$$f_x^+(-L) = \frac{1}{\mu}\left[\frac{k^2U(\mu_1 - \mu) + Sk^4}{\sigma}f(0) + \mu_1 kf(-L)\right] := (\frac{r}{\sigma} + s)f(-L). \tag{17}$$

Therefore the flow stability is governed by the problem:

$$(FS) \qquad -\mu f_{xx} + k^2\mu f = 0 + (16) + (17) \tag{18}$$

3 Estimates of the growth constant σ

We multiply with f in $(18)_1$, we use the boundary conditions (16), (17) and obtain the following formula for the growth constant:

$$\sigma = \frac{\mu[-rf^2(-L) + ef^2(0)]}{\mu[f^2(-L)s - f^2(0)q] + \mu \int_{-L}^{0} f_x^2 + k^2\mu \int_{-L}^{0} f^2} \tag{19}$$

In this paper we consider the condition

$$k^2 \leq Max\{U[\mu_2 - \mu]/T, \quad U[\mu - \mu_1]/S\}. \tag{20}$$

that means $e, -r > 0$, therefore $\sigma > 0$ (the most dangerous case).

The formula (19) of σ is depending on $f(0), f(-L)$. We have the usual inequality

$$C, D, x, y > 0 \Rightarrow Min\left\{\frac{A}{C}, \frac{B}{D}\right\} \leq \frac{Ax + By}{Cx + Dy} \leq Max\left\{\frac{A}{C}, \frac{B}{D}\right\}. \tag{21}$$

We neglect the positive integrals in the denominator of the growth constant (19), we use (21) and get the following Basic Upper Estimate (BUP)

$$(BUP) \qquad \sigma \leq MAX\left\{\frac{e}{-q}, \quad \frac{-r}{s}\right\}, \tag{22}$$

The above estimate (BUP) is not depending on L, but the exact value of σ is given in terms of L. Our problem is to find an upper estimate depending on L of the growth constant.

a) *Algebraic upper* estimates of σ. The solution of the stability equation is

$$f(x) = Ae^{kx} + Be^{-kx}, \tag{23}$$

where A, B are depending on L, μ, μ_1, μ_2, k. We use the notations:

$$a = (e/\sigma + q), \quad b = (r/\sigma + s), \tag{24}$$

therefore the boundary conditions (16), (17) and the condition $A, B \neq 0$ are giving us

$$(a - k)e^{2kL}(b + k) - (a + k)(b - k) = 0 \tag{25}$$

For very large L we get $(a - k)(b + k) = 0$, then $a = k$ *or* $b = -k$, that means

$$\sigma = \frac{e}{-q + k} \quad or \quad \sigma = \frac{-r}{s + k}, \tag{26}$$

which is an improvement of the (BUP). For a finite value of L we see that $a < 0$ *and* $b > 0$ is not possible, because in this case both terms in the left part of (25) are negative. Therefore we have $a \geq 0$ *or* $b \leq 0$, then we obtain again the *upper* (BUP) estimate (22):

$$a = e/\sigma + q \geq 0, \quad or \quad b = r/\sigma + s \leq 0 \Rightarrow \sigma \leq \{\frac{e}{-q}, \frac{-r}{s}\}. \tag{27}$$

b) *Lower - upper estimates depending on* L of the growth constant. We have

$$I = \int_{-L}^{0} f_x^2 + k^2 \int_{-L}^{0} f^2 = \frac{k}{(e^{2kL} - 1)} D \tag{28}$$

where $f_0 := f(0), \quad f_L := f(-L)$ and

$$D = f_0^2 e^{2kL} + f_L^2 + f_0^2 + f_L^2 e^{2kL} - 4 f_0 f_L e^{kL} \tag{29}$$

For this, we use the exact solution (23) of stability equation and get

$$I = 2k^2 \int_{-L}^{0} [A^2 e^{2kx} + B^2 e^{-2kx}] = k[A^2(1 - e^{-2kL}) + B^2(e^{2kL} - 1)] = \tag{30}$$

$$\frac{k(e^{2kL} - 1)}{e^{2kL}} [A^2 + e^{2kL} B^2]$$

The boundary conditions are giving $f_0 = A + B, \quad f_L = Ae^{-kL} + Be^{kL}$, therefore we obtain

$$B = f_0 - A, \quad A = \frac{f_0 e^{kL} - f_L}{e^{kL} - e^{-kL}}, \qquad B = \frac{-f_0 e^{-kL} + f_L}{e^{kL} - e^{-kL}} \tag{31}$$

We insert the above A, B in (30) and obtain the expression (28). Moreover, we have the following estimates of D given by (29):

$$D = (f_0^2 + f_L^2)(1 + e^{2kL} - 2e^{kL}) + 2(f_0 - f_L)^2 e^{kL} \geq \tag{32}$$

$$(f_0^2 + f_L^2)(1 - e^{kL})^2,$$

$$D \leq f_0^2 e^{2kL} + f_L^2 + f_0^2 + f_L^2 e^{2kL} + 2(f_0^2 + f_L^2) e^{kL} = (f_0^2 + f_L^2)(1 + e^{kL})^2. \tag{33}$$

We use (28), (32), (33) and get the lower-upper estimates for I:

$$k \frac{(e^{kL} - 1)}{(e^{kL} + 1)} (f_0^2 + f_L^2) \leq I \leq \frac{e^{kL} + 1}{e^{2kL} - 1} (f_0^2 + f_L^2) \tag{34}$$

The above integral I appears in the formula (19) of σ. We use this formula, the above estimates (34) of I, the inequalities (21) and get the following upper-lower estimates for σ, depending on L:

$$Min \left\{ \frac{e}{-q + LE}, \quad \frac{-r}{s + LE} \right\} \leq \sigma \leq MAX \left\{ \frac{e}{-q + RI}, \quad \frac{-r}{s + RI} \right\}, \tag{35}$$

$$LE = k \frac{e^{kL} + 1}{e^{kL} - 1}, \quad RI = k \frac{e^{kL} - 1}{e^{kL} + 1} \tag{36}$$

For $L \to \infty$ we obtain

$$Min \left\{ \frac{e}{-q + k}, \quad \frac{-r}{s + k} \right\} \leq \sigma \leq MAX \left\{ \frac{e}{-q + k}, \quad \frac{-r}{s + k} \right\},$$

in accord with the *algebraic estimates* (26).

4 Conclusions

In this paper we study the linear stability of the Hele-Shaw displacement of two immiscible fluids (water and oil) with constant viscosity, when a third constant viscosity fluid exists in a middle region between water and oil. The main results are following.

1) The middle region length L is a stabilizing element. The quantities LE and RI in (35) - (36) are increasing in terms of L. Then the upper bound of σ is decreasing in terms of L. The minimum value of the upper bound of σ is obtained for very large L.

2) If the surface tensions T, S are large enough, we obtain an improvement of stability, compared with the Saffman-Taylor case, *independent* of the intermediate viscosity μ. For this, we recall the Saffman-Taylor formula for the growth constant σ_{ST}, when the water (viscosity μ_1) is displacing the oil (viscosity μ_2) and the surface tension on the water-oil interface is T_0:

$$\sigma_{ST} = \frac{\alpha U(\mu_2 - \mu_1) - \alpha^3 T_0}{\mu_2 + \mu_1}. \tag{37}$$

As we pointed above, the largest upper bound of our growth constant σ is obtained for small L, then we use the (BUP) estimate (22). We compute the maximum values of (BUP) and (37) in terms of α and get

$$MAX_\alpha\{\sigma\} < MAX_\alpha\{\sigma_{ST}\} \Leftrightarrow S, T > T_0(\frac{\mu_1 + \mu_2}{\mu_2})^{3/2}. \tag{38}$$

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