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ON *L*-IMPLICATIVE-GROUPS AND ASSOCIATED ALGEBRAS OF LOGIC

Afrodita IORGULESCU[1]

Abstract

The l-implicative-group is a term equivalent definition of the group coming from algebras of logic. In this paper, we study the representability of l-implicative-groups and of associated algebras of logic. First, we find equivalent conditions for an l-implicative-group to be representable. Then, we prove that representability at l-implicative-group level is inherited by the algebras obtained by restricting the l-implicative-group operations to the negative, positive cones.

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1 Introduction

Pseudo-MV algebras, the non-commutative generalizations of Chang's MV algebras [5], were introduced in 1999 [9] and developed in [11] (see also [17]). Pseudo-MV algebras are particular cases of bounded (non-commutative) residuated lattices and are intervals [6] ([16], in the commutative case) in l-groups.

On the other hand, pseudo-Wajsberg algebras, the non-commutative generalizations of Wajsberg algebras [7], are term equivalent [3], [4] to pseudo-MV algebras. Pseudo-Wajsberg algebras are particular cases of bounded pseudo-BCK(pP) lattices [10], [13]. And (bounded) pseudo-BCK(pP) lattices are categorically equivalent to (bounded) residuated lattices [12].

Hence, pseudo-Wajsberg algebras had to be connected to (are intervals in) a notion that is term equivalent to the l-group: that notion is the *l-implicative-group*, introduced and studied in [14], [15].

Note that, usually in the literature, looking from algebraic point of view, the case of *right-pseudo-MV algebras* (the *right-algebras* in general) is considered, since in po-groups, l-groups the *positive cone* is usually considered.

[1]Faculty of Economic Cybernetics, Statistics and Informatics, *Bucharest* Academy of Economic Studies, Romania, e-mail: afrodita.iorgulescu@ase.ro

But, note also that, looking from logical point of view, where the truth is represented by 1, and not by 0, we arive to consider the case of *left-pseudo-MV algebras* (the *left-algebras* in general) and the *negative cone* of po-groups, l-groups. The reader finds more on *left-algebras* and *right-algebras* of logic in [13].

Therefore, in [14], [15] we have studied both left- and right-algebras of logic.

In this paper, we present in details some of the results from [15] announced at the Seventh Congress of Romanian Mathematicians, June 29 - July 5, 2011, Braşov, Romania, namely those concerning the representability of l-implicative-groups and of associated algebras of logic. First, in Section 3, we find equivalent conditions for an l-implicative-group to be representable (Theorem 3.2). Then, in Section 4, we prove that the representability at l-implicative-group level is inherited by the algebras obtained by restricting the l-implicative-group operations to the negative, positive cones (Theorem 4.2). Another important result here is Theorem 4.3. Some open problems are presented.

2 Preliminaries

Recall first the following notations from [14], [15] (where ^{d} means "dual"), in the case of pseudo-BCK lattices:

(pP) $\exists\ x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to^L z\} = \min\{z \mid y \leq x \rightsquigarrow^L z\}$,

(pS) $\exists\ x \oplus y \overset{notation}{=} \max\{z \mid x \geq y \to^R z\} = \max\{z \mid y \geq x \rightsquigarrow^R z\}$,

(pC) $\quad x \vee y = (x \rightsquigarrow^L y) \to^L y = (x \to^L y) \rightsquigarrow^L y$,

(pC^d) $\ x \wedge y = (x \to^R y) \rightsquigarrow^R y = (x \rightsquigarrow^R y) \to^R y$;

(pprel) (pseudo-prelinearity) $(x \to^L y) \vee (y \to^L x) = 1 = (x \rightsquigarrow^L y) \vee (y \rightsquigarrow^L x)$,

(pdiv) (pseudo-divisibility) $x \wedge y = (x \to^L y) \odot x = x \odot (x \rightsquigarrow^L y)$,

(pprel^d) $(x \to^R y) \wedge (y \to^R x) = 0 = (x \rightsquigarrow^R y) \wedge (y \rightsquigarrow^R x)$,

(pdiv^d) $x \vee y = (x \to^R y) \oplus x = x \oplus (x \rightsquigarrow^R y)$.

Recall also [13] that condition (pC) implies conditions (pprel), (pdiv) and dually, condition (pC^d) implies conditions (pprel^d), (pdiv^d).

We now recall from [14] some of the necessary results needed in the sequel concerning the (implicative-) groups.

2.1 Groups, po-groups, l-groups

• Let $\mathcal{G} = (G, +, -, 0)$ be a group, in additive notation in this paper. We introduced the new operations $\to$ and $\rightsquigarrow$ on G, called "implications", defined by: for all $x, y \in G$,

$$x \to y \overset{def.}{=} -[x + (-y)] = y + (-x), \quad x \rightsquigarrow y \overset{def.}{=} -[(-y) + x] = (-x) + y. \tag{2.1}$$

The two implications satisfy the following properties: for all $x, y, z \in G$,

$$x + y = -(x \to (-y)) = (-y) \to x, \quad x + y = -(y \rightsquigarrow (-x)) = (-x) \rightsquigarrow y, \tag{2.2}$$

$$y \to z = (z \to x) \rightsquigarrow (y \to x), \quad y \rightsquigarrow z = (z \rightsquigarrow x) \to (y \rightsquigarrow x), \tag{2.3}$$

$$(y \to x) \rightsquigarrow x = y = (y \rightsquigarrow x) \to x, \tag{2.4}$$

$$-x = x \to 0 = x \rightsquigarrow 0, \tag{2.5}$$

$$x = y \Longleftrightarrow x \to y = 0 \Longleftrightarrow x \rightsquigarrow y = 0, \tag{2.6}$$

$$x + y = z \Longleftrightarrow x = y \to z \Longleftrightarrow y = x \rightsquigarrow z \quad (\textit{see } [8],\ \textit{page } 160). \tag{2.7}$$

• Let now $\mathcal{G} = (G, \leq, +, -, 0)$ be a partially-ordered group (po-group). Then the following properties hold: for all $x, y, z \in G$,

$$(i) \quad x + y \leq z \Leftrightarrow x \leq y \to z \Leftrightarrow y \leq x \rightsquigarrow z, \qquad \textit{and dually} \tag{2.8}$$

$$(ii) \quad x + y \geq z \Leftrightarrow x \geq y \to z \Leftrightarrow y \geq x \rightsquigarrow z,$$

$$x \leq y \implies z \to x \leq z \to y \text{ and } z \rightsquigarrow x \leq z \rightsquigarrow y, \tag{2.9}$$

$$x \leq y \implies y \to z \leq x \to z \text{ and } y \rightsquigarrow z \leq x \rightsquigarrow z. \tag{2.10}$$

• Let finally $\mathcal{G} = (G, \vee, \wedge, +, -, 0)$ be a lattice-ordered group (l-group). Then we have, for all $x, y, z \in G$:

$$(x \vee z) \to y = (x \to y) \wedge (z \to y), \quad (x \vee z) \rightsquigarrow y = (x \rightsquigarrow y) \wedge (z \rightsquigarrow y) \quad \textit{and dually} \tag{2.11}$$

$$(x \wedge z) \to y = (x \to y) \vee (z \to y), \quad (x \wedge z) \rightsquigarrow y = (x \rightsquigarrow y) \vee (z \rightsquigarrow y); \tag{2.12}$$

$$y \to (x \vee z) = (y \to x) \vee (y \to z), \quad y \rightsquigarrow (x \vee z) = (y \rightsquigarrow x) \vee (y \rightsquigarrow z) \quad \textit{and dually} \tag{2.13}$$

$$y \to (x \wedge z) = (y \to x) \wedge (y \to z), \quad y \rightsquigarrow (x \wedge z) = (y \rightsquigarrow x) \wedge (y \rightsquigarrow z). \tag{2.14}$$

2.2 Implicative-groups, po-implicative-groups, l-implicative-groups

• An *implicative-group* ([14], Definition 4.1) is an algebra $\mathcal{G} = (G, \to, \rightsquigarrow, 0)$ of type $(2, 2, 0)$ such that the following axioms hold: for all $x, y, z \in G$,
(I1) $y \to z = (z \to x) \rightsquigarrow (y \to x), \quad y \rightsquigarrow z = (z \rightsquigarrow x) \to (y \rightsquigarrow x)$,
(I2) $y = (y \to x) \rightsquigarrow x, \quad y = (y \rightsquigarrow x) \to x$,
(I3) $x = y \Longleftrightarrow x \to y = 0 \Longleftrightarrow x \rightsquigarrow y = 0$,
(I4) $x \to 0 = x \rightsquigarrow 0$.

The implicative-group is said to be *commutative or abelian* if $\to = \rightsquigarrow$.

Let $\mathcal{G}$ be an implicative-group. Then, we have, for all $x, y, z \in G$:
(I7) $0 \to x = x = 0 \rightsquigarrow x$,
(I8) $z \rightsquigarrow (y \to x) = y \to (z \rightsquigarrow x)$,
(I9) $x \to x = 0 = x \rightsquigarrow x$,

$$z \to x = (y \to z) \to (y \to x), \quad z \rightsquigarrow x = (y \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow x). \tag{2.15}$$

The groups and the implicative-groups are termwise equivalent:

Theorem 2.1. *([14], Theorem 4.13)*

(1) Let $\mathcal{G} = (G, +, -, 0)$ be a group. Define $\Phi(\mathcal{G}) = (G, \to, \rightsquigarrow, 0)$ by: for all $x, y \in G$,
$x \to y \overset{def.}{=} -(x + (-y)) = -(x - y) = y - x$,

$x \rightsquigarrow y \overset{def.}{=} -((-y)+x) = -(-y+x) = -x+y$.
Then $\Phi(\mathcal{G})$ is an implicative-group.

(1') Conversely, let $\mathcal{G} = (G, \to, \rightsquigarrow, 0)$ be an implicative-group. Define $\Psi(\mathcal{G}) = (G, +, -, 0)$ by: for all $x, y \in G$,
$-x \overset{def.}{=} x \to 0 \overset{(I4)}{=} x \rightsquigarrow 0, \quad x + y \overset{def.}{=} -(x \to (-y)) = -(y \rightsquigarrow (-x))$.
Then $\Psi(\mathcal{G})$ is a group.

(2) The maps Φ and Ψ are mutually inverse.

The implicative-group is commutative if and only if the term equivalent group is commutative.

• A *partially-ordered implicative-group* (*po-implicative-group*) ([14], Definition 4.17) is a structure $\mathcal{G} = (G, \leq, \to, \rightsquigarrow, 0)$, where $(G, \to, \rightsquigarrow, 0)$ is an implicative-group and $\leq$ is a partial order on G compatible with $\to$, $\rightsquigarrow$, i.e. we have: for all $x, y, z \in G$,
(I5) $x \leq y$ implies $z \to x \leq z \to y$ and $z \rightsquigarrow x \leq z \rightsquigarrow y$.

The po-groups and the po-implicative-groups are termwise equivalent ([14], Theorem 4.23).

• If the partial order relation $\leq$ is a lattice order relation, then $\mathcal{G}$ is a *lattice-ordered implicative-group* (*l-implicative-group*) denoted $\mathcal{G} = (G, \vee, \wedge, \to, \rightsquigarrow, 0)$.

The l-groups and the l-implicative-groups are termwise equivalent ([14], Corollary 4.31).

2.3 "Vertical" connections (between group level and algebras of logic level)

Theorem 2.2. *(see [14], Theorem 5.3) Let $\mathcal{G} = (G, \vee, \wedge, \to, \rightsquigarrow, 0)$ be an l-implicative-group.*

(1) Define, for all $x, y \in G^-$:

$$x \to^L y \overset{def.}{=} (x \to y) \wedge 0, \quad x \rightsquigarrow^L y \overset{def.}{=} (x \rightsquigarrow y) \wedge 0. \tag{2.16}$$

Then, $\mathcal{G}^L = (G^-, \wedge, \vee, \to^L, \rightsquigarrow^L, \mathbf{1} = 0)$ is a left-pseudo-BCK(pP) lattice (with the pseudo-product $\odot = +$), lattice that is distributive, verifying condition (pC).

(1') Define, for all $x, y \in G^+$:

$$x \to^R y \overset{def.}{=} (x \to y) \vee 0, \quad x \rightsquigarrow^R y \overset{def.}{=} (x \rightsquigarrow y) \vee 0. \tag{2.17}$$

Then, $\mathcal{G}^R = (G^+, \vee, \wedge, \to^R, \rightsquigarrow^R, \mathbf{0} = 0)$ is a right-pseudo-BCK(pS) lattice (with the pseudo-sum $\oplus = +$), lattice that is distributive, verifying the dual condition (pC^d).

3 Representable l-groups, l-implicative-groups

Recall (see [1], for example) that an l-group is *representable* if it is a subdirect product of totally-ordered groups. Recall also the following theorem that gives characterizations of representable l-groups, some of them needed in the sequel.

Theorem 3.1. *(see [1], Theorem 4.1.1)*

Let $\mathcal{G} = (G, \vee, \wedge, +, -, 0)$ be an l-group. The following are equivalent:
(a) $\mathcal{G}$ is representable.
(b) For all $a, b \in G$, $2(a \wedge b) = 2a \wedge 2b$;
(b^d) For all $a, b \in G$, $2(a \vee b) = 2a \vee 2b$.
(c) For all $a, b \in G$, $a \wedge (-b - a + b) \leq 0$;
(c^d) For all $a, b \in G$, $a \vee (-b - a + b) \geq 0$.
(d) Each polar subgroup is normal.
(e) Each minimal prime subgroup is normal.
(f) For each $a \in G$, $a > 0$, $a \wedge (-b + a + b) > 0$, for all $b \in G$;
(f^d) For each $a \in G$, $a < 0$, $a \vee (-b + a + b) < 0$, for all $b \in G$.
Note that ^{d} means "dual".

Remark 3.1. Note that in commutative l-groups we have, for all $a, b \in G$:

$$2(a \wedge b) = 2a \wedge 2b \Longleftrightarrow (b \to a) \wedge (a \to b) \leq 0.$$
$$2(a \vee b) = 2a \vee 2b \Longleftrightarrow (b \to a) \vee (a \to b) \geq 0.$$

Indeed, for example:
$2(a \vee b) = 2a \vee 2b \Longleftrightarrow (a \vee b) + (a \vee b) = 2a \vee 2b \Longleftrightarrow$
$2a \vee 2b = [a + (a \vee b)] \vee [b + (a \vee b)] \Longleftrightarrow 2a \vee 2b = 2a \vee (a + b) \vee (b + a) \vee 2b \Longleftrightarrow$
$2a \vee 2b = 2a \vee 2b \vee (a + b) \Longleftrightarrow 2a \vee 2b \geq a + b \Longleftrightarrow (2a \vee 2b) - b \geq a \Longleftrightarrow$
$(2a - b) \vee b \geq a \Longleftrightarrow [(2a - b) \vee b] - a \geq 0 \Longleftrightarrow (a - b) \vee (b - a) \geq 0 \Longleftrightarrow (b \to a) \vee (a \to b) \geq 0.$

We obtain in the non-commutative case the following results.

Proposition 3.1. *Let $\mathcal{G} = (G, \vee, \wedge, +, -, 0)$ be an l-group. Then*

$$(b) \iff (b1) \iff (b2), \qquad (b^d) \iff (b1^d) \iff (b2^d),$$

where:
(b1) for all $a, b \in G$, $(b \to a) \wedge (a \rightsquigarrow b) \leq 0 \wedge [(b \rightsquigarrow a) \rightsquigarrow (b \to a)]$,
(b2) for all $a, b \in G$, $(b \rightsquigarrow a) \wedge (a \to b) \leq 0 \wedge [(b \to a) \to (b \rightsquigarrow a)]$;
($b1^d$) for all $a, b \in G$, $(b \to a) \vee (a \rightsquigarrow b) \geq 0 \vee [(b \rightsquigarrow a) \rightsquigarrow (b \to a)]$,
($b2^d$) for all $a, b \in G$, $(b \rightsquigarrow a) \vee (a \to b) \geq 0 \vee [(b \to a) \to (b \rightsquigarrow a)]$.

Proof. (b^d) $\Longleftrightarrow$ ($b1^d$):
$2(a \vee b) = 2a \vee 2b \Longleftrightarrow (a \vee b) + (a \vee b) = 2a \vee 2b \Longleftrightarrow$
$[a + (a \vee b)] \vee [b + (a \vee b)] = 2a \vee 2b \Longleftrightarrow 2a \vee (a + b) \vee (b + a) \vee 2b = 2a \vee 2b \Longleftrightarrow$
$2a \vee 2b \vee (a + b) \vee (b + a) = 2a \vee 2b \Longleftrightarrow 2a \vee 2b \geq (a + b) \vee (b + a) \Longleftrightarrow$
$(2a \vee 2b) - b \geq [(a + b) \vee (b + a)] - b \Longleftrightarrow (2a - b) \vee b \geq a \vee (b + a - b) \Longleftrightarrow$
$-a + [(2a - b) \vee b] \geq -a + [a \vee (b + a - b)] \Longleftrightarrow$
$(a - b) \vee (-a + b) \geq 0 \vee (-a + b + a - b) \Longleftrightarrow$
$(b \to a) \vee (a \rightsquigarrow b) \geq -a + b + [(-b + a) \vee (a - b)] = -(-b + a) + [(b \rightsquigarrow a) \vee (b \to a)] \Longleftrightarrow$
$(b \to a) \vee (a \rightsquigarrow b) \geq (b \rightsquigarrow a) \rightsquigarrow [(b \rightsquigarrow a) \vee (b \to a)] \overset{(2.13)}{=} 0 \vee [(b \rightsquigarrow a) \rightsquigarrow (b \to a)].$

(b^d) $\iff$ (b2^d):
$2(a \vee b) = 2a \vee 2b \iff \ldots \iff 2a \vee 2b \geq (b+a) \vee (a+b) \iff$
$[a \vee (2b-a)] + a \geq [b \vee (a+b-a)] + a \iff a \vee (2b-a) \geq b \vee (a+b-a) \iff$
$b + [(-b+a) \vee (b-a)] \geq b + [0 \vee (-b+a+b-a)] \iff$
$(-b+a) \vee (b-a) \geq 0 \vee (-b+a+b-a) \iff$
$(b \rightsquigarrow a) \vee (a \to b) \geq [(a-b) \vee (-b+a)] + b - a \iff$
$(b \rightsquigarrow a) \vee (a \to b) \geq [(a-b) \vee (-b+a)] - (a-b) \iff$
$(b \rightsquigarrow a) \vee (a \to b) \geq (b \to a) \to [(b \to a) \vee (b \rightsquigarrow a)] = 0 \vee [(b \to a) \to (b \rightsquigarrow a)]$.
The rest of the proof is similar. □

Remark 3.2. (see Remark 3.1)
Note that

$$(b1) \implies (b1"), \quad (b2) \implies (b2"); \qquad (b1^d) \implies (b1^{d"}), \quad (b2^d) \implies (b2^{d"}),$$

where:
(b1") for all $a, b \in G$, $(b \to a) \wedge (a \rightsquigarrow b) \leq 0$,
(b2") for all $a, b \in G$, $(b \rightsquigarrow a) \wedge (a \to b) \leq 0$;
(b1$^{d"}$) for all $a, b \in G$, $(b \to a) \vee (a \rightsquigarrow b) \geq 0$,
(b2$^{d"}$) for all $a, b \in G$, $(b \rightsquigarrow a) \vee (a \to b) \geq 0$.
Note that the converse implications are not true.
Note also that (b1") and (b2") coincide and that (b1$^{d"}$) and (b2$^{d"}$) coincide.

Proposition 3.2. *Let $\mathcal{G} = (G, \vee, \wedge, +, -, 0)$ be an l-group. Then*

$$(c) \iff (c1) \iff (c2), \qquad (c^d) \iff (c1^d) \iff (c2^d),$$

where:
(c1) for all $x, y, z, w \in G$, $(x \rightsquigarrow y) \wedge (([((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z] \to w) \to w) \leq 0$,
(c2) for all $x, y, z, w \in G$, $(x \to y) \wedge (([((y \to x) \to z) \to z] \rightsquigarrow w) \rightsquigarrow w) \leq 0$;
(c1^d) for all $x, y, z, w \in G$, $(x \rightsquigarrow y) \vee (([((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z] \to w) \to w) \geq 0$,
(c2^d) for all $x, y, z, w \in G$, $(x \to y) \vee (([((y \to x) \to z) \to z] \rightsquigarrow w) \rightsquigarrow w) \geq 0$.

Proof. (c^d) $\implies$ (c1^d): $(x \rightsquigarrow y) \vee (([((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z] \to w) \to w) =$
$(-x+y) \vee (([-(-(-y+x)+z)+z] \to w) \to w) =$
$(-x+y) \vee (([-(-x+y+z)+z] \to w) \to w) =$
$(-x+y) \vee (([-z-y+x+z] \to w) \to w) =$
$(-x+y) \vee ((w - [-z-y+x+z]) \to w) =$
$(-x+y) \vee ((w-z-x+y+z) \to w) =$
$(-x+y) \vee (w - (w-z-x+y+z)) =$
$(-x+y) \vee (w-z-y+x+z-w) =$
$(-x+y) \vee ((w-z) - (-x+y) + (z-w)) =$
$a \vee (-b-a+b) \geq 0$, by (c^d).
(c1^d) $\implies$ (c^d): Take $x=0$, $y=a$, $z=0$, $w=-b$ in (c1^d); we obtain:
$(0 \rightsquigarrow a) \vee (([((a \rightsquigarrow 0) \rightsquigarrow 0) \rightsquigarrow 0] \to -b) \to -b) \geq 0 \iff$
$a \vee ((-a \to -b) \to -b) \geq 0 \iff$

$a \vee ((-b-(-a)) \to -b) \geq 0 \iff$
$a \vee ((-b+a) \to -b) \geq 0 \iff$
$a \vee (-b-(-b+a)) \geq 0 \iff$
$a \vee (-b-a+b) \geq 0$. Thus $(c^d) \iff (c1^d)$.

$(\mathrm{c}^d) \Longrightarrow (\mathrm{c}2^d)$: $(x \to y) \vee ((\,[((y \to x) \to z) \to z] \rightsquigarrow w) \rightsquigarrow w) =$
$(y-x) \vee (([z-(z-(x-y))] \rightsquigarrow w) \rightsquigarrow w) =$
$(y-x) \vee (([z-(z+y-x)] \rightsquigarrow w) \rightsquigarrow w) =$
$(y-x) \vee (([z+x-y-z] \rightsquigarrow w) \rightsquigarrow w) =$
$(y-x) \vee ((-[z+x-y-z]+w) \rightsquigarrow w) =$
$(y-x) \vee ((z+y-x-z+w) \rightsquigarrow w) =$
$(y-x) \vee (-(z+y-x-z+w)+w) =$
$(y-x) \vee (-w+z+x-y-z+w) =$
$a \vee (-b-a+b) \geq 0$, by (c^d).

$(\mathrm{c}2^d) \Longrightarrow (\mathrm{c}^d)$: Take $x=0$, $y=a$, $z=0$, $w=b$ in $(\mathrm{c}2^d)$; we obtain:
$(0 \to a) \vee ((\,[((a \to 0) \to 0) \to 0] \rightsquigarrow b) \rightsquigarrow b) \geq 0 \iff$
$a \vee ((-a \rightsquigarrow b) \rightsquigarrow b) \geq 0 \iff$
$a \vee ((a+b) \rightsquigarrow b) \geq 0 \iff$
$a \vee (-b-a+b) \geq 0$. Thus $(c^d) \iff (c2^d)$.

The rest of the proof is similar. □

We shall say that an l-implicative-group is *representable* if it is a subdirect product of totally-ordered implicative-groups. Consequently, an l-implicative-group is representable if and only if its term equivalent l-group is representable. Then we have the following result, needed in the sequel.

Theorem 3.2. *Let $\mathcal{G} = (G, \vee, \wedge, \to, \rightsquigarrow, 0)$ be an l-implicative-group. The following are equivalent:*
(a) $\mathcal{G}$ is representable, (b1), (b2), (b1^d), (b2^d), (c1), (c2), (c1^d), (c2^d).

Proof. By Theorem 3.1 and Propositions 3.1, 3.2. □

We can put together Theorems 3.1 and 3.2 in the following resuming statement:

Theorem 3.3. *Let $\mathcal{G} = (G, \vee, \wedge, +, -, 0)$ be an l-group or, equivalently, let $\mathcal{G} = (G, \vee, \wedge, \to, \rightsquigarrow, 0)$ be an l-implicative-group. The following are equivalent:*
(a) $\mathcal{G}$ is representable.

(b) For all $a, b \in G$, $2(a \wedge b) = 2a \wedge 2b$,
(b1) For all $a, b \in G$, $(b \to a) \wedge (a \rightsquigarrow b) \leq 0 \wedge [(b \rightsquigarrow a) \rightsquigarrow (b \to a)]$,
(b2) For all $a, b \in G$, $(b \rightsquigarrow a) \wedge (a \to b) \leq 0 \wedge [(b \to a) \to (b \rightsquigarrow a)]$;
(b^d) For all $a, b \in G$, $2(a \vee b) = 2a \vee 2b$,
(b1^d) For all $a, b \in G$, $(b \to a) \vee (a \rightsquigarrow b) \geq 0 \vee [(b \rightsquigarrow a) \rightsquigarrow (b \to a)]$,
(b2^d) For all $a, b \in G$, $(b \rightsquigarrow a) \vee (a \to b) \geq 0 \vee [(b \to a) \to (b \rightsquigarrow a)]$.

(c) For all $a, b \in G$, $a \wedge (-b-a+b) \leq 0$,

(c1) For all $x, y, z, w \in G$, $(x \rightsquigarrow y) \wedge (([((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z] \to w) \to w) \leq 0$,
(c2) For all $x, y, z, w \in G$, $(x \to y) \wedge (([((y \to x) \to z) \to z] \rightsquigarrow w) \rightsquigarrow w) \leq 0$;
(c^d) For all $a, b \in G$, $a \vee (-b - a + b) \geq 0$,
($c1^d$) For all $x, y, z, w \in G$, $(x \rightsquigarrow y) \vee (([((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z] \to w) \to w) \geq 0$,
($c2^d$) For all $x, y, z, w \in G$, $(x \to y) \vee (([((y \to x) \to z) \to z] \rightsquigarrow w) \rightsquigarrow w) \geq 0$.

(d) Each polar subgroup is normal.

(e) Each minimal prime subgroup is normal.

(f) For each $a \in G$, $a > 0$, $a \wedge (-b + a + b) > 0$, *for all* $b \in G$;
(f^d) For each $a \in G$, $a < 0$, $a \vee (-b + a + b) < 0$, *for all* $b \in G$.

4 Connections between the representability at l-implicative-group level and the representability at negative, positive cones level

• Recall that in the **commutative case**:

A left-residuated lattice $\mathcal{A}^L = (A^L, \wedge, \vee, \odot, \to^L, 1)$ or, equivalently, a left-BCK(P) lattice $\mathcal{A}^L = (A^L, \wedge, \vee, \to^L, 1)$ with the product $\odot$:
(P) there exist $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \to^L z\}$, for all $x, y \in A^L$,
is *representable* if it is a subdirect product of linearly-ordered ones. It is known that representable such algebras are characterized by the prelinearity condition:

$$(prel) \qquad (x \to^L y) \vee (y \to^L x) = 1.$$

Dually, a right-residuated lattice $\mathcal{A}^R = (A^R, \vee, \wedge, \oplus, \to^R, 0)$ or, equivalently, a right-BCK(S) lattice $\mathcal{A}^R = (A^R, \vee, \wedge, \to^R, 0)$ with the sum $\oplus$:
(S) there exist $x \oplus y \overset{notation}{=} \max\{z \mid x \geq y \to^R z\}$, for all $x, y \in A^R$,
is *representable* if it is a subdirect product of linearly-ordered ones; representable such algebras are characterized by the dual prelinearity condition:

$$(prel^d) \qquad (x \to^R y) \wedge (y \to^R x) = 0.$$

Then we have the following result:

Theorem 4.1. *Let* $\mathcal{G} = (G, \vee, \wedge, \to, 0)$ *be a representable commutative l-implicative-group.*
(1) Define, for all $x, y \in G^-$:

$$x \to^L y \overset{def.}{=} (x \to y) \wedge 0. \tag{4.18}$$

Then, $\mathcal{G}^L = (G^-, \wedge, \vee, \to^L, \mathbf{1} = 0)$ *is a representable left-BCK(P) lattice.*

(1') Define, for all $x, y \in G^+$:

$$x \to^R y \stackrel{def.}{=} (x \to y) \vee 0. \tag{4.19}$$

Then, $\mathcal{G}^R = (G^+, \vee, \wedge, \to^R, \mathbf{0} = 0)$ is a representable right-BCK(S) lattice.

Proof. (1): By Theorem 2.2, $\mathcal{G}^L$ is a left-BCK(P) lattice. To prove that it is representable, we must prove that (prel) holds. Indeed, $(x \to^L y) \vee (y \to^L x) = [(x \to y) \wedge 0] \vee [(y \to x) \wedge 0] = [(x \to y) \vee (y \to x)] \wedge 0 = 0$, by Theorem 3.1 and Remark 3.1.

(1') By Theorem 2.2, $\mathcal{G}^R$ is a right-BCK(S) lattice. To prove that it is representable, we must prove that (prel^d) holds. Indeed, $(x \to^R y) \wedge (y \to^R x) = [(x \to y) \vee 0] \wedge [(y \to x) \vee 0] = [(x \to y) \wedge (y \to x)] \vee 0 = 0$, by Theorem 3.1 and Remark 3.1. □

• Recall that in the **non-commutative case**:

A non-commutative left-residuated lattice $\mathcal{A}^{\mathcal{L}} = (A^L, \wedge, \vee, \odot, \to^L, \rightsquigarrow^L, 1)$ or, equivalently, a left-pseudo-BCK(pP) lattice $\mathcal{A}^L = (A^L, \wedge, \vee, \to^L, \rightsquigarrow^L, 1)$ (with the pseudo-product $\odot$) is *representable* if it is a subdirect product of linearly-ordered ones. C.J. van Alten [2] proved that such non-commutative algebras are representable if and only if they satisfy the identity:

$$(x \rightsquigarrow^L y) \vee (([((y \rightsquigarrow^L x) \rightsquigarrow^L z) \rightsquigarrow^L z] \to^L w) \to^L w) = 1, \tag{4.20}$$

or the identity

$$(x \to^L y) \vee (([((y \to^L x) \to^L z) \to^L z] \rightsquigarrow^L w) \rightsquigarrow^L w) = 1. \tag{4.21}$$

Dually,

a non-commutative right-residuated lattice $\mathcal{A}^R = (A^R, \vee, \wedge, \oplus, \to^R, \rightsquigarrow^R, 0)$ or, equivalently, a right-pseudo-BCK(pS) lattice $\mathcal{A}^R = (A^R, \vee, \wedge, \to^R, \rightsquigarrow^R, 0)$ (with the pseudo-sum $\oplus$) is *representable* if it is a subdirect product of linearly-ordered ones. Representable such algebras are characterized then by the dual condition:

$$(x \rightsquigarrow^R y) \wedge (([((y \rightsquigarrow^R x) \rightsquigarrow^R z) \rightsquigarrow^R z] \to^R w) \to^R w) = 0, \tag{4.22}$$

or

$$(x \to^R y) \wedge (([((y \to^R x) \to^R z) \to^R z] \rightsquigarrow^R w) \rightsquigarrow^R w) = 0. \tag{4.23}$$

We shall prove the following result:

Theorem 4.2. *(see Theorem 2.2)*

Let $\mathcal{G} = (G, \vee, \wedge, \to, \rightsquigarrow, 0)$ be a representable l-implicative-group. Then,

(1) $\mathcal{G}^L = (G^-, \wedge, \vee, \to^L, \rightsquigarrow^L, \mathbf{1} = 0)$ is a representable left-pseudo-BCK(pP) lattice (with the pseudo-product $\odot = +$).

(1') $\mathcal{G}^R = (G^+, \vee, \wedge, \to^R, \rightsquigarrow^R, \mathbf{0} = 0)$ is a representable right-pseudo-BCK(pS) lattice (with the pseudo-sum $\oplus = +$).

Proof. (1): By Theorem 2.2, $\mathcal{G}^L$ is a left-pseudo-BCK(pP) lattice. To prove that $\mathcal{G}^L$ is representable, we must prove that condition (4.20), for example, holds. First denote:

$$A \overset{notation}{=} ((y \rightsquigarrow^L x) \rightsquigarrow^L z) \rightsquigarrow^L z,$$

$$B \overset{notation}{=} (A \to^L w) \to^L w,$$

$$C \overset{notation}{=} (x \rightsquigarrow^L y) \vee B.$$

We must prove, by (4.20), that $C = \mathbf{1}$. Indeed,

• **First proof**:

$A = ((y \rightsquigarrow^L x) \rightsquigarrow^L z) \rightsquigarrow^L z = ([(-y+x) \wedge 0] \rightsquigarrow^L z) \rightsquigarrow^L z =$
$[(-[(-y+x) \wedge 0] + z) \wedge 0] \rightsquigarrow^L z =$
$[([(-x+y) \vee 0] + z) \wedge 0] \rightsquigarrow^L z =$
$[[(-x+y+z) \vee z] \wedge 0] \rightsquigarrow^L z =$
$(-[[(-x+y+z) \vee z] \wedge 0] + z) \wedge 0 =$
$([-((-x+y+z) \vee z) \vee 0] + z) \wedge 0 =$
$([[(-z-y+x) \wedge (-z)] \vee 0] + z) \wedge 0 =$
$(([(-z-y+x) \wedge (-z)] + z) \vee z) \wedge 0 =$
$(((-z-y+x+z) \wedge 0) \vee z) \wedge 0 =$
$[(-z-y+x+z) \wedge 0] \vee z =$
$[(-z-y+x+z) \vee z] \wedge 0.$

$B = (A \to^L w) \to^L w =$
$[(w-A) \wedge 0] \to^L w =$
$(w - [(w-A) \wedge 0]) \wedge 0 =$
$(w + [(A-w) \vee 0]) \wedge 0 =$
$((w+A-w) \vee w) \wedge 0 =$
$[(w + ([(-z-y+x+z) \vee z] \wedge 0) - w) \vee w] \wedge 0 =$
$[([(w + [(-z-y+x+z) \vee z]) \wedge w] - w) \vee w] \wedge 0 =$
$[([[(w-z-y+x+z) \vee (w+z)] \wedge w] - w) \vee w] \wedge 0 =$
$[([(w-z-y+x+z-w) \vee (w+z-w)] \wedge 0) \vee w] \wedge 0 =$
$[[(w-z-y+x+z-w) \wedge 0] \vee [(w+z-w) \wedge 0] \vee w] \wedge 0 =$
$[(w-z-y+x+z-w) \wedge 0] \vee [(w+z-w) \wedge 0] \vee w \geq$
$(w-z-y+x+z-w) \wedge 0.$

Hence,
$C = (x \rightsquigarrow^L y) \vee B \geq$
$[(-x+y) \wedge 0] \vee [(w-z-y+x+z-w) \wedge 0] =$
$[(-x+y) \vee (w-z-y+x+z-w)] \wedge 0 =$
$[a \vee (-b-a+b)] \wedge 0$, with $a = -x+y$, $b = z-w$.
But $\mathcal{G}$ is representable, hence by Theorem 3.1 (c^d), for all $a, b \in G$, $a \vee (-b-a+b) \geq 0$. Hence $C \geq 0$ and thus $C = 0$, i.e. $C = \mathbf{1}$.

• **Second proof**: Denote

$$D \overset{notation}{=} ((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z,$$

$$E \overset{notation}{=} (D \to w) \to w.$$

By Theorem 3.2 ($c1^d$), we have

$$(x \rightsquigarrow y) \vee E \geq 0. \tag{4.24}$$

Then,

$A = ((y \rightsquigarrow^L x) \rightsquigarrow^L z) \rightsquigarrow^L z = [([(y \rightsquigarrow x) \wedge 0] \rightsquigarrow z) \wedge 0] \rightsquigarrow^L z \overset{(2.12)}{=}$
$[(((y \rightsquigarrow x) \rightsquigarrow z) \vee (0 \rightsquigarrow z)) \wedge 0] \rightsquigarrow^L z =$
$[(((y \rightsquigarrow x) \rightsquigarrow z) \vee z) \wedge 0] \rightsquigarrow^L z \overset{distrib.}{=}$
$[[((y \rightsquigarrow x) \rightsquigarrow z) \wedge 0] \vee (z \wedge 0)] \rightsquigarrow^L z$
$([[((y \rightsquigarrow x) \rightsquigarrow z) \wedge 0] \vee z] \rightsquigarrow z) \wedge 0 \overset{(2.11)}{=}$
$([((y \rightsquigarrow x) \rightsquigarrow z) \wedge 0] \rightsquigarrow z) \wedge (z \rightsquigarrow z) \wedge 0 \overset{(2.12)}{=}$
$([((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z] \vee (0 \rightsquigarrow z)) \wedge 0 = (D \vee z) \wedge 0.$

$B = (A \rightarrow^L w) \rightarrow^L w = ([(D \vee z) \wedge 0] \rightarrow w) \wedge 0] \rightarrow^L w \overset{(2.12)}{=}$
$[[((D \vee z) \rightarrow w) \vee (0 \rightarrow w)] \wedge 0] \rightarrow^L w =$
$([[((D \vee z) \rightarrow w) \vee w] \wedge 0] \rightarrow w) \wedge 0 \overset{(2.12)}{=}$
$(([((D \vee z) \rightarrow w) \vee w] \rightarrow w) \vee (0 \rightarrow w)) \wedge 0 \overset{(2.11)}{=}$
$([(((D \vee z) \rightarrow w) \rightarrow w) \wedge (w \rightarrow w)] \vee w) \wedge 0 \overset{distrib.}{=}$
$[[(((D \vee z) \rightarrow w) \rightarrow w) \vee w] \wedge (0 \vee w)] \wedge 0 =$
$[(((D \vee z) \rightarrow w) \rightarrow w) \vee w] \wedge 0 \overset{(2.11)}{=}$
$[([(D \rightarrow w) \wedge (z \rightarrow w)] \rightarrow w) \vee w] \wedge 0 \overset{(2.12)}{=}$
$[[((D \rightarrow w) \rightarrow w) \vee ((z \rightarrow w) \rightarrow w))] \vee w] \wedge 0 =$
$[E \vee ((z \rightarrow w) \rightarrow w) \vee w] \wedge 0.$

$C = (x \rightsquigarrow^L y) \vee B =$
$[(x \rightsquigarrow y) \wedge 0] \vee [(E \vee ((z \rightarrow w) \rightarrow w) \vee w) \wedge 0] \overset{distrib.}{=}$
$[(x \rightsquigarrow y) \vee E \vee ((z \rightarrow w) \rightarrow w) \vee w] \wedge 0 = 0,$
since $(x \rightsquigarrow y) \vee E \vee ((z \rightarrow w) \rightarrow w) \vee w \geq (x \rightsquigarrow y) \vee E \geq 0$, by (4.24), and hence $[(x \rightsquigarrow y) \vee E] \wedge 0 = 0$. Thus, $C = \mathbf{1}$.

(1') has a similar proof, using Theorem 3.1 (c), in the first proof, and Theorem 3.2 (c1), in the second proof. □

Finaly, we present some intermediary results and an open problem.

Theorem 4.3. *(see Theorem 2.2)*

Let $\mathcal{G} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ be a representable l-implicative-group. Then,

(1) the left-pseudo-BCK(pP) lattice $\mathcal{G}^L = (G^-, \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, \mathbf{1} = 0)$ (with the pseudo-product $\odot = +$), verifying condition (pC), verifies also the following conditions: for all $a, b \in G^-$,
(i) $(a \vee b)^2 = a^2 \vee b^2$, i.e. $(a \vee b) \odot (a \vee b) = (a \odot a) \vee (b \odot b)$,
(ii) Condition (i) is equivalent with condition

$$[b \rightarrow^L (a \rightsquigarrow^L (a \odot a))] \vee [a \rightsquigarrow^L (b \rightarrow^L (b \odot b))] = \mathbf{1}. \tag{4.25}$$

(iii) $(b \to^L a) \vee (a \rightsquigarrow^L b) = \mathbf{1}$,
(iv) Condition (iii) implies condition (4.25).

(1') the right-pseudo-BCK(pS) lattice $\mathcal{G}^R = (G^+, \vee, \wedge, \to^R, \rightsquigarrow^R, \mathbf{0} = 0)$ *(with the pseudo-sum* $\oplus = +$*), verifying the dual condition* (pC^d)*, verifies also the following conditions: for all* $a, b \in G^+$,
(i') $2(a \wedge b) = 2a \wedge 2b$, *i.e.* $(a \wedge b) \oplus (a \wedge b) = (a \oplus a) \wedge (b \oplus b)$,
(ii') Condition (i') is equivalent with condition

$$[b \to^R (a \rightsquigarrow^R (a \oplus a))] \vee [a \rightsquigarrow^R (b \to^R (b \oplus b))] = \mathbf{0}. \tag{4.26}$$

(iii') $(b \to^R a) \wedge (a \rightsquigarrow^R b) = \mathbf{0}$,
(iv') Condition (iii') implies condition (4.26).

Proof. We prove (1). We denote $\to = \to^L$ and $\rightsquigarrow = \rightsquigarrow^L$.

(i): follows obviously by Theorem 3.3 (b^d), since $\mathcal{G}$ is representable.

(ii): We shall prove that (i) $\Longleftrightarrow$ (4.25). Indeed,

(i) $\Longrightarrow$ (4.25):
(i) $(a \vee b) \odot (a \vee b) = (a \odot a) \vee (b \odot b) \Longleftrightarrow$
$[(a \vee b) \odot a] \vee [(a \vee b) \odot b] = (a \odot a) \vee (b \odot b) \Longleftrightarrow$
$a \odot a \vee b \odot a \vee a \odot b \vee b \odot b = a \odot a \vee b \odot b \Longleftrightarrow$

$$a \odot b \vee b \odot a \leq a \odot a \vee b \odot b. \tag{4.27}$$

And (4.27) $\Longrightarrow a \odot b \leq a \odot a \vee b \odot b \Longrightarrow b \to (a \odot b) \leq b \to (a \odot a \vee b \odot b) \Longrightarrow$

$$a \rightsquigarrow (b \to (a \odot b)) \leq a \rightsquigarrow (b \to (a \odot a \vee b \odot b)). \tag{4.28}$$

But $a \rightsquigarrow (b \to (a \odot b) = b \to (a \rightsquigarrow (a \odot b)) \leq b \to b = \mathbf{1}$, since $b \leq a \rightsquigarrow (a \odot b)$. Hence,
(4.28) $\Longrightarrow a \rightsquigarrow (b \to (a \odot a \vee b \odot b)) = \mathbf{1} \overset{(pprel)}{\Longleftrightarrow}$
$a \rightsquigarrow [(b \to a \odot a) \vee (b \to b \odot b)] = \mathbf{1} \overset{(pprel)}{\Longleftrightarrow}$
$[a \rightsquigarrow (b \to a \odot a)] \vee [a \rightsquigarrow (b \to b \odot b)] = \mathbf{1} \Longleftrightarrow$
$[b \to (a \rightsquigarrow (a \odot a))] \vee [a \rightsquigarrow (b \to (b \odot b))] = \mathbf{1}$, i.e.(4.25) holds.
Note that we have used an equivalent condition with (pprel) denoted ($\text{pprel}_{\Rightarrow\vee}$) in [13], pag. 386:
($\text{pprel}_{\Rightarrow\vee}$) $x \to (y \vee z) = (x \to y) \vee (x \to z)$ and $x \rightsquigarrow (y \vee z) = (x \rightsquigarrow y) \vee (x \rightsquigarrow z)$.

(4.25) $\Longrightarrow$ (i):
(4.25) $[b \to (a \rightsquigarrow (a \odot a))] \vee [a \rightsquigarrow (b \to (b \odot b))] = \mathbf{1} \Longleftrightarrow$
$[a \rightsquigarrow (b \to (a \odot a))] \vee [a \rightsquigarrow (b \to (b \odot b))] = \mathbf{1} \overset{(pprel)}{\Longleftrightarrow}$
$a \rightsquigarrow (b \to (a \odot a \vee b \odot b)) = \mathbf{1} \Longleftrightarrow$
$\mathbf{1} \leq a \rightsquigarrow (b \to (a \odot a \vee b \odot b)) \Longrightarrow a = a \odot \mathbf{1} \leq a \odot [a \rightsquigarrow (b \to (a \odot a \vee b \odot b))] \overset{(pdiv)}{\Longleftrightarrow}$
$a \leq a \wedge (b \to (a \odot a \vee b \odot b)) \leq a \Longrightarrow a = a \wedge (b \to (a \odot a \vee b \odot b)) \Longleftrightarrow$
$a \leq (b \to (a \odot a \vee b \odot b)) \Longrightarrow a \odot b \leq (b \to (a \odot a \vee b \odot b)) \odot b \overset{(pdiv)}{\Longleftrightarrow}$
$a \odot b \leq b \wedge (a \odot a \vee b \odot b) \leq a \odot a \vee b \odot b \Longrightarrow a \odot b \leq a \odot a \vee b \odot b.$

Similarly,
$b \odot a \leq b \odot b \vee a \odot a$,
i.e. $a \odot a \vee b \odot b$ is an upper bound of $a \odot b$ and $b \odot a$. It follows that
$a \odot b \vee b \odot a \leq a \odot a \vee b \odot b$, i.e. (4.27) holds. And we have seen above that (4.27) $\Longleftrightarrow$ (i).

(iii): $(b \rightarrow^L a) \vee (a \rightsquigarrow^L b) = [(b \rightarrow a) \wedge 0] \vee [(a \rightsquigarrow b) \wedge 0] =$
$[(b \rightarrow a) \vee (a \rightsquigarrow b)] \wedge 0 \geq (0 \vee [(b \rightsquigarrow a) \rightsquigarrow (b \rightarrow a)]) \wedge 0 = 0 = \mathbf{1}$, by Theorem 3.3 ((a) $\Longleftrightarrow$ (b1^d).

(iv): Condition (iii) implies condition (4.25). Indeed,
since $a \leq a \rightsquigarrow^L (a \odot a)$ and $b \leq b \rightarrow^L (b \odot b)$ by [13], condition (10.3), it follows that $b \rightarrow^L a \leq b \rightarrow^L [a \rightsquigarrow^L (a \odot a)]$ and $a \rightsquigarrow^L b \leq a \rightsquigarrow^L [b \rightarrow^L (b \odot b)]$, hence $\mathbf{1} = (b \rightarrow^L a) \vee (a \rightsquigarrow^L b) \leq (b \rightarrow^L [a \rightsquigarrow^L (a \odot a)]) \vee (a \rightsquigarrow^L [b \rightarrow^L (b \odot b)])$, hence $(b \rightarrow^L [a \rightsquigarrow^L (a \odot a)]) \vee (a \rightsquigarrow^L [b \rightarrow^L (b \odot b)]) = \mathbf{1}$.

(1') has a similar proof. □

Proposition 4.1. *(see Theorem 2.2)*

Let $\mathcal{G} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ be an l-implicative-group.

(1) If $\mathcal{G}$ verifies the condition (b1^d") from Remark 3.2:
(b1^d") for all $a, b \in G$, $(b \rightarrow a) \vee (a \rightsquigarrow b) \geq 0$,
then the left-pseudo-BCK(pP) lattice $\mathcal{G}^L = (G^-, \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, \mathbf{1} = 0)$ verifies the condition (iii) from Theorem 4.3 (1):
(iii) for all $a, b \in G^-$, $(b \rightarrow^L a) \vee (a \rightsquigarrow^L b) = \mathbf{1} = 0$.

(1') If $\mathcal{G}$ verifies the condition (b1") from Remark 3.2:
(b1") for all $a, b \in G$, $(b \rightarrow a) \wedge (a \rightsquigarrow b) \leq 0$,
then the right-pseudo-BCK(pS) lattice $\mathcal{G}^R = (G^+, \vee, \wedge, \rightarrow^R, \rightsquigarrow^R, \mathbf{0} = 0)$ verifies the condition (iii') from Theorem 4.3 (1'):
(iii') for all $a, b \in G^+$, $(b \rightarrow^R a) \wedge (a \rightsquigarrow^R b) = \mathbf{0} = 0$.

Proof. (1): $(b \rightarrow^L a) \vee (a \rightsquigarrow^L b) = [(b \rightarrow a) \wedge 0] \vee [(a \rightsquigarrow b) \wedge 0] \overset{distrib.}{=}$
$[(b \rightarrow a) \vee (a \rightsquigarrow b)] \wedge 0 \overset{(b1^{d}")}{=} 0 = \mathbf{1}$.

(1'): $(b \rightarrow^R a) \wedge (a \rightsquigarrow^R b) = [(b \rightarrow a) \vee 0] \wedge [(a \rightsquigarrow b) \vee 0] =$
$[(b \rightarrow a) \wedge (a \rightsquigarrow b)] \vee 0 \overset{(b1")}{=} 0 = \mathbf{0}$. □

Open problems 4.2.

(1) Find if there are connections between the representability of $\mathcal{G}^L = (G^-, \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, \mathbf{1} = 0)$ (or of the left-pseudo-MV algebra $[u', 0]$) and the conditions (i) $\Longleftrightarrow$ (4.25), (iii).

(1') Find if there are connections between the representability of $\mathcal{G}^R = (G^+, \vee, \wedge, \rightarrow^R, \rightsquigarrow^R, \mathbf{0} = 0)$ (or of the right-pseudo-MV algebra $[0, u]$) and the conditions (i') $\Longleftrightarrow$ (4.26), (iii').

Open problem 4.3. Find connections between the representability at l-group (l-implicative-group) G level and the representability at $[u', 0] \subset G^-$, $[0, u] \subset G^+$ level and at $G^- \cup \{-\infty\}$, $G^+ \cup \{+\infty\}$ level.

References

[1] Andersen, M. and Feil, T., *Lattice-Ordered Groups - An Introduction*, D. Reidel Publishing Company, 1988.

[2] Van Alten, C.J., *Representable biresiduated lattices*, J. Algebra **247** (2002), 672-691.

[3] Ceterchi, R., *On Algebras with Implications, categorically equivalent to Pseudo-MV Algebras*, Information Technology. Proc. of the Fourth International Symposium on Economic Informatics, Bucharest, Romania, May 1999, INFOREC Printing House, 912-916, 1999.

[4] Ceterchi, R., *Pseudo-Wajsberg Algebras*, Mult. Val. Logic (A special issue dedicated to the memory of Gr. C. Moisil) **6** (2001), no. 1-2, 67-88.

[5] Chang, C.C., *Algebraic analysis of many valued logics*, Trans. Amer. Math. Soc. **88** (1958), 467-490.

[6] Dvurečenskij, A., *Pseudo MV-algebras are intervals in l-groups*, J. of Australian Mathematical Society **72** (2002), 427-445.

[7] Font, J.M., Rodriguez, A.J. and Torrens, A., *Wajsberg algebras*, Stochastica Vol. VIII (1984), no. 1, 5-31.

[8] Galatos, N., Jipsen, P., Kowalski, T. and Ono, H., *Residuated lattices: an algebraic glimps at substrucural logic*, Studies in Logic and the Foundation of Mathematics, Vol. 151, Elsevier, 2007.

[9] Georgescu, G. and Iorgulescu, A., *Pseudo-MV Algebras: a Noncommutative Extension of MV Algebras*, Information Technology. Proc. of the Fourth International Symposium on Economic Informatics, Bucharest, Romania, May 1999, INFOREC Printing House, 961-968, 1999.

[10] Georgescu, G. and Iorgulescu, A., *Pseudo-BCK algebras: An extension of BCK algebras*, Combinatorics, Computability and Logic. Proc. of the Third International Conference on Combinatorics, Computability and Logic (DMTCS'01), Constanţa, Romania, 2001, (Eds. Calude, C.S., Dinneen, M.J., Sburlan, S.), Springer, London, 97-114, 2001.

[11] Georgescu, G. and Iorgulescu, A., *Pseudo-MV algebras*, Mult. Val. Logic (A special issue dedicated to the memory of Gr. C. Moisil) **6** (2001), no. 1-2, 95-135.

[12] Iorgulescu, A., *On pseudo-BCK algebras and porims*, Scientiae Mathematicae Japonicae **10** (2004), no. 16, 293-305.

[13] Iorgulescu, A., *Algebras of logic as BCK algebras*, Academy of Economic Studies Press, Bucharest, 2008.

[14] Iorgulescu, A., *The implicative-group - a term equivalent definition of the group coming from algebras of logic - Part I*, Preprint nr. 11/2011, Institutul de Matematică "Simion Stoilow" al Academiei Române, Bucureşti, România.

[15] Iorgulescu, A., *The implicative-group - a term equivalent definition of the group coming from algebras of logic - Part II*, Preprint nr. 12/2011, Institutul de Matematică "Simion Stoilow" al Academiei Române, Bucureşti, România.

[16] Mundici, D., *Interpretation of AF C^*-algebras in Lukasiewicz sentential calculus*, J. Funct. Anal. **65** (1986), 15-63.

[17] Rachunek, J., *A non-commutative generalization of MV-algebras*, Czechoslovak Math. J. **52** (2002), 255-273.