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EXISTENCE OF POSITIVE SOLUTIONS FOR A NONLINEAR HIGHER-ORDER MULTI-POINT BOUNDARY VALUE PROBLEM

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Abstract

We investigate the existence of positive solutions of a system of higher-order nonlinear ordinary differential equations, subject to multi-point boundary conditions.

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1 Introduction

In recent years, the multi-point boundary value problems for second-order or higher-order differential or difference equations/systems have been investigated by many authors, by using different methods such us fixed point theorems in cones, the Leray-Schauder continuation theorem and its nonlinear alternatives and the coincidence degree theory.

In this paper, we consider the system of nonlinear higher-order ordinary differential equations

$$(S)\qquad \begin{cases} u^{(n)}(t)+\lambda c(t)f(u(t),v(t))=0, \;\; t\in(0,T), \;\; n\in\mathbb{N}, \;\; n\geq 2, \\ v^{(m)}(t)+\mu d(t)g(u(t),v(t))=0, \;\; t\in(0,T), \;\; m\in\mathbb{N}, \;\; m\geq 2, \end{cases}$$

with the multi-point boundary conditions

$$(BC)\qquad \begin{cases} u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, \;\; u(T)=\displaystyle\sum_{i=1}^{p-2}a_iu(\xi_i), \;\; p\in\mathbb{N}, \;\; p\geq 3, \\ v(0)=v'(0)=\cdots=v^{(m-2)}(0)=0, \;\; v(T)=\displaystyle\sum_{i=1}^{q-2}b_iv(\eta_i), \;\; q\in\mathbb{N}, \;\; q\geq 3. \end{cases}$$

We give sufficient conditions on λ, μ, f and g such that positive solutions of $(S)-(BC)$ exist. By a positive solution of problem $(S)-(BC)$ we mean a pair of functions $(u,v)\in$

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$C^n([0,T]) \times C^m([0,T])$ satisfying (S) and (BC) with $u(t) \geq 0,\ v(t) \geq 0$ for all $t \in [0,T]$ and $\|u\| + \|v\| > 0$, where $\|u\| = \sup_{t\in[0,T]} |u(t)|$. This problem is a generalization of the one studied in [19], where $n = m,\ p = q,\ a_i = b_i,\ \xi_i = \eta_i$ for all $i = 1, \ldots, p-2$. The system (S) with $n = m,\ f(u,v) = \widetilde{f}(v),\ g(u,v) = \widetilde{g}(u)$ (denoted by $(\widetilde{S})$) and the boundary conditions (BC) with $p = q,\ a_i = b_i,\ \xi_i = \eta_i,\ i = 1, \ldots, p-2$ (denoted by $(\widetilde{BC})$) has been investigated in [16]. In [4], the authors studied the system $(\widetilde{S})$ with $T = 1$ and the boundary conditions $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0,\ u(1) = \alpha u(\eta)$, $v(0) = v'(0) = \cdots = v^{(n-2)}(0) = 0,\ v(1) = \alpha v(\eta)$, where $0 < \eta < 1,\ 0 < \alpha\eta^{n-1} < 1$. We also mention the paper [20], where the authors used the fixed point index theory to prove the existence of positive solutions for the system (S) with $\lambda = \mu = 1$ and (BC), where $\frac{1}{2} \leq \xi_1 < \xi_2 < \cdots < \xi_p < 1,\ \ \frac{1}{2} \leq \eta_1 < \eta_2 < \cdots < \eta_q < 1$.

The system (S) with $n = m = 2$ and the boundary conditions $\alpha u(0) - \beta u'(0) = 0,\ \ u(T) = \sum_{i=1}^{m} a_i u(\xi_i),\ \ m \geq 1,\ \ \gamma v(0) - \delta v'(0) = 0,\ \ v(T) = \sum_{i=1}^{n} b_i v(\eta_i),\ \ n \geq 1$, has been investigated in [2]. Some particular cases of the last problem were studied in [6], [8], [9], [17]. In [5], the authors investigated the system $(\widetilde{S})$ with $n = m = 2$ and the boundary conditions $\alpha u(0) - \beta u'(0) = 0,\ \ \alpha v(0) - \beta v'(0) = 0,\ \ \gamma u(1) + \delta u'(1) = 0,\ \ \gamma v(1) + \delta v'(1) = 0$, with $\alpha,\ \beta,\ \gamma,\ \delta \geq 0,\ \alpha + \beta + \gamma + \delta > 0$. For the discrete problem corresponding to (S) with $n = m = 2$ and various boundary conditions, we would like to mention the papers [3], [7], [10], [14], [15], [18].

In Section 2, we present some auxiliary results which investigate two boundary value problems for higher-order equations (the problems (1)-(2) and (3)-(4) below). In Section 3, we give some existence theorems for the positive solutions with respect to a cone for our problem (S)-(BC). The proofs of these results are similar to those of Theorems 3.1 and 3.2 from [1]. These theorems are based on the Krasnoselskii fixed point theorem (see [12], [13]), which we present now.

Theorem 1. *Let $(X, \|\cdot\|)$ be a normed linear space, $K \subset X$ a cone, $0 < a < b$ two given numbers and $K(a,b) = \{x \in K,\ a \leq \|x\| \leq b\}$, $K_a = \{x \in K,\ \|x\| = a\}$, $K_b = \{x \in K,\ \|x\| = b\}$. Let $T : K(a,b) \to K$ be a completely continuous operator such that one of the following conditions is satisfied:*

i) $\|Tx\| \leq \|x\|$ if $x \in K_a$ and $\|Tx\| \geq \|x\|$ if $x \in K_b$;

ii) $\|Tx\| \geq \|x\|$ if $x \in K_a$ and $\|Tx\| \leq \|x\|$ if $x \in K_b$.

Then T has a fixed point in $K(a,b)$.

Finally, some examples are presented in Section 4 to illustrate our main results.

2 Auxiliary results

In this section, we present some auxiliary results from [11] and [16], related to the following n-order differential equation with p-point boundary conditions

$$u^{(n)}(t) + y(t) = 0,\ \ t \in (0,T), \tag{1}$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \;\; u(T) = \sum_{i=1}^{p-2} a_i u(\xi_i). \tag{2}$$

Lemma 1. *([11], [16]) If* $d = T^{n-1} - \sum_{i=1}^{p-2} a_i \xi_i^{n-1} \neq 0$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$ *and* $y \in C([0,T])$, *then the solution of (1)-(2) is given by*

$$u(t) = \frac{t^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1} y(s)\,ds - \frac{t^{n-1}}{d(n-1)!}\sum_{i=1}^{p-2} a_i \int_0^{\xi_i} (\xi_i - s)^{n-1} y(s)\,ds - \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} y(s)\,ds, \;\; 0 \le t \le T.$$

Lemma 2. *([11], [16]) Under the assumptions of Lemma 1, the Green's function for the boundary value problem (1)-(2) is given by*

$$G_1(t,s) = \begin{cases} \dfrac{t^{n-1}}{d(n-1)!}\left[(T-s)^{n-1} - \displaystyle\sum_{i=j+1}^{p-2} a_i(\xi_i - s)^{n-1}\right] - \dfrac{1}{(n-1)!}(t-s)^{n-1}, \\ \qquad if \;\; \xi_j \le s < \xi_{j+1}, \;\; s \le t, \\ \dfrac{t^{n-1}}{d(n-1)!}\left[(T-s)^{n-1} - \displaystyle\sum_{i=j+1}^{p-2} a_i(\xi_i - s)^{n-1}\right], \\ \qquad if \;\; \xi_j \le s < \xi_{j+1}, \;\; s \ge t, \;\; j = 0, \dots p-3, \\ \dfrac{t^{n-1}}{d(n-1)!}(T-s)^{n-1} - \dfrac{1}{(n-1)!}(t-s)^{n-1}, \;\; if \;\; \xi_{p-2} \le s \le T, \;\; s \le t, \\ \dfrac{t^{n-1}}{d(n-1)!}(T-s)^{n-1}, \;\; if \;\; \xi_{p-2} \le s \le T, \;\; s \ge t, \;\; (\xi_0 = 0). \end{cases}$$

Using the above Green's function the solution of problem (1)-(2) is expressed as $u(t) = \int_0^T G_1(t,s) y(s)\,ds$.

Lemma 3. *([11], [16]) If* $a_i > 0$ *for all* $i = 1, \dots, p-2$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $d > 0$ *and* $y \in C([0,T])$, $y(t) \ge 0$ *for all* $t \in [0,T]$, *then the solution* u *of problem (1)-(2) satisfies* $u(t) \ge 0$ *for all* $t \in [0,T]$.

Lemma 4. *([16]) If* $a_i > 0$ *for all* $i = 1, \dots, p-2$, $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $d > 0$, $y \in C([0,T])$, $y(t) \ge 0$ *for all* $t \in [0,T]$, *then the solution of problem (1)-(2) satisfies*

$$\begin{cases} u(t) \le \dfrac{T^{n-1}}{d(n-1)!}\displaystyle\int_0^T (T-s)^{n-1} y(s)\,ds, \;\; \forall t \in [0,T], \\ u(\xi_j) \ge \dfrac{\xi_j^{n-1}}{d(n-1)!}\displaystyle\int_{\xi_{p-2}}^T (T-s)^{n-1} y(s)\,ds, \;\; \forall\, j = \overline{1, p-2}. \end{cases}$$

Lemma 5. *([11]) Assume that $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $a_i > 0$ for all $i = 1, \ldots, p-2$, $d > 0$ and $y \in C([0,T])$, $y(t) \geq 0$ for all $t \in [0,T]$. Then the solution of problem (1)-(2) satisfies $\inf\limits_{t \in [\xi_{p-2}, T]} u(t) \geq \gamma_1 \|u\|$, where*

$$\gamma_1 = \begin{cases} \min\left\{ \dfrac{a_{p-2}(T - \xi_{p-2})}{T - a_{p-2}\xi_{p-2}}, \dfrac{a_{p-2}\xi_{p-2}^{n-1}}{T^{n-1}} \right\}, & if \ \sum\limits_{i=1}^{p-2} a_i < 1, \\ \min\left\{ \dfrac{a_1 \xi_1^{n-1}}{T^{n-1}}, \dfrac{\xi_{p-2}^{n-1}}{T^{n-1}} \right\}, & if \ \sum\limits_{i=1}^{p-2} a_i \geq 1. \end{cases}$$

We can also formulate similar results as Lemma 1 - Lemma 5 above for the boundary value problem

$$v^{(m)}(t) + h(t) = 0, \ \ t \in (0,T), \tag{3}$$

$$v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \ \ v(T) = \sum_{i=1}^{q-2} b_i v(\eta_i). \tag{4}$$

If $e = T^{m-1} - \sum\limits_{i=1}^{q-2} b_i \eta_i^{m-1} \neq 0$, $0 < \eta_1 < \cdots < \eta_{q-2} < T$ and $h \in C([0,T])$, we denote by G_2 the Green's function corresponding to problem (3)-(4). Under similar assumptions as those from Lemma 5, we have the inequality $\inf\limits_{t \in [\eta_{q-2}, T]} v(t) \geq \gamma_2 \|v\|$, where v is the solution of problem (3)-(4) and γ_2 has a similar form as γ_1 from Lemma 5 with n, p and a_i replaced by m, q and b_i, respectively.

3 Main results

In this section, we give sufficient conditions on λ, μ, f and g such that positive solutions with respect to a cone for our problem $(S) - (BC)$ exist.

We present the assumptions that we shall use in the sequel.

$(H1)$ $0 < \xi_1 < \cdots < \xi_{p-2} < T$, $a_i > 0$, $i = \overline{1, p-2}$, $d = T^{n-1} - \sum\limits_{i=1}^{p-2} a_i \xi_i^{n-1} > 0$, $0 < \eta_1 < \cdots < \eta_{q-2} < T$, $b_i > 0$, $i = \overline{1, q-2}$, $e = T^{m-1} - \sum\limits_{i=1}^{q-2} b_i \eta_i^{m-1} > 0$.

$(H2)$ The functions c, $d : [0,T] \to [0,\infty)$ are continuous and there exist t_1, $t_2 \in [\theta_0, T]$ such that $c(t_1) > 0$ and $d(t_2) > 0$, where $\theta_0 = \max\{\xi_{p-2}, \eta_{q-2}\}$.

$(H2')$ The functions c, $d : [0,T] \to [0,\infty)$ are continuous and there exist $t_1 \in [\xi_{p-2}, T]$, $t_2 \in [\eta_{q-2}, T]$ such that $c(t_1) > 0$ and $d(t_2) > 0$.

$(H3)$ The functions f, $g : [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous.

Throughout this section, we let

$$f_0^s = \limsup_{(u,v)\to(0^+,0^+)} \frac{f(u,v)}{u+v},\ g_0^s = \limsup_{(u,v)\to(0^+,0^+)} \frac{g(u,v)}{u+v},$$
$$f_0^i = \liminf_{(u,v)\to(0^+,0^+)} \frac{f(u,v)}{u+v},\ g_0^i = \liminf_{(u,v)\to(0^+,0^+)} \frac{g(u,v)}{u+v},$$
$$f_\infty^s = \limsup_{(u,v)\to(\infty,\infty)} \frac{f(u,v)}{u+v},\ g_\infty^s = \limsup_{(u,v)\to(\infty,\infty)} \frac{g(u,v)}{u+v},$$
$$f_\infty^i = \liminf_{(u,v)\to(\infty,\infty)} \frac{f(u,v)}{u+v},\ g_\infty^i = \liminf_{(u,v)\to(\infty,\infty)} \frac{g(u,v)}{u+v}.$$

We consider the Banach space $X = C([0,T])$ with supremum norm $\|\cdot\|$, and the Banach space $Y = X\times X$ with the norm $\|(u,v)\|_Y = \|u\|+\|v\|$.

We define the cone $C \subset Y$ by

$$C = \{(u,v)\in Y;\ u(t)\ge 0,\ v(t)\ge 0,\ \forall\, t\in[0,T] \text{ and } \inf_{t\in[\theta_0,T]}(u(t)+v(t)) \ge \gamma\|(u,v)\|_Y\},$$

where $\gamma = \min\{\gamma_1,\gamma_2\}$ and γ_1, γ_2 are defined in Section 2.

First, for f_0^s, g_0^s, f_∞^i, $g_\infty^i \in (0,\infty)$ and positive numbers α_1, $\alpha_2>0$ such that $\alpha_1+\alpha_2 = 1$, we define the positive numbers L_1, L_2, L_3 and L_4 by

$$L_1 = \alpha_1\left(\frac{\gamma\xi_{p-2}^{n-1}}{d(n-1)!}\int_{\theta_0}^T (T-s)^{n-1}c(s)f_\infty^i\,ds\right)^{-1},$$
$$L_2 = \alpha_1\left(\frac{T^{n-1}}{d(n-1)!}\int_0^T (T-s)^{n-1}c(s)f_0^s\,ds\right)^{-1},$$
$$L_3 = \alpha_2\left(\frac{\gamma\eta_{q-2}^{m-1}}{e(m-1)!}\int_{\theta_0}^T (T-s)^{m-1}d(s)g_\infty^i\,ds\right)^{-1},$$
$$L_4 = \alpha_2\left(\frac{T^{m-1}}{e(m-1)!}\int_0^T (T-s)^{m-1}d(s)g_0^s\,ds\right)^{-1}.$$

Theorem 2. *Assume that (H1), (H2) and (H3) hold and α_1, $\alpha_2>0$ are positive numbers such that $\alpha_1+\alpha_2=1$.*

a) If f_0^s, g_0^s, f_∞^i, $g_\infty^i\in(0,\infty)$, $L_1<L_2$ and $L_3<L_4$, then for each $\lambda\in(L_1,L_2)$ and $\mu\in(L_3,L_4)$ there exists a positive solution $(u(t),v(t))$, $t\in[0,T]$ for $(S)-(BC)$.

b) If $f_0^s=g_0^s=0$, f_∞^i, $g_\infty^i\in(0,\infty)$, then for each $\lambda\in(L_1,\infty)$ and $\mu\in(L_3,\infty)$ there exists a positive solution $(u(t),v(t))$, $t\in[0,T]$ for $(S)-(BC)$.

c) If f_0^s, $g_0^s\in(0,\infty)$, $f_\infty^i=g_\infty^i=\infty$, then for each $\lambda\in(0,L_2)$ and $\mu\in(0,L_4)$ there exists a positive solution $(u(t),v(t))$, $t\in[0,T]$ for $(S)-(BC)$.

d) If $f_0^s=g_0^s=0$, $f_\infty^i=g_\infty^i=\infty$, then for each $\lambda\in(0,\infty)$ and $\mu\in(0,\infty)$ there exists a positive solution $(u(t),v(t))$, $t\in[0,T]$ for $(S)-(BC)$.

Sketch of proof. a) We suppose f_0^s, g_0^s, f_∞^i, $g_\infty^i\in(0,\infty)$, $L_1<L_2$ and $L_3<L_4$. Let P_1, $P_2 : Y\to X$ and $\mathcal{Q}: Y\to Y$ be the operators defined by

$$P_1(u,v)(t) = \lambda\int_0^T G_1(t,s)c(s)f(u(s),v(s))\,ds,\ \ t\in[0,T],$$
$$P_2(u,v)(t) = \mu\int_0^T G_2(t,s)d(s)g(u(s),v(s))\,ds,\ \ t\in[0,T],$$

and $\mathcal{Q}(u,v) = (P_1(u,v), P_2(u,v)),\ (u,v) \in Y$, where G_1, G_2 are the Green's functions defined in Section 2.

The solutions of problem $(S)-(BC)$ are the fixed points of the operator $\mathcal{Q}$.

We consider an arbitrary element $(u,v) \in C$. Because $P_1(u,v)$ and $P_2(u,v)$ satisfy the problem (1)-(2) for $y(t) = \lambda c(t) f(u(t), v(t))$, $t \in [0,T]$, and the problem (3)-(4) for $h(t) = \mu d(t) g(u(t), v(t))$, $t \in [0,T]$, respectively, then by Lemma 5, we obtain

$$\inf_{t\in[\theta_0,T]} P_1(u,v)(t) \geq \gamma_1 \|P_1(u,v)\|, \quad \inf_{t\in[\theta_0,T]} P_2(u,v)(t) \geq \gamma_2 \|P_2(u,v)\|.$$

Therefore we deduce

$$\inf_{t\in[\theta_0,T]} [P_1(u,v)(t) + P_2(u,v)(t)] \geq \gamma_1 \|P_1(u,v)\| + \gamma_2 \|P_2(u,v)\| \geq \gamma \|\mathcal{Q}(u,v)\|_Y.$$

By using Lemma 3, (H2) and (H3), we obtain that $P_1(u,v)(t) \geq 0,\ P_2(u,v)(t) \geq 0$, for all $t \in [0,T]$, and so we deduce that $\mathcal{Q}(u,v) \in C$. Hence we get $\mathcal{Q}(C) \subset C$.

By using standard arguments, we can easily show that P_1 and P_2 are completely continuous, and then $\mathcal{Q}$ is a completely continuous operator.

Now let $\lambda \in (L_1, L_2)$, $\mu \in (L_3, L_4)$, and let $\varepsilon > 0$ be a positive number such that $\varepsilon < f_\infty^i$, $\varepsilon < g_\infty^i$ and

$$\begin{aligned}
\alpha_1 \left(\frac{\gamma \xi_{p-2}^{n-1}}{d(n-1)!} \int_{\theta_0}^T (T-s)^{n-1} c(s)(f_\infty^i - \varepsilon)\, ds \right)^{-1} &\leq \lambda, \\
\alpha_1 \left(\frac{T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} c(s)(f_0^s + \varepsilon)\, ds \right)^{-1} &\geq \lambda, \\
\alpha_2 \left(\frac{\gamma \eta_{q-2}^{m-1}}{e(m-1)!} \int_{\theta_0}^T (T-s)^{m-1} d(s)(g_\infty^i - \varepsilon)\, ds \right)^{-1} &\leq \mu, \\
\alpha_2 \left(\frac{T^{m-1}}{e(m-1)!} \int_0^T (T-s)^{m-1} d(s)(g_0^s + \varepsilon)\, ds \right)^{-1} &\geq \mu.
\end{aligned}$$

By (H3), we deduce that there exists $K_1 > 0$ such that for all $u, v \in \mathbb{R}_+$, with $0 \leq u+v \leq K_1$, we have $f(u,v) \leq (f_0^s + \varepsilon)(u+v)$ and $g(u,v) \leq (g_0^s + \varepsilon)(u+v)$.

We define the ball $\Omega_1 = \{(u,v) \in Y,\ \|(u,v)\|_Y < K_1\}$. Now let $(u,v) \in C \cap \partial\Omega_1$, that is $(u,v) \in C$ with $\|(u,v)\|_Y = K_1$ or, equivalently, $\|u\| + \|v\| = K_1$. Then $u(t) + v(t) \leq K_1$ for all $t \in [0,T]$. By Lemma 4, after some computations, we deduce that $P_1(u,v)(t) \leq \alpha_1 \|(u,v)\|_Y$ for all $t \in [0,T]$. Therefore $\|P_1(u,v)\| \leq \alpha_1 \|(u,v)\|_Y$. In a similar manner, we obtain $\|P_2(u,v)\| \leq \alpha_2 \|(u,v)\|_Y$.

Then for $(u,v) \in C \cap \partial\Omega_1$ we deduce

$$\|\mathcal{Q}(u,v)\|_Y = \|(P_1(u,v), P_2(u,v))\|_Y \leq \alpha_1 \|(u,v)\|_Y + \alpha_2 \|(u,v)\|_Y = \|(u,v)\|_Y.$$

By the definitions of f_∞^i and g_∞^i, there exists $\bar{K}_2 > 0$ such that $f(u,v) \geq (f_\infty^i - \varepsilon)(u+v)$ and $g(u,v) \geq (g_\infty^i - \varepsilon)(u+v)$ for all $u, v \geq 0$, with $u+v \geq \bar{K}_2$. We consider $K_2 =$

$\max\{2K_1, \bar{K}_2/r\}$, and we define $\Omega_2 = \{(u,v) \in Y,\ \|(u,v)\|_Y < K_2\}$. Then for $(u,v) \in C$ with $\|(u,v)\|_Y = K_2$, we obtain

$$u(t) + v(t) \geq \gamma_1\|u\| + \gamma_2\|v\| \geq \gamma(\|u\| + \|v\|) = \gamma\|(u,v)\|_Y = \gamma K_2 \geq \bar{K}_2,\ \ \forall\, t \in [\theta_0, T].$$

Then by Lemma 4, after some computations, we deduce that $P_1(u,v)(\xi_{p-2}) \geq \alpha_1\|(u,v)\|_Y$. So $\|P_1(u,v)\| \geq P_1(u,v)(\xi_{p-2}) \geq \alpha_1\|(u,v)\|_Y$. In a similar manner, we obtain $\|P_2(u,v)\| \geq P_2(u,v)(\eta_{q-2}) \geq \alpha_2\|(u,v)\|_Y$.

Hence for $(u,v) \in C \cap \partial\Omega_2$ we obtain

$$\|\mathcal{Q}(u,v)\|_Y = \|P_1(u,v)\| + \|P_2(u,v)\| \geq (\alpha_1 + \alpha_2)\|(u,v)\|_Y = \|(u,v)\|_Y.$$

By using Theorem 1 i) with $T = \mathcal{Q}$, $K = C$, $a = K_1$, $b = K_2$, $K(a,b) = C \cap (\bar{\Omega}_2 \setminus \Omega_1)$, $K_a = C \cap \partial\Omega_1$, $K_b = C \cap \partial\Omega_2$, we deduce that $\mathcal{Q}$ has a fixed point $(u,v) \in C \cap (\bar{\Omega}_2 \setminus \Omega_1)$ such that $K_1 \leq \|(u,v)\|_Y \leq K_2$ or $K_1 \leq \|u\| + \|v\| \leq K_2$.

The proofs of cases b)-d) are similar to that of case a) and we shall omit them (see also the paper [1]). □

Remark 1. *The condition $L_1 < L_2$ from Theorem 2 is equivalent to*

$$f_0^s T^{n-1} \int_0^T (T-s)^{n-1} c(s)\, ds < f_\infty^i \gamma \xi_{p-2}^{n-1} \int_{\theta_0}^T (T-s)^{n-1} c(s)\, ds$$

and $L_3 < L_4$ is equivalent to

$$g_0^s T^{m-1} \int_0^T (T-s)^{m-1} d(s)\, ds < g_\infty^i \gamma \eta_{q-2}^{m-1} \int_{\theta_0}^T (T-s)^{m-1} d(s)\, ds.$$

In what follows, for f_0^i, g_0^i, f_∞^s, $g_\infty^s \in (0,\infty)$ and positive numbers α_1, $\alpha_2 > 0$ such that $\alpha_1 + \alpha_2 = 1$, we define the positive numbers $\widetilde{L}_1$, $\widetilde{L}_2$, $\widetilde{L}_3$ and $\widetilde{L}_4$ by

$$\begin{aligned}
\widetilde{L}_1 &= \alpha_1 \left(\frac{\gamma \xi_{p-2}^{n-1}}{d(n-1)!} \int_{\xi_{p-2}}^T (T-s)^{n-1} c(s) f_0^i\, ds \right)^{-1}, \\
\widetilde{L}_2 &= \alpha_1 \left(\frac{T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} c(s) f_\infty^s\, ds \right)^{-1}, \\
\widetilde{L}_3 &= \alpha_2 \left(\frac{\gamma \eta_{q-2}^{m-1}}{e(m-1)!} \int_{\eta_{q-2}}^T (T-s)^{m-1} d(s) g_0^i\, ds \right)^{-1}, \\
\widetilde{L}_4 &= \alpha_2 \left(\frac{T^{m-1}}{e(m-1)!} \int_0^T (T-s)^{m-1} d(s) g_\infty^s\, ds \right)^{-1}.
\end{aligned}$$

Theorem 3. *Assume that (H1), (H2') and (H3) hold and α_1, $\alpha_2 > 0$ are positive numbers such that $\alpha_1 + \alpha_2 = 1$.*

a) If f_0^i, g_0^i, f_∞^s, $g_\infty^s \in (0,\infty)$, $\widetilde{L}_1 < \widetilde{L}_2$ and $\widetilde{L}_3 < \widetilde{L}_4$, then for each $\lambda \in (\widetilde{L}_1, \widetilde{L}_2)$ and $\mu \in (\widetilde{L}_3, \widetilde{L}_4)$ there exists a positive solution $(u(t), v(t))$, $t \in [0,T]$ for $(S)-(BC)$.

b) If $f_\infty^s = g_\infty^s = 0$, f_0^i, $g_0^i \in (0,\infty)$, then for each $\lambda \in (\widetilde{L}_1, \infty)$ and $\mu \in (\widetilde{L}_3, \infty)$ there exists a positive solution $(u(t), v(t))$, $t \in [0,T]$ for $(S)-(BC)$.

c) If f^s_∞, $g^s_\infty \in (0,\infty)$, $f^i_0 = g^i_0 = \infty$, then for each $\lambda \in (0, \widetilde{L}_2)$ and $\mu \in (0, \widetilde{L}_4)$ there exists a positive solution $(u(t), v(t))$, $t \in [0,T]$ for $(S)-(BC)$.

d) If $f^s_\infty = g^s_\infty = 0$, $f^i_0 = g^i_0 = \infty$, then for each $\lambda \in (0,\infty)$ and $\mu \in (0,\infty)$ there exists a positive solution $(u(t), v(t))$, $t \in [0,T]$ for $(S)-(BC)$.

Sketch of proof. a) Let $\lambda \in (\widetilde{L}_1, \widetilde{L}_2)$ and $\mu \in (\widetilde{L}_3, \widetilde{L}_4)$. We select a positive number ε such that $\varepsilon < f^i_0$, $\varepsilon < g^i_0$ and

$$\alpha_1 \left(\frac{\gamma \xi_{p-2}^{n-1}}{d(n-1)!} \int_{\xi_{p-2}}^T (T-s)^{n-1} c(s)(f^i_0 - \varepsilon)\, ds \right)^{-1} \le \lambda,$$
$$\alpha_1 \left(\frac{T^{n-1}}{d(n-1)!} \int_0^T (T-s)^{n-1} c(s)(f^s_\infty + \varepsilon)\, ds \right)^{-1} \ge \lambda,$$
$$\alpha_2 \left(\frac{\gamma \eta_{q-2}^{m-1}}{e(m-1)!} \int_{\eta_{q-2}}^T (T-s)^{m-1} d(s)(g^i_0 - \varepsilon)\, ds \right)^{-1} \le \mu,$$
$$\alpha_2 \left(\frac{T^{m-1}}{e(m-1)!} \int_0^T (T-s)^{m-1} d(s)(g^s_\infty + \varepsilon)\, ds \right)^{-1} \ge \mu.$$

We also consider the operators defined in the proof of Theorem 2. By the definitions of f^i_0, $g^i_0 \in (0,\infty)$, we deduce that there exists $K_3 > 0$ such that $f(u,v) \ge (f^i_0 - \varepsilon)(u+v)$, $g(u,v) \ge (g^i_0 - \varepsilon)(u+v)$ for all u, $v \ge 0$, with $0 \le u+v \le K_3$.

We denote by $\Omega_3 = \{(u,v) \in Y;\ \|(u,v)\|_Y < K_3\}$. Let $(u,v) \in C$ with $\|(u,v)\|_Y = K_3$, that is $\|u\| + \|v\| = K_3$. Because $u(t)+v(t) \le \|u\| + \|v\| = K_3$ for all $t \in [0,T]$, then by using Lemma 4, we obtain after some computations $P_1(u,v)(\xi_{p-2}) \ge \alpha_1 \|(u,v)\|_Y$. Therefore, $\|P_1(u,v)\| \ge (P_1(u,v))(\xi_{p-2}) \ge \alpha_1 \|(u,v)\|_Y$. In a similar manner, we obtain $\|P_2(u,v)\| \ge (P_2(u,v))(\eta_{q-2}) \ge \alpha_2 \|(u,v)\|_Y$.

Thus for an arbitrary element $(u,v) \in C \cap \partial\Omega_3$ we obtain

$$\|\mathcal{Q}(u,v)\|_Y \ge (\alpha_1 + \alpha_2)\|(u,v)\|_Y = \|(u,v)\|_Y.$$

Now we define the functions f^*, $g^* : \mathbb{R}_+ \to \mathbb{R}_+$, $f^*(x) = \max\limits_{0 \le u+v \le x} f(u,v)$, $g^*(x) = \max\limits_{0 \le u+v \le x} g(u,v)$, $x \in \mathbb{R}_+$. Then $f(u,v) \le f^*(x)$, $g(u,v) \le g^*(x)$ for all (u,v), $u \ge 0$, $v \ge 0$ and $0 \le u+v \le x$. The functions f^*, g^* are nondecreasing and they satisfy the conditions

$$\limsup_{x\to\infty} \frac{f^*(x)}{x} \le f^s_\infty, \quad \limsup_{x\to\infty} \frac{g^*(x)}{x} \le g^s_\infty.$$

Therefore, for $\varepsilon > 0$ there exists $\bar{K}_4 > 0$, such that for all $x \ge \bar{K}_4$, we have

$$\frac{f^*(x)}{x} \le \limsup_{x\to\infty} \frac{f^*(x)}{x} + \varepsilon \le f^s_\infty + \varepsilon, \quad \frac{g^*(x)}{x} \le \limsup_{x\to\infty} \frac{g^*(x)}{x} + \varepsilon \le g^s_\infty + \varepsilon,$$

and so $f^*(x) \le (f^s_\infty + \varepsilon)x$ and $g^*(x) \le (g^s_\infty + \varepsilon)x$.

We now consider $K_4 = \max\{2K_3, \bar{K}_4\}$, and we denote by $\Omega_4 = \{(u,v) \in Y,\ \|(u,v)\|_Y < K_4\}$. Let $(u,v) \in C \cap \partial\Omega_4$. By definitions of f^* and g^* we have

$$f(u(t),v(t)) \le f^*(\|(u,v)\|_Y), \quad g(u(t),v(t)) \le g^*(\|(u,v)\|_Y), \ \forall\, t \in [0,T].$$

Then for all $t \in [0,T]$, after some computations, we obtain $P_1(u,v)(t) \le \alpha_1 \|(u,v)\|_Y$, and so $\|P_1(u,v)\| \le \alpha_1 \|(u,v)\|_Y$. In a similar manner, we obtain $\|P_2(u,v)\| \le \alpha_2 \|(u,v)\|_Y$.

Therefore for $(u,v) \in C \cap \partial\Omega_4$ it follows that

$$\|\mathcal{Q}(u,v)\|_Y \le (\alpha_1+\alpha_2)\|(u,v)\|_Y = \|(u,v)\|_Y.$$

By using Theorem 1 ii) with $T = \mathcal{Q}$, $K = C$, $a = K_3$, $b = K_4$, $K(a,b) = C \cap (\overline{\Omega}_4 \setminus \Omega_3)$, $K_a = C \cap \partial\Omega_3$, $K_b = C \cap \partial\Omega_4$, we deduce that $\mathcal{Q}$ has a fixed point $(u,v) \in C \cap (\bar{\Omega}_4 \setminus \Omega_3)$ such that $K_3 \le \|(u,v)\|_Y \le K_4$.

The proofs of cases b)-d) are similar to that of case a) and we shall omit them (see also the paper [1]. ☐

Remark 2. *The condition $\widetilde{L}_1 < \widetilde{L}_2$ is equivalent to*

$$f_\infty^s T^{n-1} \int_0^T (T-s)^{n-1} c(s)\, ds \le f_0^i \gamma \xi_{p-2}^{n-1} \int_{\xi_{p-2}}^T (T-s)^{n-1} c(s)\, ds$$

and the condition $\widetilde{L}_3 < \widetilde{L}_4$ is equivalent to

$$g_\infty^s T^{m-1} \int_0^T (T-s)^{m-1} d(s)\, ds \le g_0^i \gamma \eta_{q-2}^{m-1} \int_{\eta_{q-2}}^T (T-s)^{m-1} d(s)\, ds$$

4 Examples

Let $T = 1$, $n = 3$, $m = 4$, $p = 5$, $q = 4$, $c(t) = c_0 t$, $d(t) = d_0 t$, for $t \in [0,1]$, with c_0, $d_0 > 0$, $\xi_1 = \frac{1}{4}$, $\xi_2 = \frac{1}{2}$, $\xi_3 = \frac{3}{4}$, $\eta_1 = \frac{1}{3}$, $\eta_2 = \frac{2}{3}$, $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, $b_1 = 1$, $b_2 = 2$. We have $d = \frac{5}{8}$, $e = \frac{10}{27}$, $\theta_0 = \frac{3}{4}$, $\gamma_1 = \frac{1}{16}$, $\gamma_2 = \frac{1}{27}$, $\gamma = \frac{1}{27}$.

We consider the higher-order differential system

$$(S_1) \qquad \begin{cases} u^{(3)}(t) + \lambda c_0 t f(u(t), v(t)) = 0, & t \in (0,1), \\ v^{(4)}(t) + \mu d_0 t g(u(t), v(t)) = 0, & t \in (0,1), \end{cases}$$

with the boundary conditions

$$(BC_1) \qquad \begin{cases} u(0) = u'(0) = 0, \ \ u(1) = u(\frac{1}{4}) + \frac{1}{2}u(\frac{1}{2}) + \frac{1}{3}u(\frac{3}{4}), \\ v(0) = v'(0) = v''(0) = 0, \ \ v(1) = v(\frac{1}{3}) + 2v(\frac{2}{3}) \end{cases}$$

1. First we consider the functions

$$f(u,v) = \frac{(u+v)(p_1 u + 1)(q_1 + \sin v)}{u+1}, \ \ g(u,v) = \frac{(u+v)(p_2 v + 1)(q_2 + \cos u)}{v+1},$$

with p_1, $p_2 > 0$, q_1, $q_2 > 1$.

It follows that $f_0^s = f_0^i = q_1$, $g_0^s = g_0^i = q_2 + 1$, $f_\infty^s = p_1(q_1+1)$, $f_\infty^i = p_1(q_1-1)$, $g_\infty^s = p_2(q_2+1)$, $g_\infty^i = p_2(q_2-1)$.

The constants L_i, $i = \overline{1,4}$ from Section 3 are of the form

$$L_1 = \frac{184320\alpha_1}{13 c_0 p_1 (q_1-1)}, \ \ L_2 = \frac{15\alpha_1}{c_0 q_1}, \ \ L_3 = \frac{259200\alpha_2}{d_0 p_2 (q_2-1)}, \ \ L_4 = \frac{400\alpha_2}{9 d_0 (q_2+1)},$$

and the conditions $L_1 < L_2$ and $L_3 < L_4$ are equivalent to

$$\frac{q_1}{p_1(q_1-1)} < \frac{13}{12288}, \quad \frac{q_2+1}{p_2(q_2-1)} < \frac{1}{5832}.$$

We apply Theorem 2 a) for $\alpha_1, \alpha_2 > 0$ with $\alpha_1 + \alpha_2 = 1$. If the above conditions are satisfied, then for each $\lambda \in (L_1, L_2)$ and $\mu \in (L_3, L_4)$, there exists a positive solution $(u(t), v(t)),\ t \in [0,T]$ for problem $(S_1) - (BC_1)$.

2. We consider the functions

$$f(u,v) = (u+v)^{\beta_1},\ \ g(u,v) = (u+v)^{\beta_2},\ \ u,\, v \in [0,\infty),$$

with $\beta_1, \beta_2 > 1$. Then $f_0^s = f_0^i = g_0^s = g_0^i = 0$ and $f_\infty^s = f_\infty^i = g_\infty^s = g_\infty^i = \infty$. By Theorem 2 d) we deduce that for each $\lambda \in (0,\infty)$ and $\mu \in (0,\infty)$ there exists a positive solution $(u(t), v(t)),\ t \in [0,T]$ for problem $(S_1) - (BC_1)$.

3. We consider the functions

$$f(u,v) = (u+v)^{\gamma_1},\ \ g(u,v) = (u+v)^{\gamma_2},\ \ u,\, v \in [0,\infty),$$

with $\gamma_1, \gamma_2 \in (0,1)$. Then $f_0^s = f_0^i = g_0^s = g_0^i = \infty$ and $f_\infty^s = f_\infty^i = g_\infty^s = g_\infty^i = 0$. By Theorem 3 d) we deduce that for each $\lambda \in (0,\infty)$ and $\mu \in (0,\infty)$ there exists a positive solution $(u(t), v(t)),\ t \in [0,T]$ for problem $(S_1) - (BC_1)$.

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