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A GENERALIZATION OF STRICTLY CONVERGENT POWER SERIES AND APPLICATIONS

In memory of Professor Nicolae Popescu

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Abstract

A representation of strictly convergent power series as Newton interpolating series is given. In the case of one indeterminate bounded Newton interpolating series are studied as a generalization of strictly convergent power series. A method for analytic $p-$adic continuation by means of bounded Newton interpolating series is presented.

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1 Introduction

Let R be a commutative ring with identity and $\mathbf{S} = \{(\alpha_{k,1}, ..., \alpha_{k,n})\}_{k\geq 1}$ a fixed sequence of elements of R^n. In Section 2 we define the R-algebra of Newton interpolating series in n variables denoted by $R_{\mathbf{S}}[[\mathbf{X}]]$. Algebraic properties of $K_{\mathbf{S}}[[\mathbf{X}]]$, when K is a local field are presented in [5].

If R is a commutative ring with identity and $\| \ \|$ is a non-trivial non-archimedean norm on R with $\|1\| = 1$, then $(R, \| \ \|)$ is called a *normed ring.* We consider the sets (see [1], Chapter 1): $\overset{\circ}{R}= \{x \in R : \|x\| \leq 1\}$, $\overset{\vee}{R}= \{x \in R : \|x\| < 1\}$. Then $\overset{\circ}{R}$ is a commutative ring with identity and $\overset{\vee}{R}$ is an ideal in $\overset{\circ}{R}$. We denote the residue ring $\overset{\circ}{R} \ / \ \overset{\vee}{R}$ by $\widetilde{R}$. If R is an integral domain with a non-trivial non-archimedean multiplicative norm, hence an absolute value $| \ |$, then $(R, | \ |)$ is called a *valued ring* . If $(K, | \ |)$ is a valued field and $(R, | \ |)$ is a valued ring which is a K-algebra we suppose that the absolute value of R extends that of K.

Let R be a complete non-archimedean normed ring and $R < X >$ the R-algebra of

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strictly convergent (restricted formal) power series (see [1], p.35, [4] or [8]). Useful generalizations are given in [6] (so-called separated power series) and [2] (strictly analytic functions defined on a class of domains called analoid sets). If $R = \mathbb{C}_p$ endowed with the p-adic absolute value, it is known when a Mahler series may be represented as a strictly convergent power series (see [8], p.354). In Section 3, by means of an arbitrary sequence $\mathbf{S}$ of elements of $\overset{\circ}{R}{}^n$, in the case of n variables, we define $\mathcal{H}R_{\mathbf{S}}[[\mathbf{X}]]$ an R-subalgebra of $R_{\mathbf{S}}[[\mathbf{X}]]$ which is a Banach algebra with respect to the Gauss norm. Theorem 2 from Section 3 shows that the algebra of strictly convergent power series $R < \mathbf{X} >$ and $\mathcal{H}R_{\mathbf{S}}[[\mathbf{X}]]$ are isometrically isomorphic.

In Section 4, K is a complete valued field having its residue field at most countable and T is a fixed set of representatives of the residue field in the valuation ring. By means of T we construct a sequence S_T such that every element of T appears infinitely many times in S_T. In the case $n = 1$, we study the K-subalgebra $BK_{S_T}[[X]]$ of $K_{S_T}[[X]]$ which contains the series having bounded coefficients. By Theorem 2 these series are generalization of strictly convergent power series. With respect to Gauss norm $BK_{S_T}[[X]]$ is a Banach algebra such that $BK[[X]]$, the K-algebra of formal power series with bounded coefficients, is homeomorphic to a residue algebra of $BK_{S_T}[[X]]$ by a closed ideal (see Theorem 4). Moreover for every $f \in BK[[X]]$ there exists a series of $g \in BK_{S_T}[[X]]$ such that the corresponding functions defined on the maximal ideal of the valuation ring are equal (see Corollary 2). Theorem 5 with its corollary deal with properties of zeros of associated functions to the elements of $BK_{S_T}[[X]]$. Theorem 6 is Identity Theorem for the elements of $BK_{S_T}[[X]]$.

It is well known that the analytic continuation in the p-adic analysis cannot be achieved by means of Taylor expansions. By means of Krasner's method it is possible to define analytic elements on the unit open ball for a set of functions defined by bounded power series which satisfy Christol-Robba's condition but there are simple examples of functions which do not belong to this set. If $K = \mathbb{C}_p$, we define in Section 5 so-called Newton analytic elements which extend on the unit ball the usual analytic elements (see [3] or [8]). In this manner we define analytic continuation of bounded power series even in the case when the conditions of Christol-Robba's Theorem do not hold (see Remark 1).

2 Basic notations and definitions

Let n be a fixed positive integer. If $\nu = (\nu_1, \nu_2, ..., \nu_n) \in \mathbb{N}^n$, we set $N(\nu) = \nu_1 + \nu_2 + ... + \nu_n$, for every $i = 1, 2, \ldots, n$, and $\mathbf{0} = (0, ..., 0) \in \mathbb{N}^n$. For $\nu,\ \tau = (\tau_1, \tau_2, ..., \tau_n) \in \mathbb{N}^n$, $j \in \mathbb{N}$, we define $\nu + \tau = (\nu_1 + \tau_1, ..., \nu_n + \tau_n)$ and $j\nu = (j\nu_1, j\nu_2, ..., j\nu_n)$. We set $\nu <_l \tau$ if ν is less than τ with respect to the following lexicographical order: $\nu_s < \tau_s$, where s is the greatest positive integer less than n such that $\nu_s \neq \tau_s$. We order also $\mathbb{N}^n$ in the following way: $\nu <_o \tau$ if either $N(\nu) < N(\tau)$ or $N(\nu) = N(\tau)$ and $\nu <_l \tau$. We denote by ∞^n a symbol such that $\nu <_o \infty^n$ for every $\nu \in \mathbb{N}^n$. It is obvious that for a fixed $\tau \in \mathbb{N}^n$, the set $\{\nu \in \mathbb{N}^n : \ \nu \leq_o \tau\}$ is finite.

Let R be a commutative ring with identity and $\mathbf{S} = \{(\alpha_{k,1}, ..., \alpha_{k,n})\}_{k \geq 1}$ a fixed sequence of elements of R^n. In the polynomial ring $R[\mathbf{X}] = R[X_1, ..., X_n]$ we construct by recurrence, with respect to the defined order $<_o$ of $\mathbb{N}^n$, the polynomials

$$U_{\mathbf{0}} = 1, U_{(1,0,...,0)} = X_1 - \alpha_{1,1}, ..., U_{(0,0,...,1)} = X_n - \alpha_{1,n}$$

and generally for every $\tau = (\tau_1, \tau_2, ..., \tau_n) \in \mathbb{N}^n$,

$$U_\tau = \prod_{0<j\leq\pi_1(\tau)} (X_1 - \alpha_{j,1}) \prod_{0<j\leq\pi_2(\tau)} (X_2 - \alpha_{j,2}) ... \prod_{0<j\leq\pi_n(\tau)} (X_n - \alpha_{j,n}), \tag{1}$$

where $\pi_i(\tau) = \tau_i$. If, for each $\tau \in \mathbb{N}^n$, we consider the principal ideal of $R[\mathbf{X}]$ $\mathcal{I}_\tau =< U_\tau >$, then $\{\mathcal{I}_\tau\}_{\tau\in\mathbb{N}^n}$ is a system of neighborhoods of zero of the polynomial ring. Thus $R[\mathbf{X}]$ becomes a topological Hausdorff space with respect to this topology denoted by $\mathcal{T}_{\mathbf{S}}$. We consider $R_{\mathbf{S}}[[\mathbf{X}]]$ the completion of $R[\mathbf{X}]$ with respect to $\mathcal{T}_{\mathbf{S}}$. It is easy to prove that we can represent $R_{\mathbf{S}}[[\mathbf{X}]]$ as the set of formal series

$$R_{\mathbf{S}}[[\mathbf{X}]] = \left\{ f = \sum_{\tau=\mathbf{0}}^{\infty^n} a_\tau U_\tau \mid a_\tau \in R \right\}, \tag{2}$$

where in each series the order of terms are given by $<_o$, two such expressions being regarded as equal if and only if they have the same coefficients. We call an element f from $R_{\mathbf{S}}[[\mathbf{X}]]$ a (formal) *Newton interpolating series* with coefficients in R defined by the sequence $\mathbf{S}$. If $n = 1$, $R[\mathbf{X}] = R[X]$ and $S = \{\alpha_k\}_{k\geq1}$, then the polynomials u_i defined by (1) can be written in the form

$$u_0 = 1,\ u_i = \prod_{j=1}^{i} (X - \alpha_j),\ i \geq 1. \tag{3}$$

Since, for every nonnegative integer j,

$$X^j = u_j + \sum_{i=1}^{j} q_{i,j} (\alpha_1, ..., \alpha_{j-i+1})\, u_{j-i}, \tag{4}$$

where $q_{i,j}$ are homogeneous polynomials of degree i with integral coefficients (i.e. belonging to the canonical homomorphic image of $\mathbb{Z}$ in R), it follows that every polynomial $P = \sum\limits_{i=0}^{p} b_i X^i \in R[X]$ can be written uniquely in the form

$$P = \sum_{i=0}^{p} a_i u_i, \tag{5}$$

where

$$a_i = b_i + \sum_{j=i+1}^{p} b_j Q_{i,j}(\alpha_1, ..., \alpha_{i+1}), \tag{6}$$

and $Q_{i,j}$ are homogeneous polynomials with integral coefficients. Hence if $u_i,\ u_j$ are given by (3), we obtain that for every k such that $\max\{i, j\} \leq k \leq i + j$, there exist in R the elements $d_k(i,j)$ uniquely defined such that

$$u_i u_j = \sum_{k=\max\{i,j\}}^{i+j} d_k(i,j) u_k. \tag{7}$$

Now we consider $P = \sum\limits_{\nu \leq_o \tau} b_\nu \mathbf{X}^\nu \in R[\mathbf{X}]$. From (5) and (6), by induction on n, it follows that

$$P = \sum_{\nu \leq_o \tau} a_\nu U_\nu, \; a_\nu \in R. \tag{8}$$

If $f, g = \sum\limits_{\nu=\mathbf{0}}^{\infty^n} b_\nu U_\nu \in R_{\mathbf{S}}[[\mathbf{X}]]$, we define addition and multiplication of f and g as follows:

$$f + g = \sum_{\nu=\mathbf{0}}^{\infty^n} (a_\nu + b_\nu) U_\nu, \tag{9}$$

$$fg = \sum_{\nu=\mathbf{0}}^{\infty^n} p_\nu U_\nu, \tag{10}$$

where

$$p_\nu = \sum_{\mu,\theta \in I(\nu)} D_\nu(\mu,\theta) a_\mu b_\theta, \tag{11}$$

$D_\nu(\mu,\theta) = d_{\nu_1}(\mu_1,\theta_1)...d_{\nu_n}(\mu_n,\theta_n)$, $d_i(s,t)$ are defined in (7) and $I(\nu) = \{(\mu,\theta) \in \mathbb{N}^n \times \mathbb{N}^n : \max\{\mu,\theta\} \leq_o \nu, \; \mu + \theta \geq_o \nu\}$. Thus with respect to these definitions of addition and multiplication, $R_{\mathbf{S}}[[\mathbf{X}]]$ becomes a complete Hausdorff topological commutative R-algebra which contains $R[\mathbf{X}]$. Moreover by (1), (9)-(11) it follows that as R-algebras

$$R_{\mathbf{S}^{n-1}}[\mathbf{X}^{(n-1)}]_{S_n}[X_n] \cong R_{\mathbf{S}}[[\mathbf{X}]], \tag{12}$$

where $\mathbf{S}^{n-1} = \{(\alpha_{k,1},...,\alpha_{k,n-1})\}_{k\geq 1}$, $\mathbf{X}^{(n-1)} = (X_1,...,X_{n-1})$ and $S_n = \{\alpha_{k,n}\}_{k \geq 1}$.

3 A representation of strictly convergent power series

Let $(R, \| \ \|)$ be a normed ring and $\mathbf{S} = \{(\alpha_{k,1},...,\alpha_{k,n})\}_{k\geq 1}$ a fixed sequence of elements of $\overset{\circ}{R}{}^n$. We consider

$$\mathcal{H}R_{\mathbf{S}}[[\mathbf{X}]] = \left\{ f = \sum_{\nu=\mathbf{0}}^{\infty^n} a_\nu U_\nu \in R_{\mathbf{S}}[[\mathbf{X}]] : \lim_{N(\nu)\to\infty} \|a_\nu\| = 0 \right\}. \tag{13}$$

If $f = \sum\limits_{\nu=\mathbf{0}}^{\infty^n} a_\nu U_\nu \in \mathcal{H}R_{\mathbf{S}}[[\mathbf{X}]]$, then we define

$$\|f\|_{\mathcal{H}R_{\mathbf{S}}[[\mathbf{X}]]} = \sup_\nu \|a_\nu\|. \tag{14}$$

Theorem 1. *If R is a normed (resp. valued) ring and $\mathbf{S}$ is a fixed sequence of elements of $\overset{\circ}{R}{}^{n}$, then $\mathcal{H}R_{\mathbf{S}}[[\mathbf{X}]]$ is a R-subalgebra of $R_{\mathbf{S}}[[\mathbf{X}]]$ and $\| \ \|$ defined by (14) is a non-archimedean norm (resp. absolute value) on $\mathcal{H}R_{\mathbf{S}}[[\mathbf{X}]]$. Moreover if R is a complete normed (resp. valued) ring, then $\mathcal{H}R_{\mathbf{S}}[[\mathbf{X}]]$ becomes a Banach R-algebra which is the completion of $R[\mathbf{X}]$ with respect to the metric defined by the norm (resp. absolute value).*

Proof. First suppose $n = 1$. Let $f, g = \sum_{i=0}^{\infty} b_i u_i$ be elements of $\mathcal{H}R_S[[X]]$. Then, by (9) and (14), with $n = 1$, we obtain $\|f \pm g\| = \sup_i \{\|a_i \pm b_i\|\} \leq \max\{\|f\|, \|g\|\}$. Similarly, by (7), (10) and (11), since $u_i \in \overset{\circ}{R}[X]$, it follows that $d_k(i,j) \in \overset{\circ}{R}$ and $\|fg\| = \sup_i \|p_i\| \leq \|f\|\|g\|$. If R is a valued ring we choose $i(f)$ the greatest index i such that $|a_i| = |f|$, then by (7) and (11) $\left|p_{i(f)+i(g)}\right| = \left|a_{i(f)}\right| \left|b_{i(g)}\right| = |f||g|$ and $|fg| = |f||g|$. Hence $\mathcal{H}R_S[[X]]$ is a $R-$subalgebra of $R_S[[X]]$ and $\| \ \|$ defined by (14) is a non-archimedean norm (resp. absolute value) on $\mathcal{H}R_S[[X]]$.

When R is complete it follows that $\mathcal{H}R_S[[X]]$ is complete because it is isometrically isomorphic, as an R-module, to $c(R)$, the space of zero sequences over R (see [1], Proposition 6, Sec. 2.1). Now the theorem follows by induction on n by using (12). □

Theorem 2. *If R is a complete normed ring and $S = \{\alpha_k\}_{k\geq 1}$ is a fixed sequence of elements of $\overset{\circ}{R}$, then the Banach $R-$algebra $\mathcal{H}R_S[[X]]$ is isometrically isomorphic to the R-algebra $R < X >$ of strictly convergent power series.*

Proof. If $P = \sum_{i=0}^{p} b_i X^i \in R[X]$, then it can be written also in the form (5), where a_i are given in (6). Similarly we obtain

$$b_i = a_i + \sum_{j=i+1}^{p} a_j T_{i,j}(\alpha_1, ..., \alpha_j), \tag{15}$$

where $T_{i,j}$ are homogeneous polynomial with integral coefficients. Suppose $\|P\|_{\mathcal{H}R_S[[X]]} = \|a_{i_0}\|$, where i_0 is the greatest index with this property. Since $\|T_{i,j}(\alpha_1, ..., \alpha_{i+1})\| \leq 1$, it follows that $\|b_{i_0}\| = \|a_{i_0}\|$ and $\|b_i\| \leq \max_{j\geq i}\{\|a_j\|\}$. Hence $\|P\|_{R<X>} = \|P\|_{\mathcal{H}R_S[[X]]}$.

Now, by means of (6) we define $\phi : R < X > \to \mathcal{H}R_S[[X]]$ such that

$$\phi\left(\sum_{i=0}^{\infty} b_i X^i\right) = \sum_{i=0}^{\infty} a_i u_i, \tag{16}$$

where

$$a_i = b_i + \sum_{j=i+1}^{\infty} b_j Q_{i,j}(\alpha_1, ..., \alpha_{i+1}). \tag{17}$$

Similarly, by using (15), we can define $\psi : \mathcal{H}R_S[[X]] \to R < X >$ such that

$$\psi\left(\sum_{i=0}^{\infty} a_i u_i\right) = \sum_{i=0}^{\infty} b_i X^i, \tag{18}$$

where

$$b_i = a_i + \sum_{j=i+1}^{\infty} a_j T_{i,j}(\alpha_1, ..., \alpha_j). \tag{19}$$

Then the mappings ϕ and ψ are well defined and continuous with respect to the corresponding norms. By (16)-(19) we obtain that the restricted mappings ϕ and ψ are inverse to each other on $R[X]$. Since $R[X]$ is dense in $R < X >$ and $\mathcal{H}R_S[[X]]$ it follows that ϕ and ψ are inverse to each other and hence we obtain that ϕ is bijective map. In fact ϕ is the identity map on $R[X]$ so ϕ is also a $R-$algebra morphism. So we obtain that $R < X >$ and $\mathcal{H}R_S[[X]]$ are isomorphic $R-$algebras. □

Corollary 1. *If K is a complete valued field and $\mathbf{S} = \{(\alpha_{k,1}, ..., \alpha_{k,n})\}_{k\geq 1}$ is a fixed sequence of elements of $\overset{\circ}{K}{}^n$, then the algebra of strictly convergent power series $K < \mathbf{X} >$ is isometrically isomorphic to $\mathcal{H}K_{\mathbf{S}}[[\mathbf{X}]]$.*

Proof. The corollary follows from (12) and Theorem 2. □

4 Bounded Newton interpolating series

In this section K will denote a complete valued field having its residue field at most countable. For $a \in K$ and r a positive real number, we put $B^+(a, r) = \{x \in K : |x - a| \leq r\}$ and $B^-(a, r) = \{x \in K : |x - a| < r\}$. We choose $T = \{\beta_j\}_{j\geq 1}$ a fixed set, at most countable, of elements in $\overset{\circ}{K}$ and we construct a sequence $S_T = \{\alpha_i\}_{i\geq 1}$ of elements of T.

By using (3) we define the K-algebra $K_{S_T}[[X]]$ with

$$u_i = \prod_{j=1}^{i} (X - \alpha_j) = \prod_{j=1}^{m(i)} (X - \beta_j)^{\theta(i,j)}, \tag{20}$$

where $m(i)$ is the number of distinct $X - \beta_j$ which divides $u_i(X)$. We consider

$$BK_{S_T}[[X]] = \left\{ f = \sum_{i=0}^{\infty} a_i u_i \in K_{S_T}[[X]] : \exists\, M > 0, |a_i| < M, \forall\, i \right\}. \tag{21}$$

We call an element f from $BK_{S_T}[[X]]$ a *bounded Newton interpolating series* with coefficients in K defined by the sequence S_T. If $f = \sum\limits_{i=0}^{\infty} a_i u_i \in BK_{S_T}[[X]]$, the real number

$$\|f\|_{BK_{S_T}[[X]]} = \sup_i |a_i| \tag{22}$$

is well defined. As usual we call $\| \, \|_{BK_{S_T}[[X]]}$, given in (22), *the Gauss norm* on $BK_{S_T}[[X]]$. In the case when $T = \{\beta_1\}$, $BK_{S_T}[[X]]$ becomes

$$BK[[X - \beta_1]]$$

$$= \left\{ f = \sum_{i=0}^{\infty} a_i (X - \beta_1)^i \in K[[X - \beta_1]] : \exists\, M > 0, |a_i| < M, \forall\, i \right\}. \tag{23}$$

Theorem 3. *$BK_{S_T}[[X]]$ is a subalgebra of the K-algebra $K_{S_T}[[X]]$ and the Gauss norm is a K- algebra non-archimedean norm on $BK_{S_T}[[X]]$ making it into a Banach K-algebra.*

Proof. Let $f, g = \sum_{i=0}^{\infty} b_i u_i \in BK_{S_T}[[X]]$. By (9) and (22) we obtain $\|f \pm g\|_{BK_{S_T}[[X]]} = \sup_i |a_i \pm b_i| \leq \max\left\{ \|f\|_{BK_{S_T}[[X]]}, \|g\|_{BK_{S_T}[[X]]} \right\}$. Similarly, since $u_i(X) \in \overset{\circ}{K}[X]$, by (6) and (7) it follows that $d_i(s,t) \in \overset{\circ}{K}$ and (10), (11), (22) imply

$$\|fg\|_{BK_{S_T}[[X]]} \leq \sup_i \left\{ \max_{(j,k) \in I(i)} |a_j b_k| \right\} \leq \|f\|_{BK_{S_T}[[X]]} \, \|g\|_{BK_{S_T}[[X]]}. \tag{24}$$

Thus $BK_{S_T}[[X]]$ is a subalgebra of $K_{S_T}[[X]]$ and the Gauss norm is a K-algebra norm on $BK_{S_T}[[X]]$. $BK_{S_T}[[X]]$ is complete because it is isometrically isomorphic as K-vector space to $b(K)$, the space of bounded sequences over K (see [1], Proposition 6, Sec. 2.1). $\square$

Now we choose $T = \{\beta_j\}_{j \geq 1}$ a fixed set of representatives of $\widetilde{K}$ in $\overset{\circ}{K}$ and $S_T = \{\alpha_i\}_{i \geq 1}$ a sequence of elements of T such that every element of T appears infinitely many times in S_T. Similarly with the case of Tate algebra (see [1], Sec. 5.1) we prove for $BK_{S_T}[[X]]$ two results, one on continuity and other on Identity Theorem. If $D \subset K$ is the domain of convergence of the series $f \in BK_{S_T}[[X]]$, then obviously $T \subset D$. We have the following

Lemma 1. *If $T = \{\beta_j\}_{j \geq 1}$ is a fixed set of representatives of $\widetilde{K}$ in $\overset{\circ}{K}$, $S_T = \{\alpha_i\}_{i \geq 1}$ is a sequence of elements of T such that every element of T appears infinitely many times in S_T and $f = \sum_{i=0}^{\infty} a_i u_i \in BK_{S_T}[[X]]$, then*

a) $\overset{\circ}{K} \subset D$;

b) if f converges at $\bar{x} \in K$, then it converges for every $x \in K$ such that $|x| \leq |\bar{x}|$;

c) if $x \in \overset{\circ}{K}$, then $|f(x)| \leq \|f\|_{BK_{S_T}[[X]]}$.

Proof. a) If $x \in \overset{\circ}{K}$, then there is a $\beta_j \in T$ such that $|x - \beta_j| < 1$ and for every $i \neq j$, $|x - \beta_i| = 1$. Since β_j appears infinitely many times in S_T, by (20), $\lim_{i \to \infty} \theta(i,j) = \infty$ which implies $\lim_{i \to \infty} a_i u_i(x) = 0$ and f converges at x.

b) It is enough to consider $|\bar{x}| > 1$. Then for every i, $|\bar{x} - \beta_i| = |\bar{x}|$, $|a_i u_i(x)| \leq |a_i| \max\{1, |x|\}^i \leq |a_i u_i(\bar{x})|$ and this implies b).

c) If $x \in \overset{\circ}{K}$, then $|f(x)| \leq \sup_i |a_i u_i(x)| \leq \sup_i |a_i| = \|f\|_{BK_{S_T}[[X]]}$. $\square$

Proposition 1. *If $T = \{\beta_j\}_{j \geq 1}$ is a fixed set of representatives of $\widetilde{K}$ in $\overset{\circ}{K}$ and $S_T = \{\alpha_i\}_{i \geq 1}$ is a sequence of elements of T such that every element of T appears infinitely many times in S_T, then every $f = \sum_{i=0}^{\infty} a_i u_i \in BK_{S_T}[[X]]$ defines a continuous function* *on D, denoted also by f, such that $y \to f(y) = \sum_{i=0}^{\infty} a_i u_i(y) \in K$. Moreover, if $x_0 \in D$, then there exists $\beta_j \in T$ such that the series $\sum_{i=0}^{\infty} a_i u_i(x)$ converges uniformly to $f(x)$ on $B^+(\beta_j, |x_0 - \beta_j|)$.*

Proof. We may suppose $f \neq 0$. If $y \in \overset{\circ}{K}$, then $\lim_{i\to\infty} a_i u_i(y) = 0$ and the series $\sum_{i=0}^{\infty} a_i u_i(y)$ converges to some element of K.

If $y_0 \in \overset{\circ}{K}$ we consider a real number $\varepsilon > 0$. By putting $\delta = \frac{\varepsilon}{\|f\|}$ we take $y \in \overset{\circ}{K}$ such that $|y - y_0| < \delta$. Hence it follows that

$$|f(y) - f(y_0)| \leq \sup_i |a_i| \, |u_i(y) - u_i(y_0)| \leq \|f\| \sup_i |u_i(y) - u_i(y_0)| .$$

Since $u_i(y) - u_i(y_0) = (y - y_0) w_i(y, y_0)$, where $w_i(y, y_0) \in \overset{\circ}{K}$, we obtain that $|f(y) - f(y_0)| < \varepsilon$ and f gives rise to a continuous function on $\overset{\circ}{K}$.

Now, we suppose $y_0 \in D$, $|y_0| > 1$ and we choose a real number $\varepsilon > 0$. We take $y \in D$ such that $|y - y_0| < 1$. Hence it follows that $|a_i u_i(y)| = |a_i y^i| = |a_i y_0^i| = |a_i u_i(y_0)|$. Thus we can choose i_0 such that for every $y \in B^-(y_0, 1)$ $|f(y) - S_{i_0}(y)| < \varepsilon$, where S_i is the partial sum of the series f. Since $S_{i_0}(y)$ is a continuous function there is $\delta < 1$ such that for every $y \in B^-(y_0, \delta)$, $|S_{i_0}(y) - S_{i_0}(y_0)| < \varepsilon$. Then

$$|f(y) - f(y_0)| \leq \max\{|f(y) - S_{i_0}(y)|, |S_{i_0}(y) - S_{i_0}(y_0)|, |S_{i_0}(y_0) - f(y_0)|\} < \varepsilon$$

and f gives rise to a continuous function on D.

Suppose $x_0 \in D$. If $x_0 \in \overset{\circ}{K}$, we choose $\beta_j \in T$ such that $|x_0 - \beta_j| < 1$. Then for every $x \in B^+(\beta_j, |x_0 - \beta_j|)$ and $k \neq j$, $|x - \beta_k| = 1$. Hence $|a_i u_i(x)| \leq |a_i u_i(x_0)|$ and the series converges uniformly on $B^+(\beta_j, |x_0 - \beta_j|)$.

If $|x_0| > 1$, then for every $\beta_j \in T$, $|x_0 - \beta_j| = |x_0|$. Thus for every $x \in B^+(\beta_j, |x_0 - \beta_j|) = B^+(0, |x_0|)$, $|a_i u_i(x)| \leq |a_i u_i(x_0)|$, which implies the proposition. $\square$

Theorem 4. *Let $T = \{\beta_j\}_{j\geq 1}$ be a fixed set of representatives of $\widetilde{K}$ in $\overset{\circ}{K}$ and let $S_T = \{\alpha_k\}_{k\geq 1}$ be a sequence of elements of T. If there exists $\beta_k \in T$ which appears infinitely many times in S_T, then there exists a K-algebra homomorphism $\varphi : BK_{S_T}[[X]] \to BK[[X - \beta_k]]$ such that:*

a) φ is a continuous K-algebra homomorphism from $BK_{S_T}[[X]]$ onto $BK[[X - \beta_k]]$;

b) for every $g \in BK_{S_T}[[X]]$ and $x \in B^-(\beta_k, 1)$, $g(x) = \varphi(g)(x)$;

c) the induced isomorphism $\bar{\varphi} : BK_{S_T}[[X]]/Ker\varphi \to BK[[X]]$ is a homeomorphism, where $BK_{S_T}[[X]]/Ker\varphi$ is provided with the quotient topology.

Proof. a) Consider $g = \sum_{i=0}^{\infty} a_i u_i \in BK_{S_T}[[X]]$, g_n its nth partial sum and $\mu(i,k) = \max\{t : \theta(t,k) \leq i\}$. Since, for every $j \neq k$, $X - \beta_j = X - \beta_k + \beta_k - \beta_j$, by (20) it follows that, for every $n \geq \mu(i,k)$, the coefficient of $(X - \beta_k)^i$ in the polynomial g_n written as an element from $BK[[X-\beta_k]]$ has the form

$$c_{i,k} = \sum_{j=i}^{\mu(i,k)} P_{i,j,k} a_j, \tag{25}$$

where $P_{i,j,k}$ are polynomials with integral coefficients in β_j, and

$$P_{i,\mu(i,k),k} = \prod_{j=1,\ j\neq k}^{m(i)} (\beta_k - \beta_j)^{\theta(i,j)}. \tag{26}$$

Then $\sum\limits_{i=0}^{\infty} c_{i,k}(X-\beta_k)^i \in BK[[X-\beta_k]]$ and we define

$$\varphi(g) = \sum_{i=0}^{\infty} c_{i,k}(X-\beta_k)^i. \tag{27}$$

Since $K[X]$ is dense in $BK_S[[X]]$, for every S, (resp. $BK[[X-\beta_k]]$) with respect to the topology $\mathcal{T}_S$ defined by the principal ideals $< u_i >$ (resp. the corresponding topology $\mathcal{T}$ defined by the powers of $X-\beta_k$), φ is continuous with respect to $\mathcal{T}_{S_T}$ and $\mathcal{T}$ and its restriction to $K[X]$ is the identity map, it follows that it is a K-algebra homomorphism. Moreover, because (25) implies that

$$\|\varphi(g)\|_{KB[[X-\beta_k]]} \leq \|g\|_{BK_S[[X]]}, \tag{28}$$

it follows that φ is continuous.

If $f = \sum\limits_{i=0}^{\infty} b_i(X-\beta_k)^i \in BK[[X-\beta_k]]$, then choose $g = \sum\limits_{i=0}^{\infty} a_i u_i \in K_{S_T}[[X]]$ such that $a_0 = b_0$ and generally by recurrence, for $i \geq 1$,

$$a_i = \begin{cases} 0, \text{ if } i \neq \mu(j,k) \text{ for every } j \\ \dfrac{b_j - \sum\limits_{s=j}^{\mu(j,k)-1} P_{j,s,k} a_s}{P_{j,\mu(j,k),k}}, \text{ if } i = \mu(j,k) \end{cases}, \tag{29}$$

where the polynomials $P_{i,j,k}$ defined in (25) are independent of the coefficients a_j. Since $|\beta_i| \leq 1$, $g \in BK_{S_T}[[X]]$. If, for every $i \geq 1$, a_i is given in (29), by (25) we obtain $c_{i,k} = b_i$ which implies $\varphi(g) = f$.

b) Since we may choose $g \in BK_{S_T}[[X]]$ such that $\varphi(g) = f$, by (28), we obtain, for every $x \in B^-(\beta_k, 1)$,

$$|f(x) - g_n(x)| \leq \|f - g_n\|_{BK[[X-\beta_k]]} = \|\varphi(g-g_n)\|_{BK[[X-\beta_k]]} \leq \|g - g_n\|_{BK_{S_T}[[X]]}.$$

Hence it follows b).

c) Since φ is continuous and by (29) for every $f \in BK[[X-\beta_k]]$ we may choose $g \in BK_{S_T}[[X]]$ such that $\|g\|_{BK_{S_T}[[X]]} \leq \|f\|_{BK[[X-\beta_k]]}$ by Lemma 2 from [1], p. 21 φ is strict which implies the statement. $\square$

By Theorem 4 we obtain easily the following result.

Corollary 2. *If $f \in BK[[X]]$, $T = \{\beta_j\}_{j\geq 1}$, $0 \in T$, is a set of representatives of $\widetilde{K}$ in $\overset{\circ}{K}$ and $S_T = \{\alpha_i\}_{i\geq 1}$ a sequence of elements of T such that every element of T appears infinitely many times in S_T, then there exists $g \in BK_{S_T}[[X]]$ such that $f(x) = g(x)$, for every $x \in B^-(0,1)$.*

For $f \in BK_{S_T}[[X]]$ denote $Z(f) = \{a \in\overset{\circ}{K} |\ f(a) = 0\}$ the set of all zeros of f in $\overset{\circ}{K}$ without counting the multiplicities.

Theorem 5. *If $T = \{\beta_j\}_{j\geq 1}$ is a fixed set of representatives of $\widetilde{K}$ in $\overset{\circ}{K}$, $S_T = \{\alpha_i\}_{i\geq 1}$ is a sequence of elements of T such that every element of T appears infinitely many times in S_T, $f \in BK_{S_T}[[X]]$ and for a fix j, β_j is an accumulation point of $Z(f)$, then $B^-(\beta_j, 1) \subset Z(f)$.*

Proof. Suppose first $f = \sum\limits_{i=0}^{\infty} a_i u_i \in BK_{S_T}[[X]]$ and $\beta_j = 0$. Now by Theorem 4 we have a morphism $\varphi : BK_{S_T}[[X]] \to BK[[X]]$ such that $\varphi(f) = g \in BK[[X]]$ and $f(x) = g(x)$, for every $x \in B^-(0,1)$. We choose a sequence $\{\gamma_k\}_{k\geq 1}$ of distinct elements of $Z(f) \cap B^-(0,1)$ such that $\lim\limits_{k\to\infty} \gamma_k = 0$. Then $f(\gamma_k) = 0 = g(\gamma_k)$, for all k. We show that $g = 0$ in $BK[[X]]$. If $g \neq 0$ and b_t is the first nonzero coefficient of g then $g = X^t \sum\limits_{i=0}^{\infty} b_{t+i} X^i$. Now for k large enough $\left|\sum\limits_{i=0}^{\infty} b_{t+i}\gamma_k^i\right| = |b_t|$. But $g(\gamma_k) = 0$ implies $b_t = 0$. Hence $g = 0$ and $f(\gamma) = g(\gamma) = 0$ for all $\gamma \in B^-(0,1)$. The case $\beta_j \neq 0$ can be reduced easily to the previous case by replacing X with $X + \beta_j$. □

Corollary 3. *If $T = \{\beta_j\}_{j\geq 1}$, is a fixed set of representatives of $\widetilde{K}$ in $\overset{\circ}{K}$ and $S_T = \{\alpha_i\}_{i\geq 1}$ a sequence of elements of T such that every element of T appears infinitely many times in S_T, $f \in BK_{S_T}[[X]]$ and for a fix j, there exists an element $\xi_j \in B^-(\beta_j, 1)$, which is an accumulation point $Z(f)$, then $B^-(\beta_j, 1) \subset Z(f)$.*

Proof. It is enough to replace X with $X + \beta_j - \xi_j$ and to use Corollary 2 and Theorem 5 □

Now we can prove Identity Theorem for elements of $BK_{S_T}[[X]]$.

Theorem 6. *If $T = \{\beta_j\}_{j\geq 1}$, is a fixed set of representatives of $\widetilde{K}$ in $\overset{\circ}{K}$ and $S_T = \{\alpha_i\}_{i\geq 1}$ a sequence of elements of T such that every element of T appears infinitely many times in S_T, $f \in BK_{S_T}[[X]]$ and for every j, there exists an element $\xi_j \in B^-(\beta_j, 1)$, which is an accumulation point $Z(f)$, then $f = 0$.*

Proof. Because $\overset{\circ}{K}= \bigcup\limits_{\beta_j\in T} B^-(\beta_j, 1)$, by Corollary 3 it follows that $f(x) = 0$ for every $x \in\overset{\circ}{K}$. Suppose that $f = \sum\limits_{i=t}^{\infty} a_i u_i \neq 0$ and $a_t \neq 0$. If $u_{t+1}(X)/u_t(X) = X - \beta_j$ we choose a sequence $\{\gamma_k\}_{k\geq 1}$ of distinct elements of $Z(f)$ which tends to β_j. Since $f(\gamma_k) = 0$, for all k it follows that a_t =0 which implies that $f = 0$. □

Now we fixed $T = \{\beta_j\}_{j\geq 1}$ a set of representatives of $\widetilde{K}$ in $\overset{\circ}{K}$ and we construct a particular sequence $S_T = \{\alpha_i\}_{i\geq 1}$ of elements of T, such that every element of T appears infinitely many times in S_T. Thus for every positive integer i there is a unique integer $k(i)$ such that

$$\frac{(k(i)-1)k(i)}{2} < i \leq \frac{k(i)(k(i)+1)}{2} \tag{30}$$

and we put

$$s(i) = i - \frac{(k(i)-1)k(i)}{2}. \tag{31}$$

Then we take

$$\alpha_i = \begin{cases} \beta_{r(i-1)+1}, & \text{if } \widetilde{K} \text{ has } q \text{ elements} \\ \beta_{s(i)}, & \text{if } \widetilde{K} \text{ is countable,} \end{cases} \tag{32}$$

where $r(i)$ is the remainder obtained by dividing i into q. In this case we say that the pair (T, S_T) has the *standard form*.

5 Newton analytic elements

Let D be a closed subset of $\mathbb{C}_p$. The Runge theorem of complex analysis leaded Krasner to call an *analytic element* a function $f : D \to \mathbb{C}_p$ which is a uniform limit of a sequence of rational functions having no pole in D. By a result of Christol-Robba (see Theorem of Sec. 4.6 of [8]) it is known which series of $B\mathbb{C}_p[[X]]$ define analytic elements. There are simple examples of series of $B\mathbb{C}_p[[X]]$ which do not define analytic elements on $B^-(0,1)$ (see [8], p. 353).

Now we built Newton analytic elements on $B^+(0,1)$ and $B^-(0,1)$. Consider $K = \mathbb{C}_p$, a pair (T, S_T) having the standard form $D = B^+(0,1)$ and a function $f : D \to K$. We call f a *Newton analytic element* if it is the sum of a series of $B\mathbb{C}_{p_{S_T}}[[X]]$ on D. By Corollary 1 and Theorem, Sec. 4.3, Ch. 6 of [8], it follows that the Banach algebra of analytic elements on $B^+(0,1)$ is isomorphic to a subalgebra of $B\mathbb{C}_{p_{S_T}}[[X]]$. In order to define Newton analytic elements on $B^-(0,1)$ we suppose that the pair (T, S_T) has the standard form and $\beta_1 = 0$. We take $T_s \subset T$, $T_s \neq T$, such that $0 \in T_s$ and $S_{T_s^c}$ the sequence obtained from S_T by canceling all the terms equal to $\beta_j \in T_s$. We denote by v_i the corresponding polynomials defined by (3) by means of $S_{T_s^c}$.

In $BK_{S_T}[[X]]$ we denote by M the multiplicative system generated by the polynomials $X - \beta_i$, $\beta_i \in T_s^c$ and by $M^{-1}BK_{S_T}[[X]]$ the ring of fractions of $BK_{S_T}[[X]]$ with respect to M. By using an idea for power series (see [7]), we define

$$\mathcal{H}N\mathbb{C}_{p_{S_{T_s^c}}}[[X]] = \{F = \sum_{i=-\infty}^{-1} a_i v_{-i}^{-1} + f\}, \tag{33}$$

where $a_i \in \mathbb{C}_p$, $\lim\limits_{i\to-\infty} a_i = 0$ and $f \in B\mathbb{C}_{p_{S_T}}[[X]]$. If $F \in \mathcal{H}N\mathbb{C}_{p_{S_{T_s^c}}}[[X]]$ we put

$$\|F\|_{\mathcal{H}N\mathbb{C}_{p_{S_{T_s^c}}}[[X]]} = \max\{\max_{-\infty<i\leq -1}\{|a_i|\}, \|f\|_{B\mathbb{C}_{p_{S_T}}[[X]]}\}. \tag{34}$$

It can be proved that $\mathcal{H}N\mathbb{C}_{p_{S_{T_s^c}}}[[X]]$ is the completion of the algebra $M^{-1}B\mathbb{C}_{p_{S_T}}[[X]]$ with respect to the restriction of the norm given by (34).

Consider $K = \mathbb{C}_p$, (T, S_T) a pair having the standard form with $\beta_1 = 0$, $T_s = 0$, $D = B^-(0,1)$ and a function $f : D \to \mathbb{C}_p$. We call f a *Newton analytic element* if it is the sum of a series of $\mathcal{H}N\mathbb{C}_{p_{S_{T_s^c}}}[[X]]$.

Remark 1. Let (T, S_T) be a pair having the standard form with $\beta_1 = 0$. If we denote the set of all Newton analytic elements on $B^+(0,1)$ as $\mathcal{H}N(B^+(0,1)) = B\mathbb{C}_{p_{S_T}}[[X]]$ and the set of all Newton analytic elements on $B^-(0,1)$ as $\mathcal{H}N(B^-(0,1)) = \mathcal{H}N\mathbb{C}_{p_{S_{T_s^c}}}[[X]]$, with $T_s = \{0\}$, then as in the classical case that Banach K-algebra $\mathcal{H}N(B^-(0,1))$ is isomorphic to a completion of a ring of fractions of the algebra $\mathcal{H}N(B^+(0,1))$.

By Corollary 2 and Lemma 1 it follows that every $g \in B\mathbb{C}_p[[X]]$ defines a Newton analytic element on $B^-(0,1)$ which can be extended to a Newton analytic element on $B^+(0,1)$. Hence Theorem 6 implies that for a fixed family of sequences $Z_j = \{\gamma_{j,n}\}_{n\geq 1}$, $j \geq 2$ such that $\gamma_{j,n} \in B^-(\beta_j, 1)$ and each Z_j has an accumulation point, every $g \in B\mathbb{C}_{p_{S_T}}[[X]]$ can be extended to $G \in \mathcal{H}N(B^+(0,1))$ uniquely defined by its values at $\gamma_{j,n}$.

References

[1] Bosch, S., Güntzer, U. and Remert, R., *Non-Archimedean Analysis*, Springer-Verlag, Berlin, 1984.

[2] Boussaf, K., *Strictly analytic functions on p-adic analytic open sets*, Publ. Mat., Barc. **43**, (1999). 127-162.

[3] Escassut, A., *Analytic elements in p-adic analysis*, World Scientific Publishing, Singapore, 1995.

[4] Fresnel, J. and van der Put, M., *Géométrie Analytique Rigide et Applications,* Birkhäuser, Boston, 1981.

[5] Groza, G. and Haider, A., *On the ring of Newton interpolating series over a local field*, Math. Rep., Bucur. **9**(**59**) (2007), no. 4, 343-356.

[6] Lipshitz, L., Robinson, Z., *Rings of separated power series and quasi-affinoid geometry* Astérisque 264, SMF 2000.

[7] Parshin, A. N., *Abelian covering of arithemetic schemas* (in Russian), Doklad SSSR **243** (1978), no.4.

[8] Robert, A., *A course in p-adic analysis*, Springer-Verlag, New York, 2000.