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ON SOME ISOSTROPHY INVARIANTS OF BOL LOOPS

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Abstract

A loop $(Q,\cdot)$ which satisfies the identity $x(yz\setminus x)=(x/z)(y\setminus x)$ is called a middle Bol loop. It is known that a loop $(Q,\cdot)$ is middle Bol if all its loop isotopes satisfy the anti-automorphic inverse property $(xy)^{-1}=y^{-1}x^{-1}$. A. Gwaramija proved that middle Bol loops are isostrophs of left (right) Bol loops. Invariant properties under this isostrophy are studied in the present work. It is shown that two middle Bol loops are isotopic (isomorphic) if and only if the corresponding right (left) Bol loops are isotopic (isomorphic). So, the isotopy-isomorphism property is invariant. Also, it is proved that the group of autotopisms (left pseudo-automorphisms) of a middle Bol loop is isomorphic to the group of autotopisms (left (right) pseudo-automorphisms) of the corresponding right (left) Bol loop, the group of automorphisms is invariant and that every congruence (normal congruence, normal subloop) of a right Bol loop is a congruence (normal congruence, normal subloop) of the corresponding middle Bol loop.

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1 Indroduction

A loop $(Q,\cdot)$ is called middle Bol if it satisfies the identity

$$x(yz\setminus x)=(x/z)(y\setminus x), \tag{1.1}$$

where " $\setminus$ "("/") is the right (left) division in $(Q,\cdot)$. It is known that the identity (1.1) is invariant under loop isotopy, i.e. is universal, and that the universality of (1.1) implies the power-associativity of the middle Bol loops. More, the identity (1.1) is a necessary and sufficient condition for the universality of the anti-automorphic inverse property $(x\cdot y)^{-1}=y^{-1}\cdot x^{-1}$ ([1]).

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A. Gwaramija proved in [6] that a loop $(Q, \circ)$ is middle Bol if and only if there exists a right Bol loop $Q(\cdot)$ such that

$$x \circ y = (y \cdot xy^{-1})y, \tag{1.2}$$

for every $x, y \in Q$. It follows from (1.2) that $x \circ y^{-1} = (y^{-1} \cdot xy)y^{-1}$ so, replacing x by $y \cdot x$ and using the right Bol identity, have $(y \cdot x) \circ y^{-1} = [y^{-1} \cdot (yx \cdot y)] \cdot y^{-1} = xy \cdot y^{-1} = x, \forall x, y \in Q$, i.e. $y \cdot x = x//y^{-1}$, so

$$x \cdot y = y//x^{-1}, \tag{1.3}$$

$\forall x, y \in Q$, where "//" is the left division in $(Q, \circ)$. On the other hand, denoting $x \cdot y$ by z, from (1.3) follows: $z = (x \setminus z)//x^{-1} \Leftrightarrow z \circ x^{-1} = x \setminus z$, hence

$$z \circ x = x^{-1} \setminus z, \tag{1.4}$$

$\forall z, x \in Q$. So, the loops $(Q, \cdot)$ and $(Q, \circ)$ are isostrophic.

Remark 1.1. Also, it is shown in [6] that a loop $(Q, \circ)$ is middle Bol if and only if there exists a left Bol loop $(Q, \cdot)$ such that

$$x \circ y = y(y^{-1}x \cdot y), \tag{1.5}$$

$\forall x, y \in Q$, which is equivalent to $x \circ y^{-1} = y^{-1}(yx \cdot y^{-1})$. Replacing x by $x \cdot y$ in the last identity and using the left Bol identity, get: $(x \cdot y) \circ y^{-1} = y^{-1}[(y \cdot xy) \cdot y^{-1}] = y^{-1} \cdot yx = x$, which implies

$$x \cdot y = x//y^{-1}, \tag{1.6}$$

where "//" is the left division in $Q(\circ)$. Denoting $x \cdot y = z$ in (1.6), have: $z = (z/y)//y^{-1}$ so $z \circ y^{-1} = z/y$, i.e.

$$z \circ y = z/y^{-1}, \tag{1.7}$$

for every $z, y \in Q$.

Remark 1.2. If $(Q, \circ)$ is a middle Bol loop and $(Q, \cdot)$ is the corresponding left Bol loop, then $(Q, *)$, where $x * y = y \cdot x$, for every $x, y \in Q$, is the corresponding right Bol loop for $(Q, \circ)$. Indeed, so as $(Q, \cdot)$ is a left Bol loop, $(Q, *)$ is a right Bol loop and

$$x \circ y = y(y^{-1}x \cdot y) = [y * (x * y^{-1})] * y,$$

for every $x, y \in Q$.

Remark 1.3. Remind that a bijection $\varphi : Q \to Q$ is called a semi-automorphism of a loop $(Q, \cdot)$ if $\varphi(xy \cdot x) = (\varphi(x)\varphi(y)) \cdot \varphi(x)$, $\forall x, y \in Q$, and $\varphi(e) = e$, where e is the unit of $(Q, \cdot)$ [1, 3]. If $(Q, \circ)$ is a middle Bol loop and $(Q, \cdot)$ is the corresponding right Bol loop then the inversion $I : Q \to Q, I(x) = x^{-1}$, is a semi-automorphism of

$(Q, \cdot)$ ([4]) and, as was mentioned above, I is an anti-automorphism of $(Q, \circ)$, hence $y^{-1} \circ x^{-1} = (x \circ y)^{-1} = [(y \cdot xy^{-1})y]^{-1} = (y^{-1} \cdot (xy^{-1})^{-1}) \cdot y^{-1}$ which implies

$$x \circ y = (x \cdot (y^{-1} \cdot x)^{-1}) \cdot x, \tag{1.8}$$

for every $x, y \in Q$.

2 Main results

Proposition 2.1. *Two middle Bol loops are isotopic if and only if the corresponding right (left) Bol loops are isotopic.*

Proof. Let $(Q, \circ)$ and $(Q, \otimes)$ be isotopic middle Bol loops, $(\otimes) = (\circ)^{(\alpha,\beta,\gamma)}$. Denote by $(Q, \cdot)$ (resp. $(Q, *)$) the corresponding right Bol loop for $(Q, \circ)$ (resp. $(Q, \otimes)$). Then $x \circ y = I(y) \setminus x$ and $x \otimes y = I_1(y) \setminus\setminus x$, where " $\setminus$ "(" $\setminus\setminus$") is the right division in $(Q, \cdot)$ (resp. in $(Q, *)$), $I(y)$ (resp. $I_1(y)$) is the inverse of y in $(Q, \cdot)$ and $(Q, \circ)$ (resp. in $(Q, *)$ and $(Q, \otimes)$). So, we have:

$$\gamma(x \otimes y) = \alpha(x) \circ \beta(y) \Leftrightarrow \gamma(I_1(y) \setminus\setminus x) = I\beta(y) \setminus \alpha(x) \Leftrightarrow$$
$$\alpha(x) = I\beta(y) \cdot \gamma(I_1(y) \setminus\setminus x),$$

for every $x, y \in Q$. Denoting $I_1(y) \setminus\setminus x = z$, the last equality take the form: $\alpha(I_1(y) * z) = I\beta(y) \cdot \gamma(z)$, i.e. $\alpha(y * z) = I\beta I_1(y) \cdot \gamma(z)$, for every $x, y \in Q$, so $(*) = (\cdot)^{(I\beta I_1,\gamma,\alpha)}$.

The proof is analogous when $(Q, \cdot)$ (resp. $(Q, *)$) is the corresponding left Bol loop for $(Q, \circ)$ (resp. $(Q, \otimes)$). □

Corollary 2.1. *([9]) Two middle Bol loops are isomorphic if and only if the corresponding right (left) Bol loops are isomorphic.*

Remind that a loop $(Q, \cdot)$ is called a G-loop if every loop isotope of $(Q, \cdot)$ is isomorphic to $(Q, \cdot)$. At present, the characterization of G-loops is an open problem in the theory of loops [1, 2, 3].

Corollary 2.2. *Middle Bol G-loops corresponds to the right (left) Bol G-loops.*

Proof. Let $(Q, \cdot)$ be a right Bol G-loop and let $(Q, \circ)$ be the corresponding middle Bol loop. If $(Q, \otimes)$ is a loop isotope of $(Q, \circ)$ then $(Q, \otimes)$ is middle Bol. Denote by $(Q, *)$ the corresponding right Bol loop for $(Q, \otimes)$. So as $(Q, \circ)$ and $(Q, \otimes)$ are isotopic, follows that $(Q, *)$ is an isotope of $(Q, \cdot)$, so is isomorphic to $(Q, \cdot)$. Finally, so as $(Q, *) \cong (Q, \cdot)$ we get that $(Q, \otimes) \cong (Q, \circ)$, hence $(Q, \circ)$ is a G-loop. □

D.A. Robinson published in [5] an example of a right Bol G-loop which is not Moufang, so there exist middle Bol G-loops which are not Moufang. Using (1.3), D.A.Robinson's example gives a middle Bol G-loop $(Q, \circ)$ which is not Moufang, where $Q = R^5$, R is an alternative non-associative division ring of characteristic $\neq 2$, and for $x, y \in Q$, $x = (a_1, a_2, a_3, a_4, a_5), y = (b_1, b_2, b_3, b_4, b_5)$, the operation " $\circ$ " is defined as follows:

$$x \circ y = (a_1, a_2, a_3, a_4, a_5) \circ (b_1, b_2, b_3, b_4, b_5) =$$
$$(a_1 + b_1, a_2 + b_2, a_3 + b_3 + b_1 a_2, a_4 + b_4 + b_2 a_2, a_5 + b_5 + b_1 a_4 + b_3 a_2).$$

Let $(Q, \cdot)$ be a quasigroup. A bijection $\tau : Q \to Q$ is called a left (right) pseudo-automorphism, with the companion $c \in Q$, if there exists an element $c \in Q$ such that $c \cdot \tau(xy) = (c \cdot \tau(x)) \cdot \tau(y)$ (resp. $\tau(xy) \cdot c = \tau(x) \cdot (\tau(y) \cdot c)$), $\forall x, y \in Q$ ([1, 3]).

Proposition 2.2. *If $(Q, \circ)$ is a middle Bol loop and $(Q, \cdot)$ is the corresponding right (left) Bol loop, then the following sentences are true:*

1. $T = (\alpha, \beta, \gamma)$ is an autotopism of $(Q, \cdot)$ if and only if $T_1 = (\gamma, I\beta I, \alpha)$, (resp. $T_1 = (\gamma, I\alpha I, \beta)$), where $I(x) = x^{-1}, \forall x \in Q$, is an autotopism of $(Q, \circ)$.

2. A bijection $\tau \in S_Q$ is a left (right) pseudo-automorphism, with the companion c, of a right (left) Bol loop $(Q, \cdot)$ if and only if $I\tau I$ is a left pseudo-automorphism, with the companion c, of the corresponding middle Bol loop;

3. $Aut(Q, \cdot) = Aut(Q, \circ)$.

Proof. 1. Using (1.3) and denoting by "/" (resp. "//") the left division in $(Q, \cdot)$ (resp. $(Q, \circ)$) we get that $T = (\alpha, \beta, \gamma)$ is an autotopism of $(Q, \cdot)$ if and only if

$$\gamma(x \cdot y) = \alpha(x) \cdot \beta(y) \Leftrightarrow \gamma(y//x^{-1}) = \beta(y)//I\alpha(x) \Leftrightarrow \beta(y) = \gamma(y//I(x)) \circ I\alpha(x),$$

for every $x, y \in Q$. Taking $y/I(x) = z$ in the last equality we get:

$$\beta(z \circ I(x)) = \gamma(z) \circ I\alpha(x) \Leftrightarrow \beta(z \circ x) = \gamma(z) \circ I\alpha I(x),$$

for every $x, z \in Q$, i.e. $T_1 = (\gamma, I\alpha I, \beta)$ is an autotopism of $(Q, \circ)$.

The proof is analogous when $(Q, \cdot)$ is the corresponding left Bol loop.

2. A bijection $\tau : Q \to Q$ is a left pseudo-automorphism, with the companion c, of the right Bol loop $(Q, \cdot)$ if and only if $T = (L_c^{(\cdot)}\tau, \tau, L_c^{(\cdot)}\tau)$ is an autotopism of $(Q, \cdot)$, where $L_c^{(\cdot)}$ is the left translation with c in $(Q, \cdot)$. So, according to p.1, τ is a left pseudo-automorphism, with a companion c, of the right Bol loop $(Q, \cdot)$ if and only if $T_1 = (L_c^{(\cdot)}\tau, I\tau I, L_c^{(\cdot)}\tau)$ is an autotopism of $(Q, \circ)$. On the other hand, $T_1 = (L_c^{(\cdot)}\tau, I\tau I, L_c^{(\cdot)}\tau)$ is an autotopism of $(Q, \circ)$ if and only if $L_c^{(\cdot)}\tau(x \circ y) = L_c^{(\cdot)}\tau)(x) \circ I\tau I(y)$, $\forall x, y \in Q$, which (for $x = e$, where e is the unit of the loop) implies: $L_c^{(\cdot)}\tau(y) = L_c^{(\circ)}I\tau I(y)$, $\forall y \in Q$, where $L_c^{(\circ)}$ is the left translation with c in $(Q, \circ)$, so $T_1 = (L_c^{(\circ)}I\tau I, I\tau I, L_c^{(\circ)}I\tau I)$ is an autotopism of $(Q, \circ)$, i.e. $I\tau I$ is a left pseudo-automorphism of $(Q, \circ)$, with the companion c.

Analogously, if $(Q, \cdot)$ is a left Bol loop and τ is a right pseudo-automorphism of $(Q, \cdot)$, with the companion c, then $T = (\tau, R_c^{(\cdot)}\tau, R_c^{(\cdot)}\tau)$ is an autotopism of $(Q, \cdot)$, so $T_1 = (R_c^{(\cdot)}\tau, I\tau I, R_c^{(\cdot)}\tau)$ is an autotopism of $(Q, \circ)$, i.e. $R_c^{(\cdot)}\tau(x \circ y) = I\tau I(x) \circ R_c^{(\cdot)}\tau(y)$, $\forall x, y \in Q$. Now, taking $x = e$ in the last equality, get $R_c^{(\cdot)}\tau(y) = L_c^{(\circ)}I\tau I(y)$, $\forall y \in Q$, hence $T_1 = (L_c^{(\circ)}I\tau I, I\tau I, L_c^{(\circ)}I\tau I)$ is an autotopism of $(Q, \circ)$, i.e. $I\tau I$ is a left pseudo-automorphism of $(Q, \circ)$, with the companion c.

3. Let $(Q, \cdot)$ be a right Bol loop. If $\varphi \in Aut(Q, \cdot)$ then $\varphi(e) = e$ and $e = \varphi(e) = \varphi(x \cdot x^{-1}) \Rightarrow \varphi(x^{-1}) = \varphi(x)^{-1}$, $\forall x \in Q$. So, $\varphi(x \circ y) = \varphi[(y \cdot xy^{-1})y] = (\varphi(y) \cdot \varphi(x)\varphi(y^{-1})) \cdot$

$\varphi(y) = (\varphi(y) \cdot \varphi(x)\varphi(y)^{-1}) \cdot \varphi(y) = \varphi(x) \circ \varphi(y)$, $\forall x, y \in Q$, hence $\varphi \in Aut(Q, \circ)$. On the other hand, if $\varphi \in Aut(Q, \circ)$, then

$$\varphi(x \circ y) = \varphi(x) \circ \varphi(y) \Leftrightarrow \varphi(y^{-1} \setminus x) = \varphi(y)^{-1} \setminus \varphi(x) \Leftrightarrow$$
$$\varphi(y^{-1} \setminus x) = \varphi(y^{-1}) \setminus \varphi(x) \Leftrightarrow \varphi(y^{-1}) \cdot \varphi(y^{-1} \setminus x) = \varphi(x).$$

Denoting $y^{-1} \setminus x = z$, get $y^{-1} \cdot z = x$, so

$$\varphi(y^{-1}) \cdot \varphi(y^{-1} \setminus x) = \varphi(x) \Leftrightarrow \varphi(y^{-1}) \cdot \varphi(z) = \varphi(y^{-1} \cdot z),$$

$\forall y, z \in Q$, i.e. $\varphi \in Aut(Q, \cdot)$. The proof is analogous when $(Q, \cdot)$ is a left Bol loop. $\square$

Corollary 2.3. *If $(Q, \circ)$ is a middle Bol loop and $(Q, \cdot)$ is the corresponding right (left) Bol loop, then the groups of autotopisms of $(Q, \circ)$ and $(Q, \cdot)$ are isomorphic.*

Corollary 2.4. *The group of left pseudo-automorphisms of a middle Bol loop $(Q, \circ)$ is isomorphic to the group of left (right) pseudo-automorphisms of the corresponding right (left) Bol loop $(Q, \cdot)$.*

Proposition 2.3. *A middle Bol loop $(Q, \circ)$ is a G-loop if and only if, for every $c \in Q$, there exists a left pseudo-automorphism of $(Q, \circ)$, with the companion c.*

Proof. Let $(Q, \cdot)$ be the corresponding right Bol loop for $(Q, \circ)$ and let $(Q, \otimes)$ be a loop isotope of $(Q, \circ)$: $(\otimes) = (\circ)^{(\alpha,\beta,\gamma)}$. Then $(Q, \otimes)$ is middle Bol too, so there exists its corresponding right Bol loop $(Q, *)$. According to the Proposition 1. and Proposition 2, $(*) = (\cdot)^{(I\beta I_1,\gamma,\alpha)}$, and the loops $(Q, \otimes)$ and $(Q, \circ)$ are isomorphic if and only if $(Q, *)$ and $(Q, \cdot)$ are isomorphic, i.e. $(Q, \circ)$ is a G-loop if and only if $(Q, \cdot)$ is a G-loop. It is known that a right Bol loop $(Q, \cdot)$ is a G-loop if and only if, for every $c \in Q$, there exists a right pseudo-automorphism of $(Q, \cdot)$, with the companion c ([4]). $\square$

Proposition 2.4. *Let $(Q, \cdot)$ be a RIP-loop with the unit e and let $T = (\alpha, \beta, \gamma)$ be an autotopism of $(Q, \cdot)$. Then γ is an automorphism of $(Q, \cdot)$ if and only if $\gamma(e) = e$ and $\alpha(e) \in N_m^{(\cdot)}$, where $N_m^{(\cdot)}$ is the middle nucleus in $(Q, \cdot)$.*

Proof. If $T = (\alpha, \beta, \gamma)$ is an autotopism of $(Q, \cdot)$ then, taking $y = e$ (respectively, $x = e$) in the equality $\gamma(x \cdot y) = \alpha(x) \cdot \beta(y)$, get $\alpha = R_{\beta(e)}^{-1}\gamma$ (respectively, $\beta = L_{\alpha(e)}^{-1}\gamma$), hence $T = (R_{\beta(e)}^{-1}\gamma, L_{\alpha(e)}^{-1}\gamma, \gamma)$. If $\gamma \in AutQ(\cdot)$ then $T(\gamma^{-1}, \gamma^{-1}, \gamma^{-1}) = (R_{\beta(e)}^{-1}, L_{\alpha(e)}^{-1}, \varepsilon)$ is an autotopism of $(Q, \cdot)$ so $(R_{\beta(e)}, L_{\alpha(e)}, \varepsilon)$ is an autotopism of $(Q, \cdot)$, i.e. $x \cdot y = R_{\beta(e)}(x) \cdot L_{\alpha(e)}(y)$, $\forall x, y \in Q$. Taking $y = e$ in the last equality, get $R_{\alpha(e)}R_{\beta(e)} = \varepsilon$, so $R_{\beta(e)} = R_{\alpha(e)}^{-1}$ and $(R_{\beta(e)}, L_{\alpha(e)}, \varepsilon) = (R_{\alpha(e)}^{-1}, L_{\alpha(e)}, \varepsilon)$ is an autotopism of $(Q, \cdot)$, i.e. the equality $(x \cdot \alpha(e)) \cdot y = x \cdot (\alpha(e) \cdot y)$ holds for all $x, y \in Q$. Hence $\alpha(e) \in N_m^{(\cdot)}$.

Conversely, if $(Q, \cdot)$ is a RIP-loop and $\gamma(e) = e$, then $e = \gamma(e) = \gamma(e \cdot e) = \alpha(e) \cdot \beta(e)$ implies $\beta(e)^{-1} = \alpha(e)$, so $x = (x \cdot \alpha(e)) \cdot (\alpha(e))^{-1} = (x \cdot \alpha(e)) \cdot (\beta(e)) = R_{\beta(e)}R_{\alpha(e)}(x)$, $\forall x \in Q$, hence $R_{\beta(e)} = R_{\alpha(e)}^{-1}$. Now, if $\alpha(e) \in N_m^{(\cdot)}$ then $(R_{\alpha(e)}^{-1}, L_{\alpha(e)}, \varepsilon) = (R_{\beta(e)}, L_{\alpha(e)}, \varepsilon)$

is an autotopism of $(Q,\cdot)$, so $(R^{-1}_{\beta(e)}, L^{-1}_{\alpha(e)}, \varepsilon)$ is an autotopism of $(Q,\cdot)$ and $(\alpha,\beta,\gamma) = (R^{-1}_{\beta(e)}, L^{-1}_{\alpha(e)}, \varepsilon)(\gamma,\gamma,\gamma)$ implies that (γ,γ,γ) is an autotopism of $(Q,\cdot)$, i.e. γ is an automorphism of $(Q,\cdot)$. □

Corollary 2.5. *If $(Q,\circ)$ is a middle Bol loop and $T = (\alpha,\beta,\gamma)$ is an autotopism of $(Q,\circ)$ then $\alpha \in Aut(Q,\circ)$ if and only if $\alpha(e) = e$ and $\gamma(e) \in N_l^{(\circ)}$.*

Proof. According to [9] and to Remark 1 (pt.2), $N_m^{(\cdot)} = N_l^{(\circ)} = N_r^{(\circ)}$, where $N_m^{(\cdot)}$ is the middle nucleus of the corresponding right Bol loop $(Q,\cdot)$ and $N_l^{(\circ)}$ (resp. $N_r^{(\circ)}$) is the left (resp. right) nucleus of $(Q,\circ)$. Now the proof follows from the Propositions 2 and 4. □

Lemma 2.1. *Let $(Q,\circ)$ be a middle Bol loop and let $(Q,\cdot)$ be the corresponding right Bol loop. Then, for every $a \in Q$, the following equalities are true:*

1. $R_a^{(\circ)} = R_a^{(\cdot)} L_a^{(\cdot)} R_{a^{-1}}^{(\cdot)}$;
2. $L_a^{(\circ)} = R_a^{(\cdot)} L_a^{(\cdot)} I R_a^{(\cdot)} I = I L_a^{(\cdot)-1} I$;
3. $L_a^{(\cdot)} = R_{a^{-1}}^{(\circ)-1} = I L_a^{(\circ)-1} I$;
4. $R_a^{(\cdot)} = I_a^{-1} I$,

where $I_a : Q \to Q, I_a(x) = x \setminus\setminus a$, " $\setminus\setminus$" is the right division in $(Q,\circ)$, $R_a^{(\cdot)}$ (resp. $R_a^{(\circ)}$) is the right translation with the element a in $(Q,\cdot)$ (resp. $(Q,\circ)$) and $L_a^{(\cdot)}$ (resp. $L_a^{(\circ)}$) is the left translation with the element a in $(Q,\cdot)$ (resp. $(Q,\circ)$).

Proof. 1. Taking $y = a$ in (1.2), get: $x \circ a = (a \cdot xa^{-1}) \cdot a, \forall x \in Q$, so $R_a^{(\circ)}(x) = R_a^{(\cdot)} L_a^{(\cdot)} R_{a^{-1}}^{(\cdot)}(x)$, $\forall x \in Q$.

2. The first equality follows from (1.8), for $x = a$: $a \circ y = (a \cdot (y^{-1}a)^{-1}) \cdot a, \forall y \in Q, \Leftrightarrow L_a^{(\circ)}(y) = R_a^{(\cdot)} L_a^{(\cdot)} I R_a^{(\cdot)} I(y), \forall y \in Q$. On the other hand, taking $z = a$ in (1.4), have: $L_a^{(\circ)}(x) = a \circ x = x^{-1} \setminus a$ hence, denoting $x^{-1} \setminus a = y$, get: $I(x) = L_a^{(\cdot)} I(y), \Rightarrow y = I L_a^{(\cdot)-1} I(x)$, so $L_a^{(\circ)} = I L_a^{(\cdot)-1} I$.

3. For $x = a$ the equality (1.3) is equivalent to $y = (a \cdot y) \circ a^{-1}$, i.e. $y = R_{a^{-1}}^{(\circ)} L_a^{(\cdot)}(y), \forall y \in Q$, hence $L_a^{(\cdot)} = R_{a^{-1}}^{(\circ)-1}$. So as the inversion I is an anti-automorphism of $(Q,\circ)$, get:

$R_{a^{-1}}^{(\circ)-1}(x) = y \Rightarrow x = R_{a^{-1}}^{(\circ)}(y) = y \circ a^{-1} \Rightarrow x^{-1} = a \circ y^{-1} = L_a^{(\circ)} I(y) \Rightarrow y = I L_a^{(\circ)-1} I(x)$, so $R_{a^{-1}}^{(\cdot)-1} = I L_a^{(\circ)-1} I$.

4. Using (1.3), have: $R_a^{(\cdot)}(x) = x \cdot a = a // x^{-1} = I_a^{-1} I(x)$, $\forall x \in Q$. □

Corollary 2.6. *If $(Q,\circ)$ is a middle Bol loop and $(Q,\cdot)$ is the corresponding right Bol loop then $RM(Q,\circ) = LM(Q,\cdot)$, where $LM(Q,\cdot)$ (resp. $RM(Q,\circ)$) is the left multiplication group of $(Q,\cdot)$ (resp. the right multiplication group of $(Q,\circ)$).*

If $(Q,\cdot)$ is a quasigroup, then the identity $xy\cdot x = x\cdot yx$ is called the law of flexibility. Loops with universal flexibility are studied in [7, 8]. In particular, it is shown in [7] that finite loops, of order ≤ 6, with universal flexibility are middle Bol loops.

So as I is an anti-automorphism of the middle Bol loops, from the equalities $I(x)\circ(I(y)\circ I(x)) = I((x\circ y)\circ x)$ and $(I(x)\circ I(y))\circ I(x) = I(x\circ(y\circ x))$ follows that a middle Bol loop $(Q,\circ)$ satisfies the law of flexibility if and only if I is a semi-automorphism of $(Q,\circ)$.

Proposition 2.5. *A middle Bol loop $(Q,\circ)$ is flexible if and only if the corresponding right Bol loop $(Q,\cdot)$ satisfies the identity*

$$(yx)^{-1}\cdot(x^{-1}y^{-1})^{-1}x = x. \tag{2.9}$$

Proof. Using (1.2) and the right Bol identity, have:

$$\begin{gathered}(x\circ y)\circ x = x\circ(y\circ x) \Leftrightarrow\\ (x\cdot[(y\cdot xy^{-1})y]x^{-1})x = ([(x\cdot yx^{-1})x]\cdot x[(x\cdot yx^{-1})x]^{-1})\cdot[(x\cdot yx^{-1})x] \Leftrightarrow\\ [(x\cdot yx^{-1})x]^{-1}\cdot(x\cdot[(y\cdot xy^{-1})y]x^{-1})x = x \Leftrightarrow\\ [(x\cdot yx^{-1})x]^{-1}x\cdot[(y\cdot xy^{-1})y]x^{-1} = e \Leftrightarrow\\ [(x\cdot yx^{-1})x]^{-1}x = ([(y\cdot xy^{-1})y]x^{-1})^{-1}.\end{gathered}$$

So as the inversion I is a semi-automorphism of $(Q,\cdot)$, get:

$$\begin{gathered}x^{-1}(yx^{-1})^{-1} = ([(y\cdot xy^{-1})y]x^{-1})^{-1} \Leftrightarrow [x^{-1}(yx^{-1})^{-1}]^{-1}\cdot x = (y\cdot xy^{-1})y \Leftrightarrow\\ y^{-1}\cdot[x^{-1}(yx^{-1})^{-1}]^{-1}x = x \Leftrightarrow (yx)^{-1}\cdot(x^{-1}y^{-1})^{-1}x = x,\end{gathered}$$

$\forall x,y\in Q$. □

Proposition 2.6. *Every loop isotop of a middle Bol loop $(Q,\circ)$ is isomorphic to an isotope of the form*

$$(\circ)^{(\varepsilon, II_a^{-1}, R_{a^{-1}}^{(\circ)-1})},$$

for some $a\in Q$, where $I_a^{-1}(x) = a/x$, $\forall x\in Q$.

Proof. Denote by $(Q,\cdot)$ the right Bol loop, corresponding to $(Q,\circ)$. Let $(Q,\otimes)$ be a loop isotope of $(Q,\circ)$ and let $(Q,*)$ be the corresponding right Bol loop for $(Q,\otimes)$. According to [4], there exists an element $a\in Q$, such that $(Q,*)$ is isomorphic to the isotope $(\cdot)^{(R_a^{(\cdot)}, L_a^{(\cdot)-1}, \varepsilon)}$. Up to isomorphism, we may consider $(*) = (\cdot)^{(R_a^{(\cdot)}, L_a^{(\cdot)-1}, \varepsilon)}$. Using (1.3), the equality $x*y = R_a^{(\cdot)}(x)\cdot L_a^{(\cdot)-1}(y)$ implies

$$y///x^{-1} = L_a^{(\cdot)-1}(y)//IR_a^{(\cdot)}(x),$$

$\forall x,y\in Q$, where "$///$"("$//$") is the left division in $(Q,\otimes)$ (resp. $(Q,\circ)$). Now, denoting $y///x^{-1} = t = y///x^{-1} = L_a^{(\cdot)-1}(y)//IR_a^{(\cdot)}(x)$, get: $y = t\otimes x^{-1} = L_a^{(\cdot)}(t\circ IR_a^{(\cdot)}(x))$, so

$$t\otimes x = L_a^{(\cdot)}(t\circ IR_a^{(\cdot)}I(x)),$$

for every $t,x\in Q$, i.e.

$$(\otimes) = (\circ)^{(\varepsilon, IR_a^{(\cdot)}I, L_a^{(\cdot)-1})}.$$

According to Lemma 1, $IR_a^{(\cdot)}I = II_a^{-1}$ and $L_a^{(\cdot)-1} = R_{a^{-1}}^{(\circ)}$, so

$$(\otimes) = (\circ)^{(\varepsilon, II_a^{-1}, R_{a^{-1}}^{(\circ)})}$$

. ☐

Corollary 2.7. *The flexible law is universal in a middle Bol loop $(Q,\circ)$ if and only if $(Q,\circ)$ satisfies the identity*

$$[(x \circ (y \setminus \setminus a^{-1}))//a^{-1}] \circ (x^{-1} \setminus \setminus a^{-1}) = x \circ [(a \setminus \setminus ((a//x) \circ y)) \setminus \setminus a^{-1}],$$

where "//" (" \\ ") is the left (right) division in $(Q,\circ)$.

Proof. The flexible law is universal in $(Q,\circ)$ if and only if its isotopes $(Q,\otimes)$, where $(\otimes) = (\circ)^{(\varepsilon, II_a^{-1}, R_{a^{-1}}^{(\circ)})}$, satisfy the flexible law, for every $a \in Q$. Hence

$$(x \otimes y) \otimes x = x \otimes (y \otimes x) \Leftrightarrow$$

$$R_{a^{-1}}^{(\circ)-1}[R_{a^{-1}}^{(\circ)-1}(x \circ II_a^{-1}(y)) \circ II_a^{-1}(x)] = R_{a^{-1}}^{(\circ)-1}[x \circ II_a^{-1}R_{a^{-1}}^{(\circ)-1}(y \circ II_a^{-1}(x))] \Leftrightarrow$$

$$R_{a^{-1}}^{(\circ)-1}(x \circ II_a^{-1}(y)) \circ II_a^{-1}(x) = x \circ II_a^{-1}R_{a^{-1}}^{(\circ)-1}(y \circ II_a^{-1}(x)). \tag{2.10}$$

Now, using the equalities $I_a^{-1}(x) = a//x$, $II_a^{-1}(x) = I_{a^{-1}}(x^{-1}) = x^{-1}\setminus\setminus a^{-1}$ and $R_{a^{-1}}^{(\circ)-1}(x) = x//a^{-1}$, the identity (2.10) take the form:

$$[(x \circ (y^{-1} \setminus \setminus a^{-1}))//a^{-1}] \circ (x^{-1} \setminus \setminus a^{-1}) = x \circ [(a \setminus \setminus ((a//x) \circ y^{-1})) \setminus \setminus a^{-1}]$$

so, replacing y by y^{-1}, we finish the proof. ☐

Remind that the identity $(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$ in a loop $(Q,\cdot)$ is called the automorphic inverse property.

Proposition 2.7. *A middle Bol loop $(Q,\circ)$ satisfies the automorphic inverse property if and only if the corresponding right Bol loop satisfies the automorphic inverse property.*

Proof. Let $(Q,\cdot)$ be the corresponding right Bol loop. Then $(x \circ y)^{-1} = x^{-1} \circ y^{-1} \Leftrightarrow (y^{-1} \setminus x)^{-1} = y \setminus x^{-1}$. Denoting $y^{-1} \setminus x = z$, get: $z^{-1} = y \setminus (y^{-1} \cdot z)^{-1} \Leftrightarrow y \cdot z^{-1} = (y^{-1} \cdot z)^{-1} \Leftrightarrow y^{-1}z^{-1} = (y \cdot z)^{-1}$, $\forall y, z \in Q$. ☐

Remark 2.1. If $(L,\cdot)$ is a subloop of a right Bol loop $(Q,\cdot)$ then $(L,\circ)$ is a subloop of the corresponding middle Bol loop $(Q,\circ)$ so as, for any $a, b \in Q$, have: $a \circ b = (b \cdot ab^{-1})b \in L$ (resp. $a \circ b = b(b^{-1}a \cdot b) \in L$) and the equations $a \circ x = b$ and $y \circ a = b$ have their unique solutions in L (we shall use the representation (1.8) for the equation $a \circ x = b$).

Proposition 2.8. *Let $(Q,\cdot)$ be a right Bol loop. If a binary relation θ defined on Q is a congruence (normal congruence) of $(Q,\cdot)$ then it is a congruence (normal congruence) of the corresponding middle Bol loop $(Q,\circ)$.*

Proof. Let θ be a normal congruence of the right Bol loop $(Q,\cdot)$. Then the following equivalences hold:

$x\theta y \Leftrightarrow e\theta x^{-1}y \Leftrightarrow y^{-1}\theta x^{-1}y\cdot y^{-1} \Leftrightarrow y^{-1}\theta x^{-1}$. So, $x\theta y \Leftrightarrow xc^{-1}\theta yc^{-1} \Leftrightarrow c\cdot xc^{-1}\theta c\cdot yc^{-1} \Leftrightarrow (c\cdot xc^{-1})c\theta(c\cdot yc^{-1})c \Leftrightarrow x\circ c\theta y\circ c$,

$\forall c\in Q$. And, analogously,

$x\theta y \Leftrightarrow x^{-1}\theta y^{-1} \Leftrightarrow x^{-1}\cdot c\theta y^{-1}\cdot c \Leftrightarrow (x^{-1}\cdot c)^{-1}\theta(y^{-1}\cdot c)^{-1} \Leftrightarrow c\cdot(x^{-1}\cdot c)^{-1}\theta c\cdot(y^{-1}\cdot c)^{-1} \Leftrightarrow (c\cdot(x^{-1}\cdot c)^{-1})\cdot c\theta(c\cdot(y^{-1}\cdot c)^{-1})\cdot c \Leftrightarrow c\circ x\theta c\circ y$,

$\forall c\in Q$. Hence, θ is a normal congruence of $(Q,\circ)$. $\square$

Corollary 2.8. *Let $(Q,\cdot)$ be a right Bol loop and L be a nonempty subset of Q. If $(L,\cdot)$ is a normal subloop of $(Q,\cdot)$ then $(L,\circ)$ is a normal subloop of the corresponding middle Bol loop $(Q,\circ)$.*

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