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ON LINEARIZED FORMULATIONS FOR CONTROL PROBLEMS WITH PIECEWISE DETERMINISTIC MARKOV DYNAMICS

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Abstract

We provide linear programming (primal and dual) formulations of discounted, infinite horizon control problems for piecewise deterministic Markov processes (PDMP) associated to stochastic gene networks. These formulations involve an infinite-dimensional set of probability measures and are obtained using viscosity solutions theory.

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1 Introduction

Linear programming tools have been efficiently used to deal with stochastic control problems (see [2], [3], [16], [17], [19], [20] and references therein). An approach relying mainly on Hamilton-Jacobi(-Bellman) equations has been developed in [10] for deterministic control systems. In the deterministic or Brownian diffusion framework, control problems can be formulated via linear optimization problems on appropriate sets of probability measures. For further details, the reader is referred to [10] (deterministic setting), [12] ($\mathbb{L}^\infty-$cost), [4] (discounted Brownian setting), [13] (Mayer cost and optimal stopping setting). The aim of the present paper is to provide linearized formulation of the value functions associated to continuous control problems for PDMPs. The method is based on viscosity techniques and duality for some associated linearized problem. We embed the set of control processes into a set of probability measures via occupational measures. This set of constraints is explicitly given by a deterministic condition involving the coefficient functions. In the case of Lipschitz-continuous cost functionals, we provide primal and dual linear formulations for the value function (theorem 3.1). The primal value function is given with respect to the previously introduced set of constraints. Using a Hahn-Banach type argument, it is shown that, in general, this set coincides with the closed, convex hull of occupational measures (corollary 3.1). Under convexity assumptions, this set is the set of occupational measures.

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2 Preliminaries

To our best knowledge, Markovian tools have first been employed in connection to molecular biology in [8]. The natural idea was to associate to each reaction network a pure jump model. Due to the large number of molecular species involved in the reactions, direct simulation of these models turns out to be very slow. To increase proficiency, hybrid models are adopted in [6]. They distinguish the discrete components from the "continuous" ones. Using partial Kramers-Moyal expansion, the authors of [6] replace the initial pure jump process with an appropriate piecewise deterministic Markov process (PDMP). The resulting (hybrid) models can then be applied to "in silico" studies of a variety of biologic systems: Cook's model (cf. [5]) for stochastic gene expression and its implications on haploinsufficiency (as particular case of continuous PDMP with switching), Hasty et al. (cf. [14]) model for λ-phage, etc. For further biological models, the reader is referred to [6] and references therein. The construction of controlled PDMPs and basic assumptions are recalled in section 2.1. For reader's sake, we provide an elementary description of how PDMPs are associated to biochemical reactions governing biological systems in section 2.2.

2.1 Construction of controlled PDMPs

We consider U to be a compact metric space (the control space) and $\mathbb{R}^N$ be the state space, for some $N \geq 1$.

Piecewise deterministic control processes have been introduced by Davis [7]. Such processes are given by their local characteristics: a vector field $f : \mathbb{R}^N \times U \to \mathbb{R}^N$ that determines the motion between two consecutive jumps, a jump rate $\lambda : \mathbb{R}^N \times U \to \mathbb{R}_+$ and a transition measure $Q : \mathbb{R}^N \times U \to \mathcal{P}\left(\mathbb{R}^N\right)$. We denote by $\mathcal{B}\left(\mathbb{R}^N\right)$ the Borel σ-field on $\mathbb{R}^N$ and $\mathcal{P}\left(\mathbb{R}^N\right)$ the family of probability measures on $\mathbb{R}^N$. For every $A \in \mathcal{B}\left(\mathbb{R}^N\right)$, the function $(x, u) \mapsto Q(x, u, A)$ is assumed to be measurable and, for every $(x, u) \in \mathbb{R}^N \times U$, $Q(x, u, \{x\}) = 0$.

We summarize the construction of controlled piecewise deterministic Markov processes (PDMP). Whenever $u \in \mathbb{L}^0\left(\mathbb{R}^N \times \mathbb{R}_+; U\right)$ (the space of Borel measurable function) and $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^N$, we consider the ordinary differential equation

$$\begin{cases} d\Phi_t^{t_0,x_0,u} = f\left(\Phi_t^{t_0,x_0,u}, u(x_0, t-t_0)\right) dt, \ t \geq t_0, \\ \Phi_{t_0}^{t_0,x_0,u} = x_{0.} \end{cases}$$

We choose the first jump time T_1 such that the jump rate $\lambda\left(\Phi_t^{0,x_0,u}, u(x_0,t)\right)$ satisfies

$$\mathbb{P}(T_1 \geq t) = \exp\left(-\int_0^t \lambda\left(\Phi_s^{0,x_0,u}, u(x_0,s)\right) ds\right).$$

The controlled piecewise deterministic Markov processes (PDMP) is defined by

$$X_t^{x_0,u} = \Phi_t^{0,x_0,u}, \text{ if } t \in [0, T_1).$$

The post-jump location Y_1 has $Q\left(\Phi_\tau^{0,x_0,u}, u(x_0,\tau), \cdot\right)$ as conditional distribution given $T_1 = \tau$. Starting from Y_1 at time T_1, we select the inter-jump time $T_2 - T_1$ such that

$$\mathbb{P}\left(T_2 - T_1 \geq t \ / \ T_1, Y_1\right) = \exp\left(-\int_{T_1}^{T_1+t} \lambda\left(\Phi_s^{T_1,Y_1,u}, u\left(Y_1, s - T_1\right)\right) ds\right).$$

We set

$$X_t^{x_0,u} = \Phi_t^{T_1,Y_1,u}, \text{ if } t \in [T_1, T_2).$$

The post-jump location Y_2 satisfies

$$\mathbb{P}\left(Y_2 \in A \ / \ T_2, T_1, Y_1\right) = Q\left(\Phi_{T_2}^{T_1,Y_1,u}, u\left(Y_1, T_2 - T_1\right), A\right),$$

for all Borel set $A \subset \mathbb{R}^N$. And so on.

Throughout the paper, unless stated otherwise, we assume the following:

(A1) The function $f : \mathbb{R}^N \times U \longrightarrow \mathbb{R}^N$ is uniformly continuous on $\mathbb{R}^N \times U$ and there exists a positive real constant $C > 0$ such that

$$|f(x,u) - f(y,u)| \leq C|x-y|, \text{ and } |f(x,u)| \leq C, \tag{A1}$$

for all $x, y \in \mathbb{R}^N$ and all $u \in U$.

(A2) The function $\lambda : \mathbb{R}^N \times U \longrightarrow \mathbb{R}_+$ is uniformly continuous on $\mathbb{R}^N \times U$ and there exists a positive real constant $C > 0$ such that

$$|\lambda(x,u) - \lambda(y,u)| \leq C|x-y|, \text{ and } \lambda(x,u) \leq C, \tag{A2}$$

for all $x, y \in \mathbb{R}^N$ and all $u \in U$.

(A3) For each bounded uniformly continuous function $h \in BUC\left(\mathbb{R}^N\right)$, there exists a continuous function $\eta_h : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\eta_h(0) = 0$ and

$$\sup_{u\in U}\left|\int_{\mathbb{R}^N} h(z)\,Q(x,u,dz) - \int_{\mathbb{R}^N} h(z)\,Q(y,u,dz)\right| \leq \eta_h(|x-y|). \tag{A3}$$

(A4) For every $x \in \mathbb{R}^N$ and every decreasing sequence $(\Gamma_n)_{n\geq 0}$ of subsets of $\mathbb{R}^N$,

$$\inf_{n\geq 0}\sup_{u\in U} Q(x,u,\Gamma_n) = \sup_{u\in U} Q\left(x,u,\underset{n}{\cap}\Gamma_n\right) \tag{A4a}$$

and

$$\inf_{n\geq 1}\sup_{x\in\mathbb{R}^N, u\in U} Q\left(x,u,\mathbb{R}^N \smallsetminus \overline{B}(x,n)\right) = 0. \tag{A4b}$$

.

Remark 2.1. *1) Assumption (A3) can be somewhat weakened by imposing*

(A3') For each bounded uniformly continuous function $h \in BUC\left(\mathbb{R}^N\right)$, there exists a continuous function $\eta_h : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\eta_h(0) = 0$ and

$$\sup_{u\in U}\left|\lambda(x,u)\int_{\mathbb{R}^N} h(z)\,Q(x,u,dz) - \lambda(y,u)\int_{\mathbb{R}^N} h(z)\,Q(y,u,dz)\right| \leq \eta_h(|x-y|).$$

It is obvious that whenever one assumes (A3) and $\lambda(\cdot)$ is bounded, the assumption A3' holds true. Moreover, all the proofs in this paper can be obtained (with minor changes) when A3' replaces A3.

2) The assumptions (A1-A3) are quite standard when dealing with viscosity theory in PDMP. They appear under this form in [18]. The assumption (A4) is needed in the Appendix to provide stability properties of viscosity solutions. Roughly speaking, (A4b) states that the probability of exiting the ball centered at the initial point is zero as the radius increases to ∞. The main linearization result is independent of (A4) as soon as stability for the associated system is provided.

2.2 From biochemical reactions to PDMPs

The interest in such systems is motivated by their applications in stochastic gene networks. We recall some rudiments on PDMPs associated to gene networks. For further contributions on gene networks modelling the reader is referred to [6]. We suppose that the biological evolution is given by a family of genes $\mathcal{G} = \{g_i : i = 1, N\}$ interacting through a finite set of reactions $\mathcal{R}$. Every reaction $r \in \mathcal{R}$ can be represented as

$$\alpha_1^r g_1 + \alpha_2^r g_2 + ... + \alpha_N^r g_N \xrightarrow{k_r} \beta_1^r g_1 + ... + \beta_N^r g_N$$

and it specifies that α_i^r molecules of i type (with $1 \leq i \leq N$) called reactants interact in order to form the products (β_i^r molecules of i type, with $1 \leq i \leq N$). The reaction does not occur instantaneously and one needs to specify the reaction speed $k_r > 0$. Also, the presence of all species is not required (α_i^r, $\beta_i^r \in \mathbb{N}$, for all $1 \leq i \leq N$). The species are partitioned in two classes called continuous, respectively discrete component. This partition (for further considerations, see [6]) induces a partition of the reactions. In sum, we distinguish between reactions contributing to the continuous flow ($\mathcal{C} = \{1, 2, ..., M_1\}$) and jump reactions ($\mathcal{J} = \{M_1 + 1, ..., card(\mathcal{R})\}$). To every reaction $r \in \mathcal{R}$, one associates

1) a stoichiometric column vector $\theta^r = \beta^r - \alpha^r \in \mathbb{R}^N$,

2) a propensity function $\lambda_r : \mathbb{R}^N \longrightarrow \mathbb{R}_+$.

For a $\mathcal{C}$-type reaction, $\lambda_r(x) = k_r \prod_{i=1}^{N} x_i^{\alpha_i^r}$, for all $x \in \mathbb{R}^N$.

For a $\mathcal{J}$-type reaction, one should require further regularity as $x_i \to 0$. The jump mechanism will specify that the number of molecules of type i diminishes by α_i^r. Therefore, in order to insure positive components, rather then introducing $\lambda_r(x)$ as for continuous reactions, one could consider

$$\lambda_r(x) = k_r \prod_{\substack{i=1 \\ \alpha_i^r > 0}}^{N} x_i^{\alpha_i^r} \chi\left(\frac{x_i}{\alpha_i^r}\right),$$

for some regular function χ such that $0 \leq \chi \leq 1$, $\chi(y) = 0$, for $0 \leq y \leq 1$ and $\chi(y) = 1$, for $y \geq 1 + err$ (where err is a positive constant). This construction may also apply to $\mathcal{J}$-type reactions to take into account specific thresholds levels.

We associate two matrix M_1 whose columns are the vectors α^r, where $r \in \mathcal{C}$, respectively M_2 whose columns are the vectors α^r, where $r \in \mathcal{J}$. The flow is given by

$$f(x) = M_1 \times (\lambda_1(x), \lambda_2(x), ..., \lambda_{M_1}(x)),$$

the jump intensity

$$\lambda(x) = \sum_{r \in J} \lambda_r(x)$$

and, whenever $\lambda(x) > 0$, the transition measure Q is given by

$$Q(x, dz) = \sum_{r \in \mathcal{J}} \frac{\lambda_r(x)}{\lambda(x)} \delta_{x+\theta^r}(dz).$$

One can suppose that all λ_r are bounded by a reasonable constant $\lambda^{\max} > 0$, by replacing $\lambda_r(x)$ by $\lambda_r(x) \wedge \lambda^{\max}$. All the assumptions A1, A2, A3', A4 are satisfied (as is the assumption B' appearing in section 3). For further comments, the reader is referred to [11].

3 Linear formulation of control problems

We are going to introduce a slight difference in our coefficients allowing to consider a control couple. This is needed to apply the so-called "shaking coefficients" method introduced in [15] (see also [1]). To this purpose, we make the following notations: We let the vector field $\widetilde{f} : \mathbb{R}^N \times U \times \overline{B}(0,1) \longrightarrow \mathbb{R}^N$ be given by

$$\widetilde{f}\left(x, u^1, u^2\right) = f\left(x + u^2, u^1\right), \tag{3.1}$$

for all $x \in \mathbb{R}^N$, $u^1 \in U$ and $u^2 \in \overline{B}(0,1)$. Similarly, the function $\widetilde{\lambda} : \mathbb{R}^N \times U \times \overline{B}(0,1) \longrightarrow \mathbb{R}_+$ is given by

$$\widetilde{\lambda}\left(x, u^1, u^2\right) = \lambda\left(x + u^2, u^1\right), \tag{3.2}$$

and

$$\widetilde{Q}\left(x, u^1, u^2, A\right) = Q\left(x + u^2, u^1, A + u^2\right), \tag{3.3}$$

where $A + u^2 = \left\{a + u^2 : a \in A\right\}$, for all $x \in \mathbb{R}^N$, $u^1 \in U$, $u^2 \in \overline{B}(0,1)$ and all Borel set $A \subset \mathbb{R}^N$.

Remark 3.1. *1. It is obvious that, for every $h \in C_b\left(\mathbb{R}^N\right)$ and every $x \in \mathbb{R}^N$, $u^1 \in U$, $u^2 \in \overline{B}(0,1)$,*

$$\int_{\mathbb{R}^N} h(z)\, \widetilde{Q}\left(x, u^1, u^2, dz\right) = \int_{\mathbb{R}^N} h\left(z - u^2\right) Q\left(x + u^2, u^1, dz\right).$$

2. One can easily check that the assumptions (A1)-(A2) hold true for the characteristic $\left(\widetilde{f}, \widetilde{\lambda}, \widetilde{Q}\right)$ replacing (f, λ, Q) and the set of control U replaced by $U \times \overline{B}(0,1)$.

Throughout the section we are going to strengthen (A3) and assume

(B) For each bounded uniformly continuous function $h \in BUC\left(\mathbb{R}^N\right)$, there exists a continuous function $\eta_h : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\eta_h(0) = 0$ and

$$\sup_{u^1 \in U, u^2 \in \overline{B}(0,1)} \left| \int_{\mathbb{R}^N} h\left(z - u^2\right) Q\left(x + u^2, u^1, dz\right) - \int_{\mathbb{R}^N} h\left(z - u^2\right) Q\left(y + u^2, u, dz\right) \right| \leq \eta_h\left(|x - y|\right). \tag{B}$$

Remark 3.2. *Similarly to Remark 2.1, one can alternatively assume*

(B') For each bounded uniformly continuous function $h \in BUC\left(\mathbb{R}^N\right)$, there exists a continuous function $\eta_h : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\eta_h(0) = 0$ and

$$\sup_{u^1 \in U, u^2 \in \overline{B}(0,1)} \left\{ \begin{array}{l} \lambda\left(x + u^2, u^1\right) \int_{\mathbb{R}^N} h\left(z - u^2\right) Q\left(x + u^2, u^1, dz\right) \\ -\lambda\left(y + u^2, u^1\right) \int_{\mathbb{R}^N} h\left(z - u^2\right) Q\left(y + u^2, u, dz\right) \end{array} \right\} \leq \eta_h\left(|x - y|\right).$$

For every $\varepsilon > 0$, we denote by $\mathcal{E}^\varepsilon$ the class of measurable processes $u^2 : \mathbb{R}^N \times \mathbb{R}_+ \longrightarrow \overline{B}(0, \varepsilon)$. For every admissible control couple $\left(u^1, u^2\right) \in \mathbb{L}^0\left(\mathbb{R}^N \times \mathbb{R}_+; U\right) \times \mathcal{E}^\varepsilon$, we let $X^{x,u^1,u^2}_{\cdot}$ be the piecewise deterministic process associated with the characteristic $\left(\widetilde{f}, \widetilde{\lambda}, \widetilde{Q}\right)$. Obviously, $X^{x,u^1,0}_{\cdot}$ is associated with (f, λ, Q).

3.1 The set of constraints

To any $x \in \mathbb{R}^N$ and any $u \in \mathbb{L}^0\left(\mathbb{R}^N \times \mathbb{R}_+; U\right)$, we associate the discounted occupational measures

$$\gamma_{x,u}(A) = \mathbb{E}\left[\int_0^\infty e^{-t} 1_A\left(X^{x,u}(t), u(t)\right) dt\right], \tag{3.4}$$

for all Borel subsets $A \subset \mathbb{R}^N \times U$. The set of all discounted occupational measures is denoted by $\Gamma(x)$. We let $\mathcal{P}\left(\mathbb{R}^N \times U\right)$ denote the set of all probability measures on $\mathbb{R}^N \times U$ and define

$$\Theta(x) = \left\{ \gamma \in \mathcal{P}\left(\mathbb{R}^N \times U\right) : \ \forall \phi \in C_b^1\left(\mathbb{R}^n\right) : \int_{\mathbb{R}^N \times U} \left(\mathcal{U}^u \phi(y) + \phi(x) - \phi(y)\right) \gamma(dy, du) = 0 \right\}$$

where

$$\mathcal{U}^u \phi(y) = \langle \nabla \phi(y), f(y, u) \rangle + \lambda(y, u) \int_{\mathbb{R}^N} \left(\phi(z) - \phi(y)\right) Q(y, u, dz), \tag{3.5}$$

for all $u \in U$, $\phi \in C_b^1\left(\mathbb{R}^n\right)$, and all $y \in \mathbb{R}^n$.

Remark 3.3. *Using Itô's formula (cf. Theorem 31.3 in [7]), it is clear that, whenever $\phi \in C_b^1\left(\mathbb{R}^n\right)$,*

$$\begin{aligned} &\int_{\mathbb{R}^N \times U} \left(\mathcal{U}^u \phi(y) + \phi(x) - \phi(y)\right) d\gamma_{x,u}(dy, du) \\ &= \lim_{T \to \infty} \mathbb{E}\left[\int_0^T e^{-t} \left(\mathcal{U}^{u_t} \phi\left(X_t^{x,u,0}\right) - \phi\left(X_t^{x,u,0}\right)\right) dt\right] + \phi(x) = 0. \end{aligned}$$

By abuse of notation,

$$\mathcal{U}^{u_t}\phi\left(X_t^{x,u,0}\right)=\mathcal{U}^{u\left(X_{T_i}^{x,u},t-T_i\right)}\phi\left(X_t^{x,u}\right),\ \textit{whenever } T_i\leq t<T_{i+1},$$

where T_i are the jump times appearing in section 2.1. Thus,

$$\Gamma(x)\subset\Theta(x). \tag{3.6}$$

and $\Theta(x)$ is nonempty, convex and closed. We will see later on a sufficient condition allowing to consider compact sets of constraints.

3.2 Linear formulation of continuous control problems

Whenever $g:\mathbb{R}^N\longrightarrow\mathbb{R}$ is a bounded, Lipschitz continuous function, we let

$$\begin{aligned} v_g(x)&=\inf_{u^1\in\mathbb{L}^0(\mathbb{R}^N\times\mathbb{R}_+;U)}\mathbb{E}\left[\int_0^\infty e^{-t}g\left(X_t^{x,u^1,0}\right)dt\right],\\ v_g^\varepsilon(x)&=\inf_{\substack{u^1\in\mathbb{L}^0\left(\mathbb{R}^N\times\mathbb{R}_+;U\right)\\ u^2\in\in\mathbb{L}^0\left(\mathbb{R}^N\times\mathbb{R}_+;\overline{B}(0,\varepsilon)\right)}}\mathbb{E}\left[\int_0^\infty e^{-t}g\left(X_t^{x,u^1,u^2}+u_t^2\right)dt\right],\end{aligned}$$

for all $x\in\mathbb{R}^N$. Theorem 1.1 in [18] yields that v_g^ε is the unique bounded viscosity solution of the following Hamilton-Jacobi integro-differential equation:

$$\begin{aligned}0=v^\varepsilon(x)+\sup_{|u^2|\leq\varepsilon}&\left\{-g(x+u_2)+\sup_{u^1\in U}\left\{-\left\langle f\left(x+u^2,u^1\right),\nabla v^\varepsilon(x)\right\rangle\right.\right.\\ &\left.\left.-\lambda\left(x+u^2,u^1\right)\int_{\mathbb{R}^N}\left(v^\varepsilon(z)-v^\varepsilon(x)\right)\widetilde{Q}\left(x,u^1,u^2,dz\right)\right\}\right\},\end{aligned} \tag{3.7}$$

for all $x\in\mathbb{R}^N$. For the particular case $\varepsilon=0$, the value function v^0 is the unique bounded uniformly continuous viscosity solution of

$$v^0(x)-g(x)+H\left(x,\nabla v^0(x),v^0\right)=0, \tag{3.8}$$

for all $x\in\mathbb{R}^N$, where the Hamiltonian H is given by

$$H(x,p,\psi)=\sup_{u\in U}\left\{-\langle f(x,u),p\rangle-\lambda(x,u)\int_{\mathbb{R}^N}(\psi(z)-\psi(x))\,Q(x,u,dz)\right\}. \tag{3.9}$$

Remark 3.4. *As a consequence of the definition of $\widetilde{Q}$ (eq. 3.3), for every $\varepsilon>0$ and every $u^2\in\overline{B}(0,\varepsilon)$, the function $w(\cdot)=v^\varepsilon\left(\cdot-u^2\right)$ is a viscosity subsolution of (3.8).*

The following convergence result is taken from [11] (theorem 3.6).

Proposition 3.1. *There exists a decreasing function $\eta:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ that satisfies $\lim_{\varepsilon\to0}\eta(\varepsilon)=0$ and such that*

$$\sup_{x\in\mathbb{R}^N}\left|v_g^\varepsilon(x)-v_g(x)\right|\leq\eta(\varepsilon), \tag{3.10}$$

for all $\varepsilon>0$.

We introduce the dual formulation

$$\mu^*(x) = \sup\left\{\mu \in \mathbb{R} : \exists \varphi \in C_b^1\left(\mathbb{R}^N\right) \text{ such that } \forall (y,u) \in \mathbb{R}^N \times U, \right.$$
$$\left. \mu \le \mathcal{U}^u \varphi(y) + g(y) + (\varphi(x) - \varphi(y))\right\}, \tag{3.11}$$

for all $x \in \mathbb{R}^N$. The main result is similar to [4], [13].

Theorem 3.1. *We assume that (A1,2), (B) and (A4) hold true. Then, for every* $x \in \mathbb{R}^N$, *the equality*

$$v_g(x) = \inf_{\gamma \in \Theta(x)} \int_{\mathbb{R}^N \times U} g(y)\, \gamma(dy, du) = \mu^*(x) \tag{3.12}$$

holds true.

Proof. We begin by proving that

$$v_g(x) \ge \inf_{\gamma \in \Theta(x)} \int_{\mathbb{R}^N \times U} g(y)\, \gamma(dy, du) \ge \mu^*(x), \tag{3.13}$$

for all $x \in \mathbb{R}^N$. The first inequality follows from (3.6). We fix $x \in \mathbb{R}^N$ and $(\mu, \varphi) \in \mathbb{R} \times C_b^1\left(\mathbb{R}^N\right)$ such that

$$\mu \le \mathcal{U}^u \varphi(y) + g(y) + \varphi(x) - \varphi(y),$$

for all $y \in \mathbb{R}^N, u \in U$. Integrating this inequality with respect to $\gamma \in \Theta(x)$ and passing to the infimum over $\gamma \in \Theta(x)$ gives the second inequality in 3.13. In order to complete the proof of the theorem, we still have to prove that

$$\mu^*(x) \ge v_g(x). \tag{3.14}$$

We consider (ρ_ε) a sequence of standard mollifiers $\rho_\varepsilon(y) = \frac{1}{\varepsilon^N} \rho\left(\frac{y}{\varepsilon}\right)$, $y \in \mathbb{R}^N$, $\varepsilon > 0$, where $\rho \in C^\infty\left(\mathbb{R}^N\right)$ is a positive function such that

$$Supp(\rho) \subset \overline{B}(0,1) \text{ and } \int_{\mathbb{R}^N} \rho(x) dx = 1.$$

The functions

$$V^\varepsilon = v_g^\varepsilon * \rho_\varepsilon, \tag{3.15}$$

for $\varepsilon > 0$ are (viscosity) subsolutions of (3.8). The proof follows the same arguments as Lemma 2.7 in [1] (see Appendix). Using the fact that V^ε is a subsolution of (3.8), one gets

$$V^\varepsilon(x) \le \mu^*(x).$$

It follows, from (3.10) that

$$(v_g * \rho_\varepsilon)(x) \le \mu^*(x) + \eta(\varepsilon).$$

We allow $\varepsilon \to 0$ in the last inequality, and recall that v_g is continuous, to finally get (3.14). The proof of the theorem is now complete. □

We get the following characterization of the set of constraints:

Corollary 3.1. *Let* $\Gamma_1(x) = \{\gamma(dy, U) : \gamma \in \Gamma(x)\}$ *and* $\Theta_1(x) = \{\gamma(dy, U) : \gamma \in \Theta(x)\}$. *If (A1), (A2), (B) and (A4) hold true, then*

$$\Theta_1(x) = \overline{co}(\Gamma_1(x)). \tag{3.16}$$

The closure is taken w.r.t. the usual (weak) convergence of probability measures.

Proof. Let us fix $x \in \mathbb{R}^N$. Since $\Theta(x)$ is convex and closed, (3.6) implies that $\overline{co}(\Gamma_1(x)) \subset \Theta_1(x)$. The converse inclusion follows by a separation argument: if $\gamma \in \Theta_1(x) \setminus \overline{co}(\Gamma_1(x))$, there exists a regular function g such that $\int g d\gamma < \inf_{\gamma' \in \overline{co}(\Gamma_1(x))} \int g d\gamma'$ and this contradicts the previous theorem. □

Remark 3.5. *1. It is obvious that* $\Theta_1(x)$ *and* $\Gamma_1(x)$ *coincide as soon as* $\Gamma_1(x)$ *is convex and closed. This is needed in order to obtain optimal controls. The following sufficient condition can be found, for instance in [9]: the set*

$$\left\{ \left(f(x,u), \lambda(x,u) \int_{\mathbb{R}^N} \phi(y) Q(x,u,dy) \right) : u \in U \right\} \text{ is convex,}$$

for all $x \in \mathbb{R}^N$ *and all* $\phi \in C_b(\mathbb{R}^N)$.

2. These results can be used to introduce control problems with discontinuous costs and investigate the relationships with generalized viscosity solutions of associated Hamilton-Jacobi systems. For further details of the method in the diffusion setting, the reader is referred to [13]. They can also be employed to infer general dynamic programming principles (cf. [12]).

3.3 A simple application

We recall the notion of ε-viability for an arbitrary nonempty, closed set $K \subset \mathbb{R}^N$

Definition 3.1. *A nonempty, closed set* $K \subset \mathbb{R}^N$ *is said to be* ε*-viable with respect to the controlled piecewise deterministic process* X *if, for every initial point* $x \in K$ *and every* $\varepsilon > 0$*, there exists an admissible control process* $u^\varepsilon \in \mathbb{L}^0(\mathbb{R}^N \times \mathbb{R}_+; U)$ *such that*

$$\mathbb{E}\left[\int_0^\infty e^{-t}\left(d_K\left(X_t^{x,u^\varepsilon}\right) \wedge 1\right) dt\right] \le \varepsilon.$$

Here, d_K *stands for the distance function to the closed set* K.

It is obvious that the notion of ε-viability is equivalent to the value function

$$v_K(x) = \inf_{u \in \mathbb{L}^0(\mathbb{R}^N \times \mathbb{R}_+; U)} \mathbb{E}\left[\int_0^\infty e^{-t}\left(d_K\left(X_t^{x,u}\right) \wedge 1\right) dt\right],$$

for $x \in \mathbb{R}^N$ being 0 on K. Therefore, using the dual formulation in theorem 3.1, one gets the following viability criterion:

Criterion 3.1. *A nonempty, closed set $K \subset \mathbb{R}^N$ is ε-viable with respect to the controlled piecewise deterministic process X if and only if, for every $x \in K$, every $\varepsilon > 0$, and every $\varphi \in C_b^1\left(\mathbb{R}^N\right)$, there exists a couple $(y,u) \in \mathbb{R}^N \times U$ such that*

$$\mathcal{U}^u\varphi(y) + d_K(y) \wedge 1 + (\varphi(x) - \varphi(y)) \leq \varepsilon.$$

A similar criterion has been developed in the study of reachability properties in Cook's model of haploinsufficiency (cf. section 4.2 in [11]).

4 Appendix

The Proof of Theorem 3.1 relies on the fact that the functions V^ε defined by (3.15) are viscosity subsolutions of the Hamilton-Jacobi integro-differential equation (3.8). The proof adapts the arguments used in Barles, Jakobsen [1] Lemma 2.7. Following the proof of this Lemma, we introduce, for every $h > 0$, $u^2 \in \mathbb{R}^N$, $Q_h^{u^2} = u^2 + \left[-\frac{h}{2}, \frac{h}{2}\right)^N$, $\rho_\varepsilon^{h,u^2} = \int_{Q_h^{u^2}} \rho_\varepsilon(y)dy$, and $I_h(x) = \sum_{u^2 \in h\mathbb{Z}^N} \rho_\varepsilon^{h,u^2} v^\varepsilon\left(x - u^2\right)$. Thus, I_h is a convex combination of bounded, uniformly continuous viscosity subsolutions of (3.8). Moreover, by classical results, the discretization I_h converges uniformly to V^ε. To conclude, we show that viscosity subsolutions are preserved by convex combination and uniform convergence.

Proposition 4.1. *Given two bounded, uniformly continuous viscosity subsolutions v_1 and v_2 of the Equation (3.8) and two nonnegative real constants $\lambda_1, \lambda_2 \in \mathbb{R}_+$ such that $\lambda_1 + \lambda_2 = 1$, the convex combination $\lambda_1 v_1 + \lambda_2 v_2$ is still a viscosity subsolution of (3.8).*

Proof. The assertion is trivial when either $\lambda_1 = 0$ or $\lambda_2 = 0$. If $\lambda_1\lambda_2 \neq 0$, we let $\overline{x} \in \mathbb{R}^N$ and $\varphi \in C_b^1(\mathcal{N}_{\overline{x}})$ be a test function such that

$$\lambda_1 v_1(\overline{x}) + \lambda_2 v_2(\overline{x}) - \varphi(\overline{x}) \geq \lambda_1 v_1(y) + \lambda_2 v_2(y) - \varphi(y), \tag{4.17}$$

for all $y \in \mathbb{R}^N$. We may assume, without loss of generality that $\varphi \in C_b\left(\mathbb{R}^N\right)$. Indeed, whenever φ does not satisfy this assumption, one can replace it with some φ^0 defined as follows : First, notice that there exists some $r > 0$ such that $B(\overline{x}, 2r) \subset \mathcal{N}_{\overline{x}}$. We define

$$\begin{aligned}\varphi^0(y) = &\left(\varphi(y) + \lambda_1 v_1(\overline{x}) + \lambda_2 v_2(\overline{x}) - \varphi(\overline{x})\right)\chi(y) \\ &+ \left(\lambda_1 v_1(y) + \lambda_2 v_2(y)\right)(1 - \chi(y)),\end{aligned}$$

for all $y \in \mathbb{R}^N$, where χ is a smooth function such that $0 \leq \chi \leq 1$, $\chi(y) = 1$, if $y \in B(\overline{x}, r)$ and $\chi(y) = 0$, if $y \in \mathbb{R}^N \setminus B(\overline{x}, 2r)$. Then (4.17) holds true with φ^0 instead of φ. The new function φ^0 also satisfies

$$\nabla\varphi^0(\overline{x}) = \nabla\varphi(\overline{x}).$$

We introduce, for every $\varepsilon > 0$

$$\Phi_\varepsilon(x,y) = \lambda_1 v_1(x) + \lambda_2 v_2(y) - \lambda_1\varphi(x) - \lambda_2\varphi(y) - \frac{1}{\varepsilon^2}|x-y|^2 - |x - \overline{x}|^2,$$

for all $x, y \in \mathbb{R}^N$. We recall that the functions v_1, v_2 and φ are bounded and continuous. This yields the existence of a global maximum $(x_\varepsilon, y_\varepsilon)$ of Φ_ε. Moreover, by standard arguments,

$$\lim_{\varepsilon \to 0} x_\varepsilon = \lim_{\varepsilon \to 0} y_\varepsilon = \overline{x}, \quad \lim_{\varepsilon \to 0} \left| \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} \right|^2 = 0. \tag{4.18}$$

We consider the test function ψ given by

$$\psi(x) = -\lambda_2 \lambda_1^{-1} v_2(y_\varepsilon) + \varphi(x) + \lambda_2 \lambda_1^{-1} \varphi(y_\varepsilon) + \frac{\lambda_1^{-1}}{\varepsilon^2} |x - y_\varepsilon|^2 + \lambda_1^{-1} |x - \overline{x}|^2,$$

for all $x \in \mathbb{R}^N$. We recall that the function v_1 is a viscosity subsolution for (3.8). Then,

$$v_1(x_\varepsilon) + d_{\mathcal{O}^c}(x_\varepsilon) \wedge 1 + H\left(x_\varepsilon, \nabla\varphi(x_\varepsilon) + \frac{2\lambda_1^{-1}}{\varepsilon^2}(x_\varepsilon - y_\varepsilon) + 2\lambda_1^{-1}(x_\varepsilon - \overline{x}), v_1\right) \leq 0.$$

Standard estimates yield

$$\begin{aligned} 0 \geq\ & v_1(\overline{x}) + d_{\mathcal{O}^c}(\overline{x}) \wedge 1 + \sup_{u \in U} \Big\{ -\langle f(\overline{x}, u), \nabla\varphi(\overline{x}) \rangle - \frac{2\lambda_1^{-1}}{\varepsilon^2} \langle x_\varepsilon - y_\varepsilon, f(x_\varepsilon, u) \rangle \\ & -\lambda(\overline{x}, u) \int_{\mathbb{R}^N} (v_1(z) - v_1(\overline{x})) Q(\overline{x}, u, dz) \Big\} \\ & - C\left(|x_\varepsilon - \overline{x}| + |v_1(x_\varepsilon) - v_1(\overline{x})| + |\nabla\varphi(x_\varepsilon) - \nabla\varphi(\overline{x})| + \eta_{v_1}(|x_\varepsilon - \overline{x}|)\right). \end{aligned} \tag{4.19}$$

In a similar way, we get

$$\begin{aligned} 0 \geq\ & v_2(\overline{x}) + d_{\mathcal{O}^c}(\overline{x}) \wedge 1 + \sup_{u \in U} \Big\{ -\langle f(\overline{x}, u), \nabla\varphi(\overline{x}) \rangle + \frac{2\lambda_2^{-1}}{\varepsilon^2} \langle x_\varepsilon - y_\varepsilon, f(y_\varepsilon, u) \rangle \\ & -\lambda(\overline{x}, u) \int_{\mathbb{R}^N} (v_2(z) - v_2(\overline{x})) Q(\overline{x}, u, dz) \Big\} \\ & - C\left(|y_\varepsilon - \overline{x}| + |v_2(y_\varepsilon) - v_2(\overline{x})| + |\nabla\varphi(y_\varepsilon) - \nabla\varphi(\overline{x})| + \eta_{v_2}(|y_\varepsilon - \overline{x}|)\right) \end{aligned} \tag{4.20}$$

Finally, using (4.19), (4.20) and (4.18), and passing to the limit as $\varepsilon \to 0$, yields

$$(\lambda_1 v_1 + \lambda_2 v_2)(\overline{x}) + d_{\mathcal{O}^c}(\overline{x}) \wedge 1 + H(\overline{x}, \nabla\varphi(\overline{x}), \lambda_1 v_1 + \lambda_2 v_2) \leq 0.$$

$\square$

These arguments allow to obtain, by recurrence, that any convex combination of continuous, bounded viscosity subsolutions is still a subsolution for (3.8).

Proposition 4.2. *(Stability)*
Let $(v_n)_n$ be a sequence of continuous, uniformly bounded viscosity subsolutions of (3.8). Moreover, we suppose that v_n converges uniformly on compact sets to some continuous, bounded function v. Then the function v is a viscosity subsolution of (3.8).

Proof. We let $x \in \mathbb{R}^N$ and $\varphi \in C_b^1(\mathcal{N}_x)$ be a test function such that $v - \varphi$ has a global maximum at x. As in the previous proposition, one can assume, without loss of generality, that $\varphi \in C_b(\mathbb{R}^N)$. Classical arguments yield the existence of some point $x_n \in \mathbb{R}^N$ such that

$$v_n(x_n) - \varphi(x_n) - |x_n - x|^2 \geq v_n(y) - \varphi(y) - |y - x|^2,$$

for all $y \in \mathbb{R}^N$ and

$$\lim_{n \to \infty} x_n = x.$$

We assume, without loss of generality, that $|x_n - x| \leq 1$, and $x_n \in \mathcal{N}_x$, for all $n \geq 1$. Then,

$$0 \geq v_n(x_n) + d_{\mathcal{O}^c}(x_n) \wedge 1 + \sup_{u \in U} \left\{ \begin{array}{c} -\langle f(x_n, u), \nabla\varphi(x_n) + 2(x_n - x)\rangle \\ -\lambda(x_n, u) \int_{\mathbb{R}^N} (v_n(z) - v_n(x_n)) Q(x_n, u, dz) \end{array} \right\}. \quad (4.21)$$

We have

$$-\langle f(x_n, u), \nabla\varphi(x_n) + 2(x_n - x)\rangle \geq -\langle f(x, u), \nabla\varphi(x)\rangle - C(|x_n - x| + |\nabla\varphi(x_n) - \nabla\varphi(x)|), \quad (4.22)$$

where $C > 0$ is a generic constant independent of $n \geq 1$ and $u \in U$ which may change from one line to another. We also get

$$\begin{aligned} &-\lambda(x_n, u) \int_{\mathbb{R}^N} (v_n(z) - v_n(x_n)) Q(x_n, u, dz) \\ &\quad \geq -\lambda(x, u) \int_{\mathbb{R}^N} (v(z) - v(x)) Q(x, u, dz) - C(|x_n - x| + |v_n(x_n) - v(x)| + \eta_v(|x_n - x|)) \\ &\quad - C \sup_{u \in U} \int_{\mathbb{R}^N} |v_n(z) - v(z)| Q(x_n, u, dz). \end{aligned} \quad (4.23)$$

Finally, for every $m \geq 1$,

$$\begin{aligned} &\sup_u \int_{\mathbb{R}^N} |v_n(z) - v(z)| Q(x_n, u, dz) \\ &\leq \sup_{z \in \overline{B}(0, m + |x| + 1)} (|v_n(z) - v(z)|) + C \sup_{u \in U} Q\left(x_n, u, \mathbb{R}^N \setminus \overline{B}(0, m + |x| + 1)\right) \\ &\leq \sup_{z \in \overline{B}(0, m + |x| + 1)} (|v_n(z) - v(z)|) + C \sup_{u \in U} Q\left(x_n, u, \mathbb{R}^N \setminus \overline{B}(x_n, m)\right) \\ &\leq \sup_{z \in \overline{B}(0, m + |x| + 1)} (|v_n(z) - v(z)|) + C \sup_{y \in \mathbb{R}^N, u \in U} Q\left(y, u, \mathbb{R}^N \setminus \overline{B}(y, m)\right). \end{aligned} \quad (4.24)$$

We substitute (4.22)-(4.24) in (4.21) and allow $n \to \infty$ to have

$$\begin{aligned} 0 \geq\ & v(x) + d_{\mathcal{O}^c}(x) \wedge 1 + \sup_{u \in U} \left\{ -\langle f(x, u), \nabla\varphi(x)\rangle - \lambda(x, u) \int_{\mathbb{R}^N} (v(z) - v(x)) Q(x, u, dz) \right\} \\ & - C \sup_{y \in \mathbb{R}^N, u \in U} Q\left(y, u, \mathbb{R}^N \setminus \overline{B}(y, m)\right), \end{aligned} \quad (4.25)$$

for all $m \geq 1$. We conclude using the Assumption A4b. $\square$

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