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NUMERICAL ANALYSIS OF THE STOKES/DARCY COUPLING

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Abstract

We consider a differential system based on the coupling of the Stokes and Darcy equations for modeling the interaction between surface and porous-media flows. We formulate the problem as an interface equation, we analyze the associated Steklov-Poincar operators, and we prove its well-posedness. We propose an iterative method to solve the coupling of the Stokes and Darcy equations.

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1 Indroduction

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded domain, decomposed into two non intersecting subdomains Ω_f and Ω_p separated by an interface Γ, i.e. $\overline{\Omega} = \overline{\Omega}_f \cup \overline{\Omega}_p$, $\Omega_f \cap \Omega_p = \varnothing$ and $\overline{\Omega}_f \cap \overline{\Omega}_p = \Gamma$. We suppose the boundaries $\partial\Omega_f$ and $\partial\Omega_p$ to be Lipschitz continuous. From the physical point of view, Γ is a surface separating the domain Ω_f filled by a fluid, from a domain Ω_p formed by a porous medium. We assume that the fluid contained in Ω_f has a fixed surface (i.e. we do not consider the free surface fluid case) and can filtrate through the adjacent porous medium.
We introduce the Stokes equations: $\forall t > 0$,

$$-\nabla \cdot T(u_f, p_f) = f \quad \text{in} \quad \Omega_f$$

$$\nabla \cdot u_f = 0 \quad \text{in} \quad \Omega_f \tag{1.1}$$

where $T(u_f, p_f) = v(\nabla u_f + \nabla^T u_f) - p_f I$ is the Cauchy stress tensor, $v > 0$ is the kinematic viscosity of the fluid, while u_f and p_f are the fluid velocity and pressure;
We define the piezometric head φ where z is the elevation from a reference level, p_p is the pressure of the fluid in Ω_p, ρ_f its density and g is the gravity acceleration.
The fluid motion in Ω_p is described by the equations:

$$u_p = -K\nabla\varphi \quad \Omega_p$$

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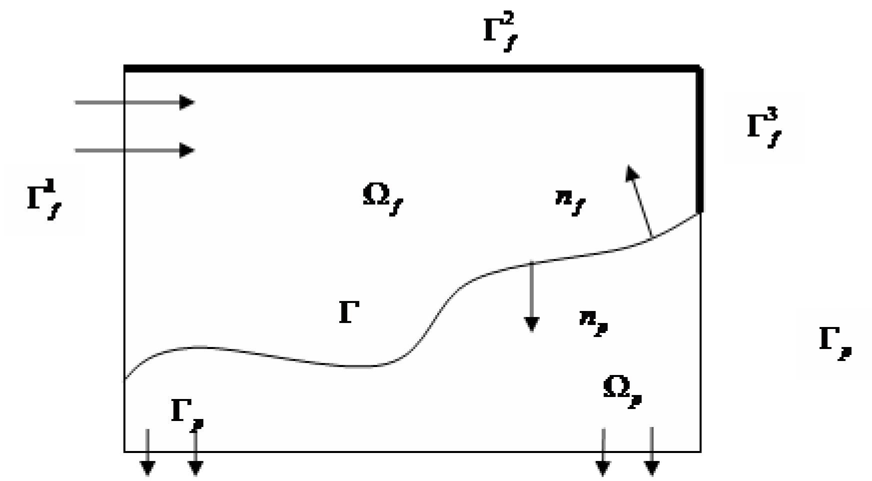

Figure 1: Schematic representation of a 2D section of possible computational domains: the surface-groundwater

$$\nabla \cdot u_p = 0 \qquad \Omega_p \tag{1.2}$$

where u_p is the fluid velocity, and $\mathbf{K}$ is the hydraulic conductivity tensor $\mathbf{K} = diag(\mathbf{K_1}, \mathbf{K_2}, \mathbf{K_3})$ with $\mathbf{K} = (\mathbf{K_{ij}})_{\mathbf{i,j=1,...,d}} \in \mathbf{L}^\infty(\mathbf{\Omega_p})$. In the following we shall denote $K = \mathbf{K}/\mathbf{n}$.

In our analysis we shall adopt homogeneous boundary conditions. In particular, for the Stokes problem we impose the no-slip condition $u_f = 0$ on $\partial\Omega_f \setminus \Gamma$, while for the Darcy problem, we set the piezometric head $\varphi = 0$ on Γ_p^D and we require the normal velocity to be null, $u_p \cdot n_p = 0$ on $\Gamma_p^N \cdot \mathbf{n}_p$ and $\mathbf{n}_f$ denote the unit outward normal vectors to the surfaces Ω_f and Ω_p and we have $\mathbf{n}_f = -\mathbf{n}_p$ on Γ. We suppose $\mathbf{n}_p$ and $\mathbf{n}_f$ to be regular enough. In the following we shall indicate $n = \mathbf{n}_p$ for simplicity of notation.

We supplement with the following conditions on Γ :

$$u_f \cdot n = u_p \cdot n \tag{1.3}$$

$$-n \cdot T(u_f, p_f) \cdot n = g\varphi \tag{1.4}$$

$$\frac{\upsilon \alpha_{BJ}}{\sqrt{K}} u_f \tau_j - \tau_j \cdot T(u_f, p_f) \cdot n = 0 \tag{1.5}$$

where τ_j $(j = 1, ..., d-1)$ are linear independent unit tangential vectors to the boundary Γ, and α_{BJ} is the characteristic length of the porous medium.

Conditions (1.3) and (1.4) impose the continuity of the normal velocity on Γ, as well as that of the normal component of the normal stress, however they allow pressure to be discontinuous across the interface. The so-called Beavers-Joseph-Saffman condition (1.5) does not yield any coupling.

Indeed, it provides a boundary condition for the Stokes problem since it involves only quantities in the domain Ω_f.

The coupled Stokes/Darcy model is as follows:

$$
\begin{aligned}
-\nabla \cdot T(u_f, p_f) = f \qquad \Omega_f \\
\nabla \cdot u_f = 0 \qquad \Omega_f \\
u_p = -K\nabla\varphi \qquad \Omega_p \\
\nabla \cdot u_p = 0 \qquad \Omega_p \\
u_f \cdot n = u_p \cdot \qquad \text{on} \qquad \Gamma \\
-n \cdot T(u_f, p_f) \cdot n = g\varphi \qquad \text{on} \qquad \Gamma \\
\frac{\upsilon\alpha_{BJ}}{\sqrt{K}} u_f\tau_j - \tau_j \cdot T(u_f, p_f) \cdot n = 0 \qquad \text{on} \qquad \Gamma.
\end{aligned}
\tag{1.6}
$$

We define the following functional spaces:

$$
\begin{aligned}
H_{\Gamma_f} = \{v \in H^1(\Omega_f) : v = 0 \qquad \text{on} \qquad \Gamma_f\} \\
H_{\Gamma_f \cup \Gamma_f^{in}} = \{v \in H_{\Gamma_f}(\Omega_f) : v = 0 \qquad \text{on} \qquad \Gamma_f^{in}, \qquad H_f = (H_{\Gamma_f \cup \Gamma_f^{in}})^d \\
H_f^0 = \{v \in H_f(\Omega_f) : v \cdot n_f = 0 \qquad \text{on} \qquad \Gamma\} \\
\tilde{H}_f = \{v \in H^1(\Omega_f)^d : v = 0 \quad \text{on} \quad \Gamma \cup \Gamma_f\} \\
Q = L^2(\Omega_f), \quad Q_0 = \{q \in Q : \int_{\Omega_f} q = 0\}
\end{aligned}
\tag{1.7}
$$

$$
H_p = \{\Psi \in H^1(\Omega_p) : \Psi = 0 \quad \text{on} \quad \Gamma_p\}, \quad H_p^0 = \{\Psi \in H_p : \Psi = 0 \quad \text{on} \quad \Gamma\}.
$$

We denote by $|\cdot|_1$ and $\| \cdot \|_1$ the H1-seminorm and norm and by $\| \cdot \|_2$ the L2-norm; it will always be clear form the context whether we are referring to spaces on Ω_f and Ω_p . Finally, we consider the trace space $\Lambda = H_{00}^{\frac{1}{2}}(\Gamma)$.
Then, we introduce the bilinear forms

$$
\begin{aligned}
a_f(v, w) = \int_{\Omega_f} fracv2(\nabla v + \nabla^T v) \cdot (\nabla w + \nabla^T w) \qquad \forall v, w \in (H^1(\Omega_f))^d \\
b_f(v, q) = -\int_{\Omega_f} q\nabla \cdot v \qquad \forall v \in (H^1(\Omega_f))^d \qquad \forall q \in Q \\
a_p(\varphi, \Psi) = \int_{\Omega_p)} \nabla\Psi \cdot K\nabla\varphi \qquad \forall\varphi, \Psi \in H^1(\Omega_p).
\end{aligned}
$$

The weak formulation of Stokes/Darcy reads from [5]:
find $u_f \in H_f$, $p_f \in Q$, $\varphi \in H_p$ such that

$$
a_f(u_f, v) + b_f(v, p_f) + \int_\Gamma g\varphi(v \cdot n) + \int_\Gamma \sum_{i=1}^{d-1} \frac{\upsilon\alpha_{BJ}}{\sqrt{K}} [(u_f) \cdot \tau_j](v \cdot \tau_j) = \int_{\Omega_f} fv \tag{1.8}
$$

$$
b_f(u_f, q) = 0 \tag{1.9}
$$

$$
a_p(\varphi, \Psi) = \int_\Gamma \Psi(u_f \cdot n)
$$

for all $v \in H_f$, $q \in Q$, $\Psi \in H_p$.

2 The Steklov-Poincar interface equation associated to the coupled problem

In this section we apply domain decomposition techniques at the differential level to study the Stokes/Darcy problem. The Stokes/Darcy problem can be rewritten in a multi-domain formulation

Proposition 2.1. *Let Λ be the space of traces introduced in Section 1. Problem (1.8)-(1.9) can be reformulated in an equivalent way as follows:*
Find $u_f^0 in H_f$, $p_f \in Q$, $\varphi \in H_p$ such that

$$a_f(u_f^0 + E_f u_{in}, w) + b_f(w, p_f) + \int_\Gamma \sum_{j=1}^{d-1} \frac{v\alpha_{BJ}}{\sqrt{K}}[(u_f^0 + E_f u_{in}) \cdot \tau_j](R_1\mu \cdot \tau_j)$$

$$= \int_{\Omega_f} fw \qquad \forall w \in H_f^0 \tag{2.10}$$

$$b_f(u_f^0 + E_f u_{in}, q) = 0 \qquad \forall q \in Q \tag{2.11}$$

$$a_p(\varphi_0 + E_p\varphi_p, \Psi) = 0 \qquad \forall \Psi \in H_p^0 \tag{2.12}$$

$$\int_\Gamma (u_f^0 \cdot n)\mu = a_p(\varphi_0 + E_p\varphi, R_2\mu) \qquad \forall \mu \in \Lambda \tag{2.13}$$

$$\int_\Gamma g\varphi\mu = \int_{\Omega_f} f(R_1\mu) - a_p(u_f^0 + E_f u_{in}, R_1\mu) - b_f(R_1\mu, p_f)$$

$$- \int_\Gamma \sum_{j=1}^{d-1} \frac{v\alpha_{BJ}}{\sqrt{K}}[(u_f^0 + E_f u_{in}) \cdot \tau_f](R_1\mu \cdot \tau_j) \tag{2.14}$$

where R_1 is any possible extension operator from Λ to H_p such that $(R_1\mu) \cdot n = \mu$ on Γ for all $\mu \in \Lambda$ and R_2 is any possible extension operator from $H^{\frac{1}{2}}(\Gamma)$ to H_p to $R_2\mu = \mu$ on Γ for all $\mu \in H^{\frac{1}{2}}(\Gamma)$.

The proof is made by direct inspection considering $u_f = u_f^0 + E_f u_{in}$ and $\varphi = \varphi_0 + E_p\varphi_p$ with $u_f^0 \in H_f$ and $\varphi \in H_p$.

We choose now a suitable governing variable on the interface . We set the interface variable λ as the trace of the normal velocity on the interface:

$$\lambda = u_f \cdot n = K\partial_n\varphi.$$

Remark that using the simplified condition $u_f \cdot \tau_j = 0$, , the multi-domain formulation of the Stokes/ Darcy problem (2.10)-(2.14) becomes:
Find $u_f^0 \in H_f^\tau$, $p_f \in Q$, $\varphi \in H_p$ such that

$$a_f(u_f^0 + E_f u_{in}, w) + b_f(w, p_f) = \int_{\Omega_f} fw \qquad \forall w \in (w \in (H_0^1(\Omega_f))^d \tag{2.15}$$

$$b_f(u_f^0 + E_f u_{in}, q) = 0 \qquad \forall q \in Q \tag{2.16}$$

$$a_p(\varphi_0 + E_p\varphi_p, \Psi) = 0 \qquad \forall \Psi \in H_p^0 \tag{2.17}$$

$$\int_\Gamma (u_f^0 \cdot n)\mu = a_p(\varphi_0 + E_p\varphi_p, R_2\mu) \qquad \forall \mu \in \Lambda \tag{2.18}$$

$$\int_\Gamma g\varphi_0\mu = \int_{\Omega_f} f(R_1^\tau\mu) - a_f(u_f^0 + E_f u_{in}, R_1^\tau\mu, p_f) \qquad \forall \mu \in \Lambda \tag{2.19}$$

with R_2 defined as in Proposition 1 and $R_1^\tau : \Lambda \to H_f^\tau$ is any possible continuos extension operator so $R_1^\tau\mu \cdot n = \mu$ pe Γ for all $\mu \in \Lambda$ with $H_f^\tau = \{v \in H_f : v \cdot \tau_j = 0 \quad \text{on} \quad \Gamma\}$.
For this simplification model (2.15)-(2.19) Discciatti in [9] made the analysis. We will do the analysis for the multi-domain formulation of the Stokes/ Darcy problem (2.10)-(2.14). We define the continuous extension operator

$$E_\Gamma : H^{\frac{1}{2}} \to H_f^\tau, \quad \eta \to E_\Gamma\eta \qquad \text{such that} \qquad E_\Gamma\eta \cdot \eta = \eta \qquad \text{on} \qquad \Gamma.$$

We consider the interface variable $\lambda = u_f \cdot n$ on Γ, $\lambda \in \Lambda$ with $\lambda = \lambda_0 + \lambda_*$ where λ_* satisfies

$$\int_\Gamma \lambda_* = -\int_{\Gamma_f^{in}} u_{in} \cdot n \tag{2.20}$$

and $\lambda_0 \in \Lambda_0$ with

$$\Lambda_0 = \{\mu \in \Lambda : \int_\Gamma \mu = 0\} \subset \Lambda.$$

Then we introduce two auxiliary problems whose solutions are related to the global problem (2.15)-(2.19):
(P1) Find $w_0^* \in (H_0^1(\Omega_f))^d$, $\pi \in Q_0$ such that

$$a_f(w_0^* + E_f u_{in} + E_\Gamma\lambda_*, v) + b_f(v, \pi^*) + \int_\Gamma \sum_{i=1}^{d-1} \frac{v\alpha_{BJ}}{\sqrt{K}} + [(w_0^* + E_f u_{in}) \cdot \tau_j](v \cdot \tau_j)$$

$$= \int_{\Omega_f} fv \qquad \forall v \in (H_0^1(\Omega_f))^d$$

where $Q_0 = \{q \in Q : \quad \int_{\Omega_f} q = 0\}$.
(P2) Find $\varphi^* \in H_p$ such that

$$a_p(\varphi^* + E_p\varphi_p, \Psi) = \int_\Gamma \lambda_*\Psi \qquad \forall \Psi \in H_p.$$

We define the following extension operators:

$$R_f : \Lambda_0 \to H_f^\tau \times Q_0, \quad \eta \to R_f\eta = (R_f^1\eta, R_f^2\eta) \quad \text{such that} \quad (R_f^1\eta) \cdot \eta = \eta \quad \text{on} \quad \Gamma \quad \textit{and}$$

$$a_f(R_f^1\eta, v) + b_f(v, R_f^2\eta) + \int_\Gamma \sum_{j=1}^{d-1} \frac{v\alpha_{BJ}}{\sqrt{K}}(R_f^1\eta \cdot \tau_j)(v \cdot \tau_j) = 0 \qquad \forall v \in (H_0^1(\Omega_f))^d \tag{2.21}$$

$$b_f(R_f^1\eta, q)0 \qquad \forall q \in Q_0 \tag{2.22}$$

$$R_p : \Lambda \to H_p \quad \text{with} \quad \eta \to R_p\eta \quad \text{such that}$$

$$a_p(R_p\eta, R_2\mu) = \int_\Gamma \eta\mu \qquad \forall \mu \in H^{\frac{1}{2}}(\Gamma) \tag{2.23}$$

We define the following extension operators:
For all $\eta \in \Lambda_0$, $\mu \in \Lambda$

$$< S\eta, \mu >= a_f(R_f^1\eta, v) + b_f(v, R_f^2\eta) + \int_\Gamma \sum_{j=1}^{d-1} \frac{v\alpha_{BJ}}{\sqrt{K}}(R_f^1\eta \cdot \tau_j)(v \cdot \tau_j) + \int_\Gamma g(R_p\eta)\mu.$$

Which can be split as the sum of two sub-operators $S = S_f + S_p$

$$< S_f\eta, \mu >= a_f(R_f^1\eta, v) + b_f(v, R_f^2\eta) + \int_\Gamma \sum_{j=1}^{d-1} \frac{v\alpha_{BJ}}{\sqrt{K}}(R_f^1\eta \cdot \tau_j)(v \cdot \tau_j) \tag{2.24}$$

$$< S_p\eta, \mu >= \int_\Gamma g(R_p\eta)\mu \tag{2.25}$$

We define the functional $\chi : \Lambda \to \mathbb{R}$

$$< \chi, \mu > \int_{\Omega_f} fv - a_f(w_0^* + E_f u_{in} + E_\Gamma\lambda_*, v) - b_f(v, \pi^*)$$

$$- \int_\Gamma \sum_{j=1}^{d-1} \frac{v\alpha_{BJ}}{\sqrt{K}}((w_0^* + E_f u_{in} + E_\Gamma\lambda_*) \cdot \tau_j)(v \cdot \tau_j) - \int_\Gamma g\varphi_0^*\mu \qquad \forall \mu \in \Lambda. \tag{2.26}$$

We can express the solutin of the coupled problem in terms of the interface variable λ_0.

Theorem 2.1. *The solutions to (2.15)-(2.19) can be characterized as follows:*

$$u_f^0 = w_0^* + R_f^1\lambda_0 + E_\Gamma\lambda_*, \quad p_f = \pi^* + R_f^2\lambda_0 + \widehat{p}_f, \quad \varphi_0 = \varphi_0^* + R_p\lambda_0 \tag{2.27}$$

where $\widehat{p}_f = (meas(\Omega_f))^{-1}\int_{\Omega_f} p_f$ *and* $\lambda_0 \in \Lambda_0$ *is the solution of the following Steklov-Poincare problem*

$$< S\lambda_0, \mu_0 >=< \chi, \mu_0 > \qquad \forall \mu_0 \in \Lambda_0. \tag{2.28}$$

Moreover, $\widehat{p}_f$ *can be obtenied from* λ_0 *by solving the algebric equation*

$$\widehat{p}_f = \frac{1}{meas(\Gamma)} < S\lambda_0 - \chi, \zeta > \tag{2.29}$$

where $\zeta \in \Lambda$ *is a fixed function such that*

$$\frac{1}{meas(\Gamma)} \int_\Gamma \zeta = 1.$$

For the proof, we refer to Badea L., Discacciati M., Quarteroni A (2008) in [3] with minor modification.

3 Iterative finite element solution of the coupled problem

In this section, we introduce and analyze an iterative method to compute the solution of a conforming finite element approximation of (2.16)-(2.18). For the easiness of notation, we will write the algorithms in continuous form. However, they can be straightforwardly translated into a discrete setting considering conforming internal Galerkin approximations of the spaces (1.7).

Moreover, the convergence results that we will present hold in the discrete case without any dependence of the convergence rate on the grid parameter h, since they are established by using the properties of the operators in the continuous case.

We consider a triangulation T_h of the domain $\overline{\Omega}_f \cup \overline{\Omega}_p$, depending on a positive parameter $h > 0$, made up of triangles if $d = 2$, or tetrahedra in the three-dimensional case. We assume that the triangulations induced on the subdomains Ω_f and Ω_p are compatible on Γ, that is they share the same edges (if $d = 2$) or faces (if $d = 3$) therein.

The crucial issue concerning the finite dimensional spaces, say V_h and Q_h, approximating the spaces of velocity and pressure is that they must satisfy the discrete compatibility condition:
there exists a positive constant $\beta^* > 0$, independent of h, such that

$$\forall q_h \in Q_h, \quad \exists v_h \in V_h, \quad v_h \neq 0 : \quad b_h(v_h, q_h) \geq \beta^* \parallel v_h \parallel_1 \parallel q_h \parallel_0 \tag{3.30}$$

Spaces satisfying (3.30) are said inf-sup stable.

The following error estimates hold (see Girault V., Raviart P(1986) in [10] and Quarteroni A., Valli A.(1994) in [11]). There exist two positive constants C_1 and C_2 such that

$$E_S^h \leq C_1 H^r (\parallel u_f \parallel_{r+1} + \parallel p_f \parallel_r), \quad r = 1, 2.$$

If $u_f \in H^{r+1}(\Omega_f)$ and $p_f \in H^r(\Omega_f)$ where

$$E_S^h = \parallel \nabla u_f - \nabla u_{fh} \parallel_0$$

while

$$E_D^h \leq C_2 h^l \parallel \varphi \parallel_{l+1}, \quad l = min(2, s-1).$$

If $\varphi \in H^S(\Omega_p)$, $s \geq 2$ with $E_D^h = \parallel \varphi - \varphi_h \parallel_l$.

The iterative method we propose to compute the solution of the Stokes/Darcy problem (2.10)-(2.14) consists in solving first Darcy problem in Ω_p imposing the continuity of the normal velocities across Γ. Then, we solve the Stokes problem imposing the continuity of the normal stresses across the interface, using the value of φ on Γ that we have just computed in the porous media domain.

The iterative scheme reads as follows:
Given u_{in}, construct λ_* as (2.20). Then, let $\lambda^0 \in \Lambda_0$ be the initial guess, and for $k \geq 0$:
(i) Find $\varphi_0^{k+1} \in H_p$ such that, for all $\Psi \in H_p$

$$a_p(\varphi_0^{k+1}, \Psi) - \int_\Gamma n\Psi\lambda_0^k = -a_p(E_p\varphi_p, \Psi) + \int_\Gamma n\Psi\lambda_* \tag{3.31}$$

(ii) Find $(u_f^0)^{k+1} \in H_f^\tau, \quad p_f^{k+1} \in Q$:

$$a_f((u_f^0)^{k+1}, w)+b_f(w, p_f^{k+1})-\int_\Gamma \sum_{j=1}^{d-1} \frac{v\alpha_{BJ}}{\sqrt{K}}((u_f^0)^{k+1}\cdot\tau_j)(v\cdot\tau_j) = \int_{\Omega_f} fw$$

$$-a_f(E_f u_{in}, w) - \int_\Gamma \sum_{j=1}^{d-1} \frac{v\alpha_{BJ}}{\sqrt{K}}((E_f u_{in}) \cdot \tau_j) \cdot \tau_j)(v \cdot \tau_j) \quad \forall w \in H_f^\tau \tag{3.32}$$

$$b_f((u_f^0)^{k+1}, q) = -b_f(E_f u_{in}, q) \tag{3.33}$$

with

$$\varphi^{k+1} = \varphi_0^{k+1} + E_p\varphi_p.$$

(iii) Update λ_0^k :

$$\lambda_0^{k+1} = \theta(u_f^{k+1} \cdot n - \lambda_*)_\Gamma + (1-\theta)\lambda_0^k \tag{3.34}$$

θ being a positive relaxation parameter and $u_f^{k+1} = (u_f^0)^{k+1} + E_f u_{in}$.

Lemma 3.1. *The iterative substructuring scheme (3.32)-(3.34) to compute the solution of the finite element approximation of the coupled problem Stokes/Darcy (2.10)-(2.14) is equivalent to a preconditioned Richardson method for the discrete Steklov-Poincare equation (2.28), the preconditioner being the operator S_f introduced in (2.24).*

Therefore we can conclude that (3.31)-(3.34) is equivalent to the preconditioned Richardson scheme: let $\lambda_0^0 \in \Lambda_0$ be given; for $k \geq 0$, find

$$\lambda_0^{k+1} \in \Lambda_0, \quad \lambda_0^{k+1} = \lambda_0^k + \theta S_f^{-1}(\chi - S\lambda_0^k) \tag{3.35}$$

The formulation (3.35) is very convenient for the analysis of convergence of the iterative scheme (3.31)-(3.34). Indeed, with this aim we can apply the following abstract convergence result (see A. Quarteroni and A. Valli(1999) in [11].

Lemma 3.2. *Let X be a (real) Hilbert space and X' be its dual. We consider a linear invertible continuous operator $Q : X \to X'$, which can be split as $Q = Q_1 + Q_2$ where both linear operators. Taken $Z \in X'$, let $x \in X$ be the unknown solution to the equation*

$$Qx = Z \tag{3.36}$$

and consider for its solution the preconditioned Richardson method

$$Q_2(x^{k+1} - x^k) = \theta(Z - Qx^k), \quad k \geq 0$$

θ being a positive relaxation parameter.

Suppose that the following conditions are satisfied:
(i) Q_2 is symmetric, continuous with constant β_2 and coercive with constant α_2;
(ii) Q_1 is continuous with constant β_1;

(iii) Q is coercive with constant α_Q.
Then, for any given $x^0 \in X$ and for any $0 < \theta < \theta_{max}$ with

$$\theta_{max} = \frac{2\alpha_f^3}{\widehat{\beta}_f(\widehat{\beta}_f + \beta_p)^2}$$

the sequence

$$x^{k+1} = x^{k+1} + \theta Q_2^{-1}(Z - Qx^k)$$

converges in X to the solution of problem (3.36).

We can now prove the main result of this section.

Theorem 3.1. *For any choice of the initial guess $\lambda_0^0 \in \Lambda_0$ and for suitable values of the relaxation parameter θ the iterative method (3.31)-(3.34) converges to the solution $(u_f^0, p_f, \varphi_0) \in H_f^\tau \times Q \times H_p$ the coupled Stokes/Darcy problem (2.10)-(2.14).*

Proof. Upon setting $X = \Lambda_0$, $Q = S$, $Q_1 = S_p$, $Q_2 = S_f$ and $Z = \chi$ the proof follows from Lemma 2, whose hypotheses are satisfied thanks to Corollary 1. In fact, for any initial guess $\lambda_0^0 \in \Lambda_0$ with $0 < \theta < \theta_{max}$

$$\theta_{max} = \frac{2\alpha_f^3}{\widehat{\beta}_f(\widehat{\beta}_f + \beta_p)^2}$$

the sequence defined in (**??**) converges to the solution of the Steklov-Poincare equation (2.28).
Taking the limit $k \to \infty$ n the iterative procedure (3.31)-(3.34), it follows that

$$\{((u_f^0)^k, p_p^k, \varphi_0^k)\}_k \to (u_f^0, p_f, \varphi_0).$$

We give a schematic overview of the numerical algorithm
Algorithm 1. Choose an initial guess $(u_f)^0 \cdot n$ on Γ. Then, for $k = 0, 1, ...$ until convergence, do
1. Solve Darcy equation with boundary condition $-K\partial_n \varphi^{k+1} = u_f^u \cdot n$ on Γ.
2. Solve Stokes problem imposing $-n \cdot T(u_f^{k+\frac{1}{2}}, p_f^{k+\frac{1}{2}}) \cdot n = g\varphi^{k+1}$ on Γ.
3. Update: $u_f^{k+1} \cdot n = \theta u_f^{k+\frac{1}{2}} \cdot n + (1-\theta) u_f^k \cdot n$ on Γ, with $\theta \in (0, 1)$.

□

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