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# ON SOME APPLICATIONS OF THE BLASIUS PROBLEM

**Aurel CHIRIŢĂ[1], Horia ENE[2] and Bogdan N. NICOLESCU[3]**

**Abstract**

In this paper, after a brief overview of Blasisus problem, we deal with a numerical approach for the solutions of the boundary layer problem for the free convection in a porous medium. Are also presented and some comparisons between numerical results obtained by us and those given in the literature.

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## 1 Introduction

H. Blasius (1908) found a celebrated solution for laminar boundary layer flow past a flat plate in [3], which is of great historical importance in fluid mechanics. This solution is referred to as the first exact solution of Navier-Stokes equations and these is regarded as an exact one since Balsius transformed the PDE equations of Navier-Stokes to a nonlinear ODE equation [10]. Among many phenomena governed by Balasius's problem, in this paper we are concerned with the free convection near a vertical impermeable surface embedded in a porous medium. This kind of problems belong to a family of heat transfer phenomena which have a wide range of applications in many geophysical and industrial fields [7], [11].

Through the free convection problem we mean the flows of Newtonian viscous fluids arising from the density differences due to temperature gradients, which are describe by various mathematical modelling, which consist of the continuity equation and motion equations (system of partial differential equations) coupled with heat transport equation through porous media (thermal effects) and with the fluid flow through this (based on Darcy's laws) [5], [9].

The thermal stratification problems may arise when there is a continuous discharge of the thermal boundary layer into the medium, for example, a heated vertical surface

[1]High School *Ion Minulescu*, Slatina, Romania, e-mail: chiritaaurel61@yahoo.com
[2]Faculty of Mathematics and Informatics, University of Piteşti, Romania, e-mail: horiaene@yahoo.com
[3]Faculty of Mathematics and Informatics, University of Piteşti, Romania, e-mail: nicolescubogdan81@yahoo.com

embedded in a porous bed which is of limited extent in the direction of the plate. Thermal stratification is a characteristic of all porous media surrounded by differentially heated side walls and enclosed regions of porous structures. These kinds of problems were studied by many researchers [1], [5], [8].

## 2 Derivation of the model

We consider the heat transfer problem from a semi-infinite flat plate embedded in an isotropic homogeneous porous medium through which a viscous Newtonian fluid is flowing. In the framework of a Cartesian co-ordinate system with its origin fixed at the leading edge of the vertical plate, the $x$-axis directed upward along the plate and the $y$-axis normal to it, the governing equations for convective flow in a porous medium under Darcy-Boussinesq law are

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0, \tag{2.1}$$

$$u = -\frac{k}{\mu}\left(\frac{\partial p}{\partial x} + \rho g\right), \tag{2.2}$$

$$v = -\frac{k}{\mu}\frac{\partial p}{\partial y}, \tag{2.3}$$

$$\frac{\partial T}{\partial t} + \frac{1}{\sigma}\left(u\frac{\partial T}{dx} + v\frac{\partial T}{dy}\right) = \frac{\chi}{(\rho c)_m}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right), \tag{2.4}$$

$$\sigma = \frac{n\rho_f C_f + (1-n)\,\rho_s C_s}{\rho_f C_f}, \qquad (\rho c)_m = n\rho_f C_f + (1-n)\,\rho_s C_s, \tag{2.5}$$

$$\rho = \rho_\infty\left[1 - \alpha\left(T - T_\infty\right)\right], \tag{2.6}$$

where $u$ and $v$ are Darcy velocities in the $x$ and $y$ directions, $\rho$ is the density of fluid, $\mu$ the viscosity, $\alpha$ the thermal expansion coefficient, $k$ the permeability of saturated porous medium, $\lambda = \frac{\chi}{\rho_f C_f}$ the equivalent thermal diffusivity and $\chi$ the thermal conductivity of the saturated porous medium.

Eliminating $p$ from (2.2) and (2.3) by cross differentiation, and using (2.6) we obtain

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + \frac{\alpha\rho_\infty k g}{\mu}\frac{\partial T}{\partial y} = 0. \tag{2.7}$$

In the hypothesis of the Prandl theory, if $\delta$ is the boundary thickness, we suppose that: $x \sim 1$, $y \sim \delta$, $u \sim 1$, $v \sim \delta$, $T \sim 1$, $p \sim 1$, and $C_j \sim 1$, $j = f, s$, where $\sim$ denotes the order of magnitude, we find that $\frac{\partial}{\partial x} \sim 1$, $\frac{\partial}{\partial y} \sim \delta^{-1}$ and $\frac{\partial^2}{\partial y^2} \sim \delta^{-2}$. With these, from (2.1), (2.7) from (2.4) we obtain

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.8}$$

$$\frac{\partial u}{\partial y} = \frac{\alpha v_\infty k g}{\mu}\frac{\partial T}{\partial y}, \tag{2.9}$$

$$\frac{\partial T}{\partial t}+\frac{1}{\sigma}\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=\frac{\chi}{(\rho c)_m}\frac{\partial^2 T}{\partial y^2}. \tag{2.10}$$

The system of equations (2.8) –(2.10) is known like the thermal boundary-layer approximation in porous medium. In the steady case, this system is reduced to

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.11}$$

$$\frac{\partial u}{\partial y}=\frac{\alpha \upsilon_\infty kg}{\mu}\frac{\partial T}{\partial y}, \tag{2.12}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\lambda\frac{\partial^2 T}{\partial y^2}. \tag{2.13}$$

For a adopted characteristic length $L$ , the number Rayleigh is given as

$$Ra=\frac{\alpha\rho_\infty kg\left(T_w-T_\infty\right)L}{\mu\lambda}, \tag{2.14}$$

where $T_w$ is the wall temperature. We adopted the following dimensionless variable and unknowns

$$\bar{x}=\frac{x}{L},\quad \bar{y}=\frac{y}{L}Ra^{\frac{1}{2}},\quad \bar{t}=\frac{\lambda Ra}{\sigma L^2}t, \tag{2.15}$$

$$\bar{u}=\frac{Lu}{\lambda Ra},\quad \bar{v}=\frac{Lv}{\lambda Ra^{\frac{1}{2}}},\quad \bar{T}=\frac{T-T_\infty}{T_w-T_\infty}, \tag{2.16}$$

which involve the new form of the system equations (2.8) – (2.10)

$$\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\partial \bar{v}}{\partial \bar{y}}=0, \tag{2.17}$$

$$\frac{\partial \bar{u}}{\partial \bar{y}}=\frac{\partial \bar{T}}{\partial \bar{y}}, \tag{2.18}$$

$$\frac{\partial \bar{T}}{\partial \bar{t}}+\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{T}}{\partial \bar{y}}=\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}. \tag{2.19}$$

If we integrate equation (2.19) with respect to *bary* and using the divergent form and (2.17), we obtain its new form

$$\frac{\partial}{\partial \bar{t}}\int_0^\infty \bar{T}d\bar{y}+\frac{\partial}{\partial \bar{x}}\int_0^\infty \bar{u}\bar{T}d\bar{y}=-\left.\frac{\partial \bar{T}}{\partial \bar{y}}\right|_{\bar{y}=0}. \tag{2.20}$$

In the potential case, we introduce the stream function $\psi$ and so we can express the velocity in the classical form

$$\bar{u}=\frac{\partial \psi}{\partial \bar{y}},\quad \bar{v}=-\frac{\partial \psi}{\partial \bar{x}}, \tag{2.21}$$

and then the equations (2.18) – (2.19) become

$$\frac{\partial \psi}{\partial \bar{y}}=\bar{T}, \tag{2.22}$$

$$\frac{\partial \bar{T}}{\partial \bar{t}} + \frac{\partial \psi}{\partial \bar{y}} \frac{\partial \bar{T}}{\partial \bar{x}} - \frac{\partial \psi}{\partial \bar{x}} \frac{\partial \bar{T}}{\partial \bar{y}} = \frac{\partial^2 \bar{T}}{\partial \bar{y}^2}. \tag{2.23}$$

From the equation (2.22) we deduce that vertical velocity ant temperature distribution are of the same shape. More, this equation may by used to show that at the wall the vertical velocity varies in the same way as the prescribed wall temperature.

In particular, for an isothermal wall, vertical velocity is constant along the wall. This conclusion is a consequence of the conditions imposed at infinity: the vertical velocity is vanishing and the temperature field too. In addition, $\bar{v}$ must be zero at $\bar{y} = 0$.

In the following, we examine some phenomena which are arising in the thermal boundary layer and which are taking in account in some typical problems.

# 3 Some applications of the Blasius equation

## 3.1 The case of a longitudinal line heat source embedding in a porous medium

We consider an unbounded porous medium in a gravitational field, saturated with a fluid at temperature $T_\infty$ at rest containing a longitudinal line heat source. In a plane normal to its length, let the trace of the source by the origin of the Cartesian coordinates $\bar{x}$ and $\bar{y}$, oriented like above and the flow is stationary.

We adopted the following transformation of the variables

$$\eta = \frac{\bar{y}}{\bar{x}^{2/3}}, \tag{3.24}$$

$$\psi = \bar{x}^{1/3} f(\eta), \tag{3.25}$$

$$\bar{T} = \bar{x}^{-1/3} \theta(\eta), \tag{3.26}$$

where $f$ and $\theta$ are the similarity unknowns and $\eta$ is the similarity variable.

From (3.24) –(3.26) the equations (2.22) and (2.23) become

$$f' = \theta, \tag{3.27}$$

$$-(f\theta)' = 3\theta'', \tag{3.28}$$

which are ordinary differential equations, with the derivate to respect to $\eta$. At the system (3.27) – (3.28) are attached the following boundary conditions

$$f(0) = 0, \quad f''(0) = 0, \quad f'(+\infty) = 0, \quad \theta(+\infty) = 0. \tag{3.29}$$

We note that $f''(0) = 0$ in (3.29) also implies that $\theta'(0) = 0$, as a consequence of the fact that (2.18) yields $f'' = \theta'$. A first integration of (3.28) implies that

$$-f\theta = 3\theta', \tag{3.30}$$

where the constants of integration are zero by virtue of the conditions (3.29), so equations (3.30) is written in the new form

$$-ff' = 3f'', \tag{3.31}$$

and form its integration we obtain

$$6f' = C^2 - f^2, \tag{3.32}$$

where $C^2$ is a constant of integration.
If we take $f = CF$, than (3.32) can be integrated directly to give

$$F(\eta) = \tanh\left(\frac{C}{6}\eta\right). \tag{3.33}$$

The constant $C$ is determined from (2.20), which in this case is of the form

$$\int_0^\infty f'\theta d\eta = \text{const.} \tag{3.34}$$

and it represents the convective part of the strength of the source. In [7] the flow pattern is given for the various constant values of $\psi$.

## 3.2 The case of the heated flat plane embedding in an unbounded porous medium

In the case of the heated flat plane embedding in an unbounded porous medium, for the fluid stationary flow, if we adopt the following similarity transformations

$$\eta = \frac{\bar{y}}{\bar{x}^{1/2}}, \tag{3.35}$$

$$\psi = \bar{x}^{1/2} f(\eta), \tag{3.36}$$

$$\bar{T} = \bar{x}^{-1/2}\theta(\eta), \tag{3.37}$$

then from (2.22) and (2.23) we have

$$f' = \theta, \tag{3.38}$$

$$\theta'' = -\frac{1}{2}\theta' f, \tag{3.39}$$

and the boundary conditions become

$$f(0) = 0, \quad \theta(0) = 1, \quad f'(+\infty) = 0, \quad \theta(+\infty) = 0. \tag{3.40}$$

After an obvious calculus in (3.38) and (3.39), we obtain Blasius equations written in $f$

$$f''' + \frac{1}{2}ff'' = 0, \tag{3.41}$$

with following boundary conditions

$$f(0) = 0, \quad f'(0) = 1, \quad f'(+\infty) = 0. \tag{3.42}$$

The equations (3.41) and (3.42) are known as Blasius problem.

### 3.3 The case of the heated flat plane embedding in an unbounded stratified porous medium

For the case of the steady mixed convection flow over a heated semi-infinite vertical flat plate, which is embedded in a thermally stratified fluid-saturated porous medium of variable ambient temperature $T_\infty(x)$, where $T_w(x) > T_\infty(x)$ (heated plate), we assum the following hypotesys:

1. the plate is aligned parallel to a free stream velocity $U(x)$ oriented in the upward or downward directions, where $x$ and $y$ are the Cartesian coordinates along the plate and normal to it, respectively;
2. the convecting fluid and the porous medium are in local thermodynamic equilibrium;
3. the viscous dissipation is neglected;
4. the physical properties of the fluid except the density are constant;
5. the Boussinesq approximation holds.

Under these assumptions, the basic boundary layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.43}$$

$$u = U(x) \pm \frac{gk\beta}{\nu}(T - T_\infty), \tag{3.44}$$

$$u\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial y} = \alpha_m \frac{\partial^2 T}{\partial y^2}. \tag{3.45}$$

subject to the boundary conditions

$$v = 0, \quad T = T_w(x) \quad \text{at} \quad y = 0, \quad T = T_\infty(x) \quad \text{as} \quad y \to \infty, \tag{3.46}$$

where $k$ is the permeability of the porous medium, $v$ is the kinematic viscosity, $\beta$ is the thermal expansion coefficient, $\alpha_m$ is the effective thermal diffusivity, $g$ is the acceleration due to gravity, and $+$ and $-$ signs in Eq. (3.44) correspond to the assisting and opposing flows, respectively (fig.1.)

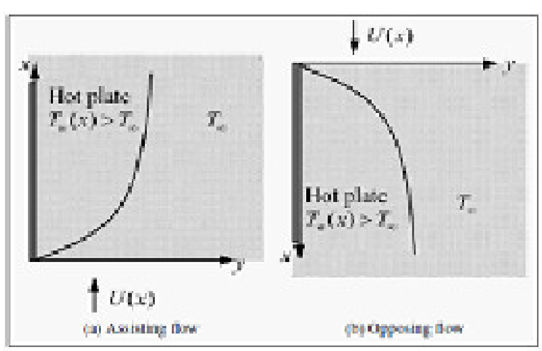


**Fig. 1**. Physical model and coordinate system [1]

In addition, we adopt the following laws for velocity and temperature variation

$$U(x) = ax^m, \quad T_w(x) = T_0 + bx^m, \quad T_\infty(x) = T_0 + cx^m, \tag{3.47}$$

where $a$, $b$ and $c$ are positive constants, $m$ is a parameter and $T_0$ is the ambient temperature at the leading edge. Obvious, the stratified porous medium is stable when $\frac{dT_\infty}{dx} > 0$. If we applied following similarity variables

$$\begin{array}{l} \psi = \alpha_m (2Pe_x)^{1/2} f(\eta), \\ \theta(\eta) = (T - T_\infty) / (T_w - T_0), \ , \\ \eta = (Pe_x/2)^{1/2} (y/x), \end{array} \tag{3.48}$$

where $Pe_x = U(x)\, x/\alpha_m$ is the local Peclet number. That Eqs. (3.43) – (3.45) become

$$f' = 1 + \lambda\theta, \tag{3.49}$$

$$\theta'' + (1+m) f\theta' - 2mf'(S+\theta) = 0, \tag{3.50}$$

and the boundary conditions (3.46) become

$$f(0) = 0, \quad \theta(0) = 1 - S, \quad \theta(0) \to 0, \tag{3.51}$$

where $\lambda$ is the constant mixed convection parameter and $S$ is the constant stratification parameter, which are defined by

$$\lambda = \pm \frac{Ra_x}{Pe_x}, \quad S = \frac{c}{b}, \tag{3.52}$$

and local Rayleigh number for the porous medium given by

$$Ra_x = gk\beta (T_w - T_0)\, x/\alpha_m \nu$$

From the Eqs. (3.49) and (3.50) we can eliminate the unknown $\theta$ and obtain a single equation

$$f''' + (1+m) ff'' - 2m\left(1 - f' - \lambda S\right) f' = 0, \tag{3.53}$$

subject to the boundary conditions

$$f(0) = 0, \quad f'(0) = 1 + \lambda(1-S), \quad f'(\infty) \to 1 \tag{3.54}$$

In the case when $m = 0$ and $\lambda(1-S) = -1$, Eq. (3.53) subjected to (3.54) reduces to the classical Blasius equation, while the case $m = 1$ corresponds to mixed convection flow near the stagnation point on the vertical surface embedded in a porous medium [1]. In this problem is the local Nusselt is written as

$$Nu_x (Pe_x/2)^{-1/2} = -\theta'(0) = -\frac{f''(0)}{\lambda}, \tag{3.55}$$

and it represents the parameter of the problem interest. Also, in [1] there are obtained some remarcable numerical results for the differently combinations of the amonts of the parameters $\lambda$, $S$ and $m$

## 4 Some numerical results

The ordinary differential equation Eq.(3.53) subject to the boundary conditions Eq.(3.54) has been solved numerical by us too, by means Runge-Kutta method, shooting method [6]. Here, we give our numerical results, Fig. 2-3, for the same cases presented in [1].

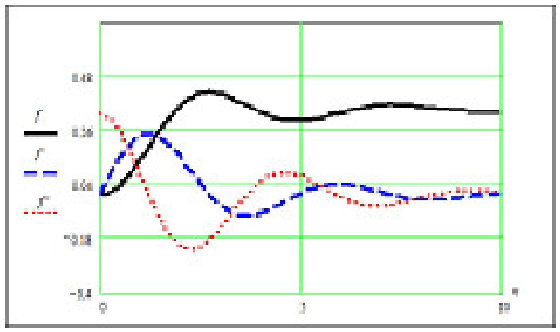

**Fig. 2**. The numerical results by the shooting method for the case $m = 1,\ S = 0,$ $\lambda = -1.$

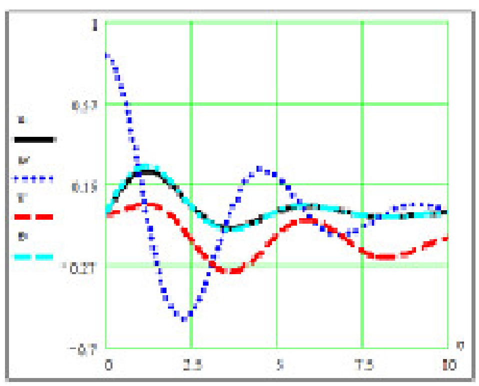


**Fig. 3.** The numerical results by Runge-Kutta method for the case $m = 1,\ S = 0,$ $\lambda = -1.$

From the directly comparisons between our numerical results and the other numerical results given in the literature [1], [2], [3], [4], [10], we can say that there is a good agreement.

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