

A NOTE ON THE SOLUTION SET OF A FRACTIONAL DIFFERENTIAL INCLUSION

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Abstract

We consider a Cauchy problem for a fractional differential inclusion of order $\alpha \in (1, 2]$ involving a nonconvex set-valued map and we prove that the set of selections corresponding to the solutions of the problem considered is a retract of the space of integrable functions on unbounded interval.

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1 Introduction

Differential equations with fractional order have recently proved to be strong tools in the modelling of many physical phenomena. As a consequence there was an intensive development of the theory of differential equations of fractional order ([2, 16, 17, 19] etc.). The study of fractional differential inclusions was initiated by El-Sayed and Ibrahim ([14]). Recently several qualitative results for fractional differential inclusions were obtained in [1, 3, 7-11, 15, 18] etc.. Applied problems require definitions of fractional derivative allowing the utilization of physically interpretable initial conditions. Caputo's fractional derivative, originally introduced in [6] and afterwards adopted in the theory of linear visco elasticity, satisfies this demand. For a consistent bibliography on this topic, historical remarks and examples we refer to [1,16,17,19].

In this paper we study fractional differential inclusions of the form

$$D_c^\alpha x(t) \in F(t, x(t)) \quad a.e. ([0, \infty)), \quad x(0) = x_0, \quad x'(0) = x_1, \quad (1.1)$$

where $\alpha \in (1, 2]$, D_c^α is the Caputo fractional derivative, $F : [0, \infty) \times \mathbf{R} \rightarrow \mathcal{P}(\mathbf{R})$ is a set-valued map and $x_0, x_1 \in \mathbf{R}$, $x_0, x_1 \neq 0$.

The aim of this paper is to prove, when the set-valued map is Lipschitz in the second variable, that the set of selections of the set-valued map F that correspond to the

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solutions of problem (1.1) is a retract of $L^1_{loc}([0, \infty), \mathbf{R})$. The result is essentially based on Bressan and Colombo results ([4]) concerning the existence of continuous selections of lower semicontinuous multifunctions with decomposable values.

We note that in the classical case of differential inclusions topological properties of solution set are obtained using various methods and tools ([5,13,20-22] etc.). On one hand, our result is an extension of Theorem 3.4 in [12] obtained in the case when the interval is bounded and on the other hand, the result in the present paper extends to fractional differential inclusions the main result in [20] obtained in the case of classical differential inclusions.

The paper is organized as follows: in Section 2 we present the notations, definitions and the preliminary results to be used in the sequel and in Section 3 we prove our main result.

2 Preliminaries

Let $T > 0$, $I := [0, T]$ and denote by $\mathcal{L}(I)$ the σ -algebra of all Lebesgue measurable subsets of I . Let X be a real separable Banach space with the norm $|\cdot|$. Denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X . If $A \subset I$ then $\chi_A(\cdot) : I \rightarrow \{0, 1\}$ denotes the characteristic function of A . For any subset $A \subset X$ we denote by $cl(A)$ the closure of A .

The distance between a point $x \in X$ and a subset $A \subset X$ is defined as usual by $d(x, A) = \inf\{|x - a|; a \in A\}$. We recall that Pompeiu-Hausdorff distance between the closed subsets $A, B \subset X$ is defined by $d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}$, $d^*(A, B) = \sup\{d(a, B); a \in A\}$.

As usual, we denote by $C(I, X)$ the Banach space of all continuous functions $x : I \rightarrow X$ endowed with the norm $|x|_C = \sup_{t \in I} |x(t)|$ and by $L^1(I, X)$ the Banach space of all (Bochner) integrable functions $x : I \rightarrow X$ endowed with the norm $|x|_1 = \int_0^T |x(t)| dt$.

We recall first several preliminary results we shall use in the sequel.

A subset $D \subset L^1(I, X)$ is said to be *decomposable* if for any $u, v \in D$ and any subset $A \in \mathcal{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I \setminus A$.

We denote by $\mathcal{D}(I, X)$ the family of all decomposable closed subsets of $L^1(I, X)$.

Next (S, d) is a separable metric space; we recall that a multifunction $G : S \rightarrow \mathcal{P}(X)$ is said to be lower semicontinuous (l.s.c.) if for any closed subset $C \subset X$, the subset $\{s \in S; G(s) \subset C\}$ is closed.

Lemma 1. *Let $F^* : I \times S \rightarrow \mathcal{P}(X)$ be a closed-valued $\mathcal{L}(I) \otimes \mathcal{B}(S)$ -measurable multifunction such that $F^*(t, \cdot)$ is l.s.c. for any $t \in I$.*

Then the multifunction $G : S \rightarrow \mathcal{D}(I, X)$ defined by

$$G(s) = \{v \in L^1(I, X); \quad v(t) \in F^*(t, s) \quad a.e. (I)\}$$

is l.s.c. with nonempty closed values if and only if there exists a continuous mapping $p : S \rightarrow L^1(I, X)$ such that

$$d(0, F^*(t, s)) \leq p(s)(t) \quad a.e. (I), \quad \forall s \in S.$$

Lemma 2. Let $G : S \rightarrow \mathcal{D}(I, X)$ be a l.s.c. multifunction with closed decomposable values and let $\phi : S \rightarrow L^1(I, X)$, $\psi : S \rightarrow L^1(I, \mathbf{R})$ be continuous such that the multifunction $H : S \rightarrow \mathcal{D}(I, X)$ defined by

$$H(s) = cl\{v(\cdot) \in G(s); \quad |v(t) - \phi(s)(t)| < \psi(s)(t) \quad a.e. (I)\}$$

has nonempty values.

Then H has a continuous selection, i.e. there exists a continuous mapping $h : S \rightarrow L^1(I, X)$ such that $h(s) \in H(s) \quad \forall s \in S$.

The proofs of Lemmas 1 and 2 may be found in [4].

Definition 1. ([16]). a) The fractional integral of order $\alpha > 0$ of a Lebesgue integrable function $f : (0, \infty) \rightarrow \mathbf{R}$ is defined by

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds,$$

provided the right-hand side is pointwise defined on $(0, \infty)$ and $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$.

b) The Caputo fractional derivative of order $\alpha > 0$ of a function $f : [0, \infty) \rightarrow \mathbf{R}$ is defined by

$$D_c^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{-\alpha+n-1} f^{(n)}(s) ds,$$

where $n = [\alpha] + 1$. It is assumed implicitly that f is n times differentiable whose n -th derivative is absolutely continuous.

We recall (e.g., [16]) that if $\alpha > 0$ and $f \in C(I, \mathbf{R})$ or $f \in L^\infty(I, \mathbf{R})$ then $(D_c^\alpha I^\alpha f)(t) \equiv f(t)$.

Definition 2. A function $x \in C([0, \infty), \mathbf{R})$ is called a solution of problem (1.1) if there exists a function $f \in L^1_{loc}([0, \infty), \mathbf{R})$ with $f(t) \in F(t, x(t))$, a.e. $[0, \infty)$ such that $D_c^\alpha x(t) = f(t)$ a.e. $[0, \infty)$ and $x(0) = x_0, x'(0) = x_1$.

In this case $(x(\cdot), f(\cdot))$ is called a trajectory-selection pair of problem (1.1).

We shall use the following notations for the solution sets and for the selection sets of problem (1.1).

$$\mathcal{S}(x_0, x_1) = \{x \in C([0, \infty), \mathbf{R}); \quad x \text{ is a solution of (1.1)}\}, \tag{2.1}$$

$$\begin{aligned} \mathcal{T}(x_0, x_1) = \{f \in L^1_{loc}([0, \infty), \mathbf{R}); \\ f(t) \in F(t, x_0 + tx_1 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds) \quad a.e. [0, \infty)\}. \end{aligned} \tag{2.2}$$

3 The main result

In order to prove our topological property of the solution set of problem (1.1) we need the following hypotheses.

Hypothesis 1. i) $F(.,.) : [0, \infty) \times \mathbf{R} \rightarrow \mathcal{P}(\mathbf{R})$ has nonempty compact values and is $\mathcal{L}([0, \infty)) \otimes \mathcal{B}(\mathbf{R})$ measurable.

ii) There exists $L \in L^1_{loc}([0, \infty), \mathbf{R})$ such that, for almost all $t \in [0, \infty)$, $F(t, .)$ is $L(t)$ -Lipschitz in the sense that

$$d_H(F(t, x), F(t, y)) \leq L(t)|x - y| \quad \forall x, y \in \mathbf{R}.$$

iii) There exists $p \in L^1_{loc}([0, \infty), \mathbf{R})$ such that

$$d_H(\{0\}, F(t, 0)) \leq p(t) \quad a.e. [0, \infty).$$

In what follows $I = [0, T]$ and we use the notations

$$\tilde{u}(t) = x_0 + tx_1 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) ds, \quad u \in L^1(I, \mathbf{R}) \quad (3.1)$$

and

$$p_0(u)(t) = |u(t)| + p(t) + L(t)|\tilde{u}(t)|, \quad t \in I \quad (3.2)$$

Let us note that

$$d(u(t), F(t, \tilde{u}(t))) \leq p_0(u)(t) \quad a.e. (I) \quad (3.3)$$

and, since for any $u_1, u_2 \in L^1(I, \mathbf{R})$

$$|p_0(u_1) - p_0(u_2)|_1 \leq (1 + |I^\alpha L(T)|)|u_1 - u_2|_1$$

the mapping $p_0 : L^1(I, \mathbf{R}) \rightarrow L^1(I, \mathbf{R})$ is continuous.

Also define

$$\mathcal{T}_I(x_0, x_1) = \{f \in L^1(I, \mathbf{R}); \quad f(t) \in F(t, x_0 + tx_1 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds) \\ a.e. (I)\}.$$

The next result is proved in [12].

Lemma 3. Assume that Hypothesis 1 is satisfied and let $\phi : L^1(I, \mathbf{R}) \rightarrow L^1(I, \mathbf{R})$ be a continuous map such that $\phi(u) = u$ for all $u \in \mathcal{T}_I(x_0, x_1)$. For $u \in L^1(I, \mathbf{R})$, we define

$$\Psi(u) = \{u \in L^1(I, \mathbf{R}); \quad u(t) \in F(t, \widetilde{\phi(u)}(t)) \quad a.e. (I)\},$$

$$\Phi(u) = \begin{cases} \{u\} & \text{if } u \in \mathcal{T}_I(x_0, x_1), \\ \Psi(u) & \text{otherwise.} \end{cases}$$

Then the multifunction $\Phi : L^1(I, \mathbf{R}) \rightarrow \mathcal{P}(L^1(I, \mathbf{R}))$ is lower semicontinuous with closed decomposable and nonempty values.

In what follows we shall use the following notations

$$I_k = [0, k], \quad k \geq 1, \quad |u|_{1,k} = \int_0^k |u(t)| dt, \quad u \in L^1(I_k, \mathbf{R}).$$

We are able now to prove the main result of this paper.

Theorem 1. Assume that Hypothesis 1 is satisfied, there exists $|I^\alpha L| := \sup_{t \in [0, \infty)} |I^\alpha L(t)| < 1$ and $x_0, x_1 \in \mathbf{R}$.

Then there exists a continuous mapping $G : L^1_{loc}([0, \infty), \mathbf{R}) \rightarrow L^1_{loc}([0, \infty), \mathbf{R})$ such that

- (i) $G(u) \in \mathcal{T}(x_0, x_1), \quad \forall u \in L^1_{loc}([0, \infty), \mathbf{R}),$
- (ii) $G(u) = u, \quad \forall u \in \mathcal{T}(x_0, x_1).$

Proof. We shall prove that for every $k \geq 1$ there exists a continuous mapping $g^k : L^1(I_k, \mathbf{R}) \rightarrow L^1(I_k, \mathbf{R})$ with the following properties

- (I) $g^k(u) = u, \quad \forall u \in \mathcal{T}_{I_k}(x_0, x_1)$
- (II) $g^k(u) \in \mathcal{T}_{I_k}(x_0, x_1), \quad \forall u \in L^1(I_k, \mathbf{R})$
- (III) $g^k(u)(t) = g^{k-1}(u|_{I_{k-1}})(t), \quad \forall t \in I_{k-1}$

If the sequence $\{g^k\}_{k \geq 1}$ is constructed, we define $G : L^1_{loc}([0, \infty), \mathbf{R}) \rightarrow L^1_{loc}([0, \infty), \mathbf{R})$ by

$$G(u)(t) = g^k(u|_{I_k})(t), \quad \forall k \geq 1$$

From (III) and the continuity of each $g^k(\cdot)$ it follows that $G(\cdot)$ is well defined and continuous. Moreover, for each $u \in L^1_{loc}([0, \infty), \mathbf{R})$, according to (II) we have

$$G(u)|_{I_k}(t) = g^k(u|_{I_k})(t) \in \mathcal{T}_{I_k}(x_0, x_1), \quad \forall k \geq 1$$

and thus $G(u) \in \mathcal{T}(x_0, x_1)$.

Fix $\varepsilon > 0$ and for $m \geq 0$ set $\varepsilon_m = \frac{m+1}{m+2}\varepsilon$. For $u \in L^1(I_1, \mathbf{R})$ and $m \geq 0$ define

$$p_0^1(u)(t) = |u(t)| + p(t) + L(t)|\tilde{u}(t)|, \quad t \in I_1$$

and

$$p_{m+1}^1(u) = |I^\alpha L|^m \left(\frac{1}{\Gamma(\alpha)} |p_0^1(u)|_{1,1} + \varepsilon_{m+1} \right).$$

By the continuity of the map $p_0^1(\cdot) = p_0(\cdot)$, already proved, we obtain that $p_m^1 : L^1(I_1, \mathbf{R}) \rightarrow L^1(I_1, \mathbf{R})$ is continuous.

We define $g_0^1(u) = u$ and we shall prove that for any $m \geq 1$ there exists a continuous map $g_m^1 : L^1(I_1, \mathbf{R}) \rightarrow L^1(I_1, \mathbf{R})$ that satisfies

$$(a_1) \quad g_m^1(u) = u, \quad \forall u \in \mathcal{T}_{I_1}(x_0, x_1),$$

$$(b_1) \quad g_m^1(u)(t) \in F(t, \widetilde{g_{m-1}^1(u)}(t)) \quad a.e. (I_1),$$

$$(c_1) \quad |g_1^1(u)(t) - g_0^1(u)(t)| \leq p_0^1(u)(t) + \varepsilon_0 \quad a.e. (I_1),$$

$$(d_1) \quad |g_m^1(u)(t) - g_{m-1}^1(u)(t)| \leq L(t)p_{m-1}^1(u) \quad a.e. (I_1), \quad m \geq 2.$$

For $u \in L^1(I_1, \mathbf{R})$, we define

$$\Psi_1^1(u) = \{v \in L^1(I_1, \mathbf{R}); v(t) \in F(t, \tilde{u}(t)) \quad a.e.(I_1)\},$$

$$\Phi_1^1(u) = \begin{cases} \{u\} & \text{if } u \in \mathcal{T}_{I_1}(x_0, x_1), \\ \Psi_1^1(u) & \text{otherwise.} \end{cases}$$

and by Lemma 3 (with $\phi(u) = u$) we obtain that $\Phi_1^1 : L^1(I_1, \mathbf{R}) \rightarrow \mathcal{D}(I_1, \mathbf{R})$ is lower semicontinuous. Moreover, due to (3.3) the set

$$H_1^1(u) = cl\{v \in \Phi_1^1(u); \quad |v(t) - u(t)| < p_0^1(u)(t) + \varepsilon_0 \quad a.e. (I_1)\}$$

is not empty for any $u \in L^1(I_1, \mathbf{R})$. So applying Lemma 2, we find a continuous selection g_1^1 of H_1^1 that satisfies (a₁)-(c₁).

Suppose we have already constructed $g_i^1(\cdot)$, $i = 1, \dots, m$ satisfying (a₁)-(d₁). Then from (b₁), (d₁) and Hypothesis 1 we get

$$\begin{aligned} d(g_m^1(u)(t), F(t, \widetilde{g_m^1(u)}(t))) &\leq L(t)(|\widetilde{g_{m-1}^1(u)}(t) - \widetilde{g_m^1(u)}(t)|) \\ &\leq |I^\alpha L| p_m^1(u) = L(t)(p_{m+1}^1(u) - r_m) < L(t)p_{m+1}^1(u), \end{aligned} \quad (3.4)$$

where $r_m := |I^\alpha L|^m(\varepsilon_{m+1} - \varepsilon_m) > 0$.

For $u \in L^1(I_1, \mathbf{R})$, we define

$$\Psi_{m+1}^1(u) = \{v \in L^1(I_1, \mathbf{R}); \quad v(t) \in F(t, \widetilde{g_m^1(u)}(t)) \quad a.e. (I_1)\},$$

$$\Phi_{m+1}^1(u) = \begin{cases} \{u\} & \text{if } u \in \mathcal{T}_{I_1}(x_0, x_1), \\ \Psi_{m+1}^1(u) & \text{otherwise.} \end{cases}$$

We apply Lemma 3 (with $\phi(u) = g_m^1(u)$) and obtain that $\Phi_{m+1}^1(\cdot)$ is a lower semicontinuous multifunction with closed decomposable and nonempty values. Moreover, by (3.4), the set

$$H_{m+1}^1(u) = cl\{v \in \Phi_{m+1}^1(u); \quad |v(t) - g_{m+1}^1(u)(t)| < L(t)p_{m+1}^1(u) \quad a.e. (I_1)\}$$

is nonempty for any $u \in L^1(I_1, \mathbf{R})$. With Lemma 2, we find a continuous selection g_{m+1}^1 of H_{m+1}^1 , satisfying (a₁)-(d₁).

Therefore we obtain that

$$|g_{m+1}^1(u) - g_m^1(u)|_{1,1} \leq |I^\alpha L|^m \left(\frac{1}{\Gamma(\alpha)} |p_0^1(u)|_{1,1} + \varepsilon \right)$$

and this implies that the sequence $\{g_m^1(u)\}_{m \in \mathbf{N}}$ is a Cauchy sequence in the Banach space $L^1(I_1, \mathbf{R})$. Let $g^1(u) \in L^1(I_1, \mathbf{R})$ be its limit. The function $s \rightarrow |p_0^1(u)|_{1,1}$ is continuous, hence it is locally bounded and the Cauchy condition is satisfied by $\{g_m^1(u)\}_{m \in \mathbf{N}}$ locally uniformly with respect to u . Hence the mapping $g^1(\cdot) : L^1(I_1, \mathbf{R}) \rightarrow L^1(I_1, \mathbf{R})$ is continuous.

From (a₁) it follows that $g^1(u) = u$, $\forall u \in \mathcal{T}_{I_1}(x_0, x_1)$ and from (b₁) and the fact that F has closed values we obtain that

$$g^1(u)(t) \in F(t, \widetilde{g^1(u)}(t)), \quad a.e. (I_1) \quad \forall u \in L^1(I_1, \mathbf{R}).$$

In the next step of the proof we suppose that we have already constructed the mappings $g^i(\cdot) : L^1(I_i, \mathbf{R}) \rightarrow L^1(I_i, \mathbf{R})$, $i = 2, \dots, k-1$ with the properties (I)-(III) and we shall construct a continuous map $g^k(\cdot) : L^1(I_k, \mathbf{R}) \rightarrow L^1(I_k, \mathbf{R})$ satisfying (I)-(III).

Let $g_0^k : L^1(I_k, \mathbf{R}) \rightarrow L^1(I_k, \mathbf{R})$ be defined by

$$g_0^k(u)(t) = g^{k-1}(u|_{I_{k-1}})(t)\chi_{I_{k-1}} + u(t)\chi_{I_k \setminus I_{k-1}}(t) \quad (3.5)$$

Let us note, first, that $g_0^k(\cdot)$ is continuous. Indeed, if $u_0, u \in L^1(I_k, \mathbf{R})$ one has

$$|g_0^k(u) - g_0^k(u_0)|_{1,k} \leq |g^{k-1}(u|_{I_{k-1}}) - g^{k-1}(u_0|_{I_{k-1}})|_{1,k-1} + \int_{k-1}^k |u(t) - u_0(t)| dt$$

So, using the continuity of $g^{k-1}(\cdot)$ we get the continuity of $g_0^k(\cdot)$.

On the other hand, since $g^{k-1}(u) = u, \forall u \in \mathcal{T}_{I_{k-1}}(x_0, x_1)$ from (3.5) it follows that

$$g_0^k(u) = u, \quad \forall u \in \mathcal{T}_{I_k}(x_0, x_1).$$

For $u \in L^1(I_k, \mathbf{R})$, we define

$$\Psi_1^k(u) = \{w \in L^1(I_k, \mathbf{R}); w(t) = g^{k-1}(u|_{I_{k-1}})(t)\chi_{I_{k-1}}(t) + v(t)\chi_{I_k \setminus I_{k-1}}(t), v(t) \in F(t, \widetilde{g_0^k(u)(t)}) \text{ a.e. } ([k-1, k])\},$$

$$\Phi_1^k(u) = \begin{cases} \{u\} & \text{if } u \in \mathcal{T}_{I_k}(x_0, x_1), \\ \Psi_1^k(u) & \text{otherwise.} \end{cases}$$

We apply Lemma 3 (with $\phi(u) = g_0^k(u)$) and we obtain that $\Phi_1^k(\cdot) : L^1(I_k, \mathbf{R}) \rightarrow \mathcal{D}(I_k, \mathbf{R})$ is lower semicontinuous. Moreover, for any $u \in L^1(I_k, \mathbf{R})$ one has

$$d(g_0^k(t), F(t, \widetilde{g_0^k(u)(t)})) = d(u(t), F(t, \widetilde{g_0^k(u)(t)})\chi_{I_k \setminus I_{k-1}}) \leq p_0^k(u)(t) \text{ a.e. } (I_k), \quad (3.6)$$

where

$$p_0^k(u)(t) = |u(t)| + p(t) + L(t)|\widetilde{g_0^k(u)(t)}|.$$

Obviously, $p_0^k : L^1(I_k, \mathbf{R}) \rightarrow L^1(I_k, \mathbf{R})$ is continuous. For $m \geq 0$ set

$$p_{m+1}^k(u) = |I^\alpha L|^m \left(\frac{k^{\alpha-1}}{\Gamma(\alpha)} |p_0^k(u)|_{1,k} + \varepsilon_{m+1} \right).$$

and by the continuity of $p_0^k(\cdot)$ we infer that $p_m^k : L^1(I_k, \mathbf{R}) \rightarrow L^1(I_k, \mathbf{R})$ is continuous.

We shall prove, next, that for any $m \geq 1$ there exists a continuous map $g_m^k : L^1(I_k, \mathbf{R}) \rightarrow L^1(I_k, \mathbf{R})$ such that

$$(a_k) \quad g_m^k(u)(t) = g^{k-1}(u|_{I_{k-1}})(t) \quad \forall t \in I_{k-1},$$

$$(b_k) \quad g_m^k(u) = u \quad \forall u \in \mathcal{T}_{I_k}(x_0, x_1),$$

$$(c_k) \quad g_m^k(u)(t) \in F(t, \widetilde{g_{m-1}^k(u)(t)}) \text{ a.e. } (I_k),$$

$$(d_k) \quad |g_1^k(u)(t) - g_0^k(u)(t)| \leq p_0^k(u)(t) + \varepsilon_0 \text{ a.e. } (I_k),$$

$$(e_k) \quad |g_m^k(u)(t) - g_{m-1}^k(u)(t)| \leq L(t)p_{m-1}^k(u) \quad a.e. (I_k), \quad m \geq 2.$$

Define

$$H_1^k(u) = cl\{v \in \Phi_1^k(u); |v(t) - g_0^k(u)(t)| < p_0^k(u)(t) + \varepsilon_0 \quad a.e. (I_k)\}.$$

From (3.6), $H_1^k(u) \neq \emptyset \quad \forall u \in L^1(I_1, \mathbf{R})$. Using the continuity of g_0^k, p_0^k and Lemma 2, we obtain a continuous selection g_1^k of H_1^k that satisfies (a_k) - (d_k) .

Assume we have constructed $g_i^k(\cdot)$, $i = 1, \dots, m$ satisfying (a_k) - (e_k) . Then from (e_k) we have

$$\begin{aligned} d(g_m^k(u)(t), F(t, \widetilde{g_m^k(u)(t)})) &\leq L(t)(|\widetilde{g_{m-1}^k(u)(t)} - \widetilde{g_m^k(u)(t)}|) \\ &\leq |I^\alpha L| p_m^k(u) = L(t)(p_{m+1}^k(u) - r_m) < L(t)p_{m+1}^k(u), \end{aligned} \quad (3.7)$$

where $r_m := |I^\alpha L|^m(\varepsilon_{m+1} - \varepsilon_m) > 0$.

For $u \in L^1(I_k, \mathbf{R})$, we define

$$\begin{aligned} \Psi_{m+1}^k(u) &= \{w \in L^1(I_k, \mathbf{R}); w(t) = g^{k-1}(u|_{I_{k-1}})(t)\chi_{I_{k-1}}(t) + \\ &v(t)\chi_{I_k \setminus I_{k-1}}(t), \quad v(t) \in F(t, \widetilde{g_m^k(u)(t)}) \quad a.e. ([k-1, k])\}, \end{aligned}$$

$$\Phi_{m+1}^k(u) = \begin{cases} \{u\} & \text{if } u \in \mathcal{T}_{I_k}(x_0, x_1), \\ \Psi_{m+1}^k(u) & \text{otherwise.} \end{cases}$$

With Lemma 3 we infer that $\Phi_{m+1}^k(\cdot) : L^1(I_k, \mathbf{R}) \rightarrow \mathcal{P}(L^1(I_k, \mathbf{R}))$ is lower semicontinuous with closed decomposable and nonempty values. By (3.7) the set

$$H_{m+1}^k(u) = cl\{v \in \Phi_{m+1}^k(u); |v(t) - g_{m+1}^k(u)(t)| < L(t)p_{m+1}^k(u) \quad a.e. (I_k)\}$$

is nonempty for any $u \in L^1(I_k, \mathbf{R})$. So, applying Lemma 2, we deduce a continuous selection g_{m+1}^k of H_{m+1}^k , satisfying (a_k) - (e_k) .

By (e_k) one has

$$|g_{m+1}^k(u) - g_m^k(u)|_{1,k} \leq |I^\alpha L|^m \left[\frac{k^{\alpha-1}}{\Gamma(\alpha)} |p_0^k(u)|_{1,1} + \varepsilon \right].$$

Therefore, with a similar proof as in the case $k = 1$, we find that the sequence $\{g_m^k(u)\}_{m \in \mathbf{N}}$ converges to some $g^k(u) \in L^1(I_k, \mathbf{R})$ and the mapping $g^k(\cdot) : L^1(I_k, \mathbf{R}) \rightarrow L^1(I_k, \mathbf{R})$ is continuous.

By (a_k) we have that

$$g^k(u)(t) = g^{k-1}(u|_{I_{k-1}})(t) \quad \forall t \in I_{k-1},$$

by (b_k) $g^k(u) = u$, $\forall u \in \mathcal{T}_{I_k}(x_0, x_1)$ and from (c_k) and the fact that F has closed values we obtain that

$$g^k(u)(t) \in F(t, \widetilde{g^k(u)(t)}), \quad a.e. (I_k) \quad \forall u \in L^1(I_k, \mathbf{R}).$$

Therefore $g^k(\cdot)$ satisfies the properties (I), (II) and (III).

Remark 1. We recall that if Y is a Hausdorff topological space, a subspace X of Y is called retract of Y if there is a continuous map $h : Y \rightarrow X$ such that $h(x) = x$, $\forall x \in X$.

Therefore, by Theorem 1, for any $x_0, x_1 \in \mathbf{R}$, the set $\mathcal{T}(x_0, x_1)$ of selections that correspond to solutions of (1.1) is a retract of the Banach space $L_{loc}^1([0, \infty), \mathbf{R})$.

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