

ON THE ASYMPTOTIC BEHAVIOR OF ELLIPTIC PROBLEMS IN PERIODICALLY PERFORATED DOMAINS WITH MIXED-TYPE BOUNDARY CONDITIONS

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Abstract

Using the periodic unfolding method, we analyze the asymptotic behavior of a class of elliptic equations with highly oscillating coefficients, in a perforated periodic domain. We consider, in each period, two types of holes and we impose, on their boundaries, different conditions of Neumann and/or Signorini types. The limit problems contain additional terms which capture both the influence of the size of the holes and the effect of the conditions imposed on their boundaries.

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1 Introduction

In this paper, the homogenization of a class of elliptic second-order equations with highly oscillating coefficients, in a perforated periodic domain, is addressed. We consider an ε -periodic perforated structure, with two holes of different sizes in each period. We analyze three distinct problems. In the first one, we impose a Signorini and, respectively, a Neumann condition on the boundary of the holes and we deal with the so-called “critical case” for both of them. The homogenized problem, written as a variational equation, contains two additional terms coming from the particular geometry. These new terms, a right hand side term and a “strange” one, capture the two sources of oscillations involved in this problem, i.e. those arising from the special size of the holes and those due to the periodic heterogeneity of the medium. In the second problem, we keep the same boundary conditions, but we change the magnitude of the Signorini holes, considering them of being of the same size as the period of the domain, i.e. “big holes”. In this case, we obtain, at

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the limit, an obstacle problem, expressed as a variational inequality, containing again the extra term generated by the Neumann holes. In the last problem we analyze in this paper, the boundary conditions imposed on both holes are of Neumann type and the size of the holes is critical. We prove that the limit problem is given by a variational equation with two additional right hand side terms. Let us mention that in all the above problems, on the exterior fixed boundary of the perforated domain an homogeneous Dirichlet condition is prescribed.

The homogenization of elliptic problems in perforated domains, with various boundary conditions, including the mixed or nonlinear ones, was addressed in the literature by many authors. We mention here the seminal work of D. Cioranescu, F. Murat [15], which deals with the homogenization of the Poisson equation with a Dirichlet condition in perforated domains, putting into evidence, in the case of critical holes, the appearance of a “strange” term. Their results were extended later, using different techniques, to heterogeneous media by N. Ansini, A. Braides [1], G. Dal Maso, F. Murat [18] and D. Cioranescu, A. Damlamian, G. Griso, D. Onofrei [8]. For mixed and nonlinear boundary conditions, the interested reader is referred to C. Conca, P. Donato [16], D. Onofrei [20], D. Cioranescu, P. Donato [9], D. Cioranescu, P. Donato, H. Ene [10], D. Cioranescu, P. Donato, R. Zaki [11], A. Capatina, H. Ene [3], C. Conca, F. Murat, C. Timofte [17], D. Cioranescu, Hammouda [14].

Our approach is based on the periodic unfolding method introduced, for fixed domains, by D. Cioranescu, A. Damlamian, G. Griso [6], [7], A. Damlamian [19] and extended later, to perforated domains, by D. Cioranescu, P. Donato, R. Zaki [11], [12], D. Cioranescu, A. Damlamian, G. Griso, D. Onofrei [8], D. Onofrei [20].

The structure of this paper is the following one: in Section 2, we give the geometrical setting and the formulations of our three problems, while in Section 3 we present the corresponding convergence results.

2 Notation and formulations of the problems

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set such that $|\partial\Omega| = 0$ and let $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^n$ be the reference cell.

Let ϵ be a real parameter taking values in a sequence of positive numbers converging to zero. We shall consider an ϵY periodic perforated structure with two kind of holes: some of size $\epsilon\delta_1$ and the other ones of size $\epsilon\delta_2$, with δ_1 and δ_2 depending on ϵ and going to zero as ϵ goes to zero. More precisely, we consider two open sets T_1 and T_2 with smooth boundaries such that $T_1 \subset\subset Y$, $T_2 \subset\subset Y$ and $T_1 \cap T_2 = \emptyset$ and we denote the above mentioned holes by

$$T_1^{\epsilon\delta_1} = \bigcup_{\xi \in \mathbb{Z}^n} \epsilon(\xi + \delta_1 T_1),$$

$$T_2^{\epsilon\delta_2} = \bigcup_{\xi \in \mathbb{Z}^n} \epsilon(\xi + \delta_2 T_2).$$

Let $Y_{\delta_1\delta_2} = Y \setminus (\overline{\delta_1 T_1} \cup \overline{\delta_2 T_2})$ be the part occupied by the material in the cell and suppose that it is connected.

The perforated domain $\Omega_{\epsilon,\delta_1\delta_2}$ with holes of size of order $\epsilon\delta_1$ and of size of order $\epsilon\delta_2$ at the same time is defined by

$$\Omega_{\epsilon,\delta_1\delta_2} = \Omega \setminus (T_1^{\epsilon\delta_1} \cup T_2^{\epsilon\delta_2}) = \left\{ x \in \Omega \mid \left\{ \frac{x}{\epsilon} \right\}_Y \in Y_{\delta_1\delta_2} \right\}.$$

Let $A \in L^\infty(\Omega)^{n \times n}$ be a Y -periodic symmetric matrix. We suppose that there exist two positive constants α and β , with $0 < \alpha < \beta$, such that

$$\alpha|\xi|^2 \leq A(y)\xi \cdot \xi \leq \beta|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \quad \forall y \in Y.$$

Moreover, we assume that A is continuous at the point 0.

Given a function $f \in L^2(\Omega)$, we consider, in the first step, the following problem

$$\left\{ \begin{array}{l} -\operatorname{div}(A^\epsilon \nabla u_{\epsilon,\delta_1\delta_2}) = f \quad \text{in } \Omega_{\epsilon,\delta_1\delta_2}, \\ u_{\epsilon,\delta_1\delta_2} \geq 0, \quad A^\epsilon \nabla u_{\epsilon,\delta_1\delta_2} \cdot \nu_{T_1} \geq 0, \quad u_{\epsilon,\delta_1\delta_2} A^\epsilon \nabla u_{\epsilon,\delta_1\delta_2} \cdot \nu_{T_1} = 0 \quad \text{on } \partial T_1^{\epsilon\delta_1}, \\ A^\epsilon \nabla u_{\epsilon,\delta_1\delta_2} \cdot \nu_{T_2} = g^{\epsilon\delta_2} \quad \text{on } \partial T_2^{\epsilon\delta_2}, \\ u_{\epsilon,\delta_1\delta_2} = 0 \quad \text{on } \partial_{\text{ext}} \Omega_{\epsilon,\delta_1\delta_2}, \end{array} \right. \quad (1)$$

where

$$A^\epsilon(x) = A\left(\frac{x}{\epsilon}\right)$$

and

$$g^{\epsilon\delta_2}(x) = g\left(\frac{1}{\delta_2} \left\{ \frac{x}{\epsilon} \right\}_Y\right) \quad \text{a.e. } x \in \partial T_2^{\epsilon\delta_2},$$

$g \in L^2(\partial T_2)$ being a Y -periodic given function.

In (1), ν_{T_1} and ν_{T_2} are the unit exterior normals to T_1 and, respectively, T_2 .

The variational formulation of (1) is the following inequality:

$$\left\{ \begin{array}{l} \text{Find } u_{\epsilon,\delta_1\delta_2} \in K_{\delta_1\delta_2}^\epsilon \text{ such that} \\ \int_{\Omega_{\epsilon,\delta_1\delta_2}} A^\epsilon \nabla u_{\epsilon,\delta_1\delta_2} \cdot (\nabla v - \nabla u_{\epsilon,\delta_1\delta_2}) dx \geq \int_{\Omega_{\epsilon,\delta_1\delta_2}} f_\epsilon(v - u_{\epsilon,\delta_1\delta_2}) dx \\ + \int_{\partial T_2^{\epsilon\delta_2}} g^{\epsilon\delta_2}(v - u_{\epsilon,\delta_1\delta_2}) ds \quad \forall v \in K_{\delta_1\delta_2}^\epsilon, \end{array} \right. \quad (2)$$

where

$$V_{\delta_1\delta_2}^\epsilon = \{v \in H^1(\Omega_{\epsilon,\delta_1\delta_2}) \mid v = 0 \text{ on } \partial_{\text{ext}} \Omega_{\epsilon,\delta_1\delta_2}\}$$

and

$$K_{\delta_1 \delta_2}^\epsilon = \{v \in V_{\delta_1 \delta_2}^\epsilon \mid v \geq 0 \text{ on } \partial T_1^{\epsilon \delta_1}\}.$$

In this first step, we are interested in the following two cases:

$$(i) \quad \begin{cases} k_1 = \lim_{\epsilon \rightarrow 0} \frac{\delta_1^{\frac{n}{2}-1}}{\epsilon} \in (0, +\infty), \\ k_2 = \lim_{\epsilon \rightarrow 0} \frac{\delta_2^{n-1}}{\epsilon} \in (0, +\infty), \end{cases}$$

which means that we are dealing with the case of critical size, both for the Signorini holes and, respectively, the Neumann ones, and

$$(ii) \quad \begin{cases} \delta_1 = \mathcal{O}(1), \\ k_2 = \lim_{\epsilon \rightarrow 0} \frac{\delta_2^{n-1}}{\epsilon} \in (0, +\infty), \end{cases}$$

which means that we are dealing with the case in which the Signorini holes are of the same size as the period, while the Neumann holes are still of the critical size.

In the second step, we shall consider the case in which both holes are of Neumann type and of the same size, the critical one. In the above notation, this means that $\delta_1 = \delta_2$ and, for simplicity, their common value will be denoted by δ . So, in this third situation,

$$(iii) \quad k = \lim_{\epsilon \rightarrow 0} \frac{\delta^{n-1}}{\epsilon} \in (0, +\infty)$$

and the problem is

$$\begin{cases} -\operatorname{div}(A^\epsilon \nabla u_{\epsilon, \delta}) = f & \text{in } \Omega_{\epsilon, \delta}, \\ A^\epsilon \nabla u_{\epsilon, \delta} \cdot \nu_{T_1} = g_1^{\epsilon \delta} & \text{on } \partial T_1^{\epsilon \delta}, \\ A^\epsilon \nabla u_{\epsilon, \delta} \cdot \nu_{T_2} = g_2^{\epsilon \delta} & \text{on } \partial T_2^{\epsilon \delta}, \\ u_{\epsilon, \delta} = 0 & \text{on } \partial_{ext} \Omega_{\epsilon, \delta}, \end{cases} \quad (3)$$

where $\Omega_{\epsilon, \delta} = \Omega_{\epsilon, \delta \delta}$ and

$$g_i^{\epsilon \delta}(x) = g_i \left(\frac{1}{\delta} \left\{ \frac{x}{\epsilon} \right\}_Y \right) \quad \text{a.e. } x \in \partial T_i^{\epsilon \delta},$$

$g_i \in L^2(\partial T_i)$ ($i = 1, 2$) being Y -periodic given functions.

The variational formulation of this problem is

$$\begin{cases} \text{Find } u_{\epsilon, \delta} \in V_\delta^\epsilon \text{ such that} \\ \int_{\Omega_{\epsilon, \delta}} A^\epsilon \nabla u_{\epsilon, \delta} \cdot \nabla v \, dx = \int_{\Omega_{\epsilon, \delta}} f_\epsilon v \, dx + \int_{\partial T_1^{\epsilon \delta}} g_1^{\epsilon \delta} v \, ds + \int_{\partial T_2^{\epsilon \delta}} g_2^{\epsilon \delta} v \, ds \\ \forall v \in V_\delta^\epsilon, \end{cases} \quad (4)$$

where $V_\delta^\epsilon = V_{\delta\delta}^\epsilon$.

Classical results (see, for example, [21], [2]) ensure the existence and the uniqueness of weak solutions of the problems (2) and (4).

3 Homogenization results

All the convergence results stated in this paper are obtained using the periodic unfolding method (see [6], [7]). In their formulations, \mathcal{T}_ϵ is the classical unfolding operator.

We first discuss the problem (2), case (i). In order to state the convergence result, let us introduce the following functional space

$$K_{T_1} = \{v \in L^{2^*}(\mathbb{R}^n) ; \nabla v \in L^2(\mathbb{R}^n), v = \text{ct. on } T_1\}$$

where 2^* is the Sobolev exponent $\frac{2n}{n-2}$ associated to 2.

Also, for $i = \overline{1, n}$, let us consider χ_i the solution of the cell problem

$$\begin{cases} \chi_i \in H_{per}^1(Y), \\ \int_Y A \nabla(\chi_i - y_i) \cdot \nabla \phi \, dy = 0 \quad \forall \phi \in H_{per}^1(Y) \end{cases} \quad (5)$$

and θ the solution of the problem

$$\begin{cases} \theta \in K_{T_1}, \quad \theta(T_1) = 1, \\ \int_{\mathbb{R}^n \setminus T_1} A(0) \nabla \theta \cdot \nabla v \, dz = 0 \quad \forall v \in K_{T_1} \text{ with } v(T_1) = 0. \end{cases} \quad (6)$$

The main result in this case is stated in the following theorem:

Theorem 1. *Let $u_{\epsilon, \delta_1 \delta_2} \in K_{\delta_1 \delta_2}^\epsilon$ be the solution of the variational inequality (2) in the hypothesis (i). Then, there exists $u \in H_0^1(\Omega)$ such that*

$$\mathcal{T}_\epsilon(u_{\epsilon, \delta_1 \delta_2}) \rightharpoonup u \quad \text{weakly in } L^2(\Omega; H^1(Y)) \quad (7)$$

and u is the unique solution of the homogenized problem

$$\begin{cases} u \in H_0^1(\Omega), \\ \int_\Omega A^{hom} \nabla u \cdot \nabla \varphi \, dx - k_1^2 \int_\Omega \mu u^- \varphi \, dx = \int_\Omega f \varphi \, dx \\ + k_2 |\partial T_2| \mathcal{M}_{\partial T_2}(g) \int_\Omega \varphi \, dx \quad \forall \varphi \in H_0^1(\Omega). \end{cases} \quad (8)$$

In (8), A^{hom} is the classical homogenized matrix defined, in terms of χ_i solution of (5), as

$$A_{ij}^{\text{hom}} = \int_Y \left(a_{ij}(y) - \sum_{k=1}^n a_{ik}(y) \frac{\partial \chi_j}{\partial y_k}(y) \right) dy \quad (9)$$

and μ is the capacity of the set T_1 , given by

$$\mu = \int_{\mathbb{R}^n \setminus T_1} A(0) \nabla_z \theta \cdot \nabla_z \theta \, dz,$$

where θ verifies (6).

As we already mentioned, in the limit problem (8) we can see the presence of two extra terms generated by the suitable sizes of our holes. The strange term, depending on the matrix A , charges only the negative part u^- of the solution.

Proof. Let us sketch the proof. The variational inequality (2) is equivalent to the following minimization problem

$$\begin{cases} \text{Find } u_{\epsilon, \delta_1 \delta_2} \in K_{\delta_1 \delta_2}^\epsilon \text{ such that} \\ J_{\delta_1 \delta_2}^\epsilon(u_{\epsilon, \delta_1 \delta_2}) \leq J_{\delta_1 \delta_2}^\epsilon(v) \quad \forall v \in K_{\delta_1 \delta_2}^\epsilon, \end{cases} \quad (10)$$

where

$$J_{\delta_1 \delta_2}^\epsilon(v) = \frac{1}{2} \int_{\Omega_{\epsilon, \delta_1 \delta_2}} A_\epsilon \nabla v \cdot \nabla v \, dx - \int_{\Omega_{\epsilon, \delta_1 \delta_2}} f v \, dx - \int_{\partial T_2^{\epsilon \delta_2}} g^{\epsilon \delta_2} v \, ds. \quad (11)$$

From the problem (2), it follows that there exists a constant C such that

$$\|u_{\epsilon, \delta_1 \delta_2}\|_{H^1(\Omega_{\epsilon, \delta_1 \delta_2})} \leq C. \quad (12)$$

Since $u_{\epsilon, \delta_1 \delta_2} \in V_{\delta_1 \delta_2}^\epsilon$, we can assume that, up to a subsequence, there exists $u \in H_0^1(\Omega)$ such that

$$\mathcal{T}_\epsilon(u_{\epsilon, \delta_1 \delta_2}) \rightharpoonup u \quad \text{weakly in } L^2(\Omega; H^1(Y)). \quad (13)$$

Using a suitable test function, namely $v^{\epsilon \delta_1} = h_\epsilon^+ - w^{\epsilon \delta_1} h_\epsilon^-$, where

$$w^{\epsilon \delta_1}(x) = 1 - \theta \left(\frac{1}{\delta_1} \left\{ \frac{x}{\epsilon} \right\}_Y \right) \quad \forall x \in \mathbb{R}^n,$$

$$h_\epsilon(x) = \varphi(x) - \epsilon \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(x) \chi_i \left(\frac{x}{\epsilon} \right),$$

with θ and χ_i given by (6), respectively, (5) and $\varphi \in \mathcal{D}(\Omega)$, we prove that

$$\lim_{\epsilon \rightarrow 0} J_{\delta_1 \delta_2}^\epsilon(u_{\epsilon, \delta_1 \delta_2}) = J_0(u) = \min_{\varphi \in H_0^1(\Omega)} J_0(\varphi). \quad (14)$$

In (14), J_0 is the functional defined by

$$\begin{aligned}
 J_0(\varphi) = & \frac{1}{2} \int_{\Omega} A^{hom} \nabla \varphi \cdot \nabla \varphi \, dx + \frac{1}{2} k_1^2 \int_{\Omega} \mu(\varphi^-)^2 \, dx \\
 & - \int_{\Omega} f \varphi \, dx + k_2 |\partial T_2| \mathcal{M}_{\partial T_2}(g) \int_{\Omega} \varphi \, dx.
 \end{aligned} \tag{15}$$

As μ , the capacity of T_1 , is non-negative, by Lax-Milgram theorem, it follows that the minimum point for the functional J_0 is unique. This means that the whole sequence $\mathcal{T}_\epsilon(u_{\epsilon, \delta_1 \delta_2})$ converges to u .

The minimization problem

$$J_0(u) = \min_{\varphi \in H_0^1(\Omega)} J_0(\varphi) \tag{16}$$

is equivalent with the problem (8) and this ends the proof of Theorem 1 (for more details, we refer to [4]).

□

In the sequel, we shall analyze the problem (2) in the hypothesis (ii), i.e. the case in which the Signorini holes are of the same size as the period and the Neumann holes are critical. The homogenized problem is a variational inequality corresponding to an obstacle problem with an additional term coming from the influence of the Neumann holes. The convergence result is the following one:

Theorem 2. *Let $u_{\epsilon, \delta_1 \delta_2} \in K_{\delta_1 \delta_2}^\epsilon$ be the solution of the variational inequality (2) in the hypothesis (ii). Then, there exists $u \in H_0^1(\Omega)$ such that*

$$\mathcal{T}_\epsilon(u_{\epsilon, \delta_1 \delta_2}) \rightharpoonup u \quad \text{weakly in } L^2(\Omega; H^1(Y^*)) \tag{17}$$

and u is the unique solution of the homogenized problem

$$\left\{ \begin{array}{l} u \in H_0^1(\Omega), u \geq 0 \text{ in } \Omega, \\ \int_{\Omega} A^0 \nabla u \nabla (v - u) \, dx \geq \int_{\Omega} f(v - u) \, dx + k_2 |\partial T_2| \mathcal{M}_{\partial T_2}(g) \int_{\Omega} (v - u) \, dx \\ \forall v \in H_0^1(\Omega), v \geq 0 \text{ in } \Omega. \end{array} \right.$$

Here, $A^0 = (a_{ij}^0)$ is the classical homogenized matrix, whose entries are defined as follows:

$$a_{ij}^0 = \frac{1}{|Y^*|} \int_{Y^*} \left(a_{ij}(y) + a_{ik}(y) \frac{\partial \chi_j}{\partial y_k}(y) \right) dy,$$

in terms of the functions χ_j , $j = 1, \dots, n$, solutions of the cell problems

$$\begin{cases} \chi_j \in H_{per}^1(Y^*), \int_{Y^*} \chi_j = 0, \\ -\operatorname{div}_y A(y)(D_y \chi_j + \mathbf{e}_j) = 0 \quad \text{in } Y^*, \\ A(y)(D \chi_j + \mathbf{e}_j) \cdot \boldsymbol{\nu}_{T_1} = 0 \quad \text{on } \partial T_1, \end{cases}$$

where $Y^* = Y \setminus T_1$ and \mathbf{e}_i , $1 \leq i \leq n$, are the elements of the canonical basis in \mathbb{R}^n .

Proof. For proving the above result, we use the test function

$$v^\varepsilon(x) = \left(\varphi(x) + \sum_i \varepsilon \frac{\partial \varphi}{\partial x_i}(x) \chi_i\left(\frac{x}{\varepsilon}\right) + \delta \right) \psi(x),$$

with $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geq 0$, $\psi \in \mathcal{D}(\Omega)$, $0 \leq \psi \leq 1$ such that $\psi \equiv 1$ on $\operatorname{supp}(\varphi)$ and δ chosen such that $v^\varepsilon \in K_{\delta_1 \delta_2}^\varepsilon$. For details, see [5]. □

In the last problem we consider, i.e. the problem (4) with the condition (iii), the limit problem is expressed as a variational equality with two additional terms generated by the two critical Neumann holes. In this case, we take as test function

$$h_\varepsilon(x) = \varphi(x) - \varepsilon \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(x) \chi_i\left(\frac{x}{\varepsilon}\right),$$

with χ_i given by (5) and $\varphi \in \mathcal{D}(\Omega)$.

Arguing as in [5], we are led to the following result:

Theorem 3. *Let $u_{\varepsilon, \delta} \in V_\delta^\varepsilon$ be the solution of the variational inequality (4). Under the hypothesis (iii), there exists $u \in H_0^1(\Omega)$ such that*

$$\mathcal{T}_\varepsilon(u_{\varepsilon, \delta_1 \delta_2}) \rightharpoonup u \quad \text{weakly in } L^2(\Omega; H^1(Y)), \quad (18)$$

u being the unique solution of the homogenized problem

$$\begin{cases} u \in H_0^1(\Omega), \\ \int_{\Omega} A^{hom} \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx + k |\partial T_1| \mathcal{M}_{\partial T_1}(g_1) \int_{\Omega} v dx + k |\partial T_2| \mathcal{M}_{\partial T_2}(g_2) \int_{\Omega} v dx \\ \forall v \in H_0^1(\Omega), \end{cases}$$

with A^{hom} defined by (9).

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