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SOME PROPERTIES OF A GENERAL INTEGRAL OPERATOR

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Abstract

In this paper, we consider a general integral operator $G_n(z)$. Considering the classes $\mathcal{S}(\alpha)$, $\mathcal{K}(\alpha)$, and G_b, we derive some properties for this integral operator.

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1 Indroduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$$

and satisfy the following normalization condition

$$f(0) = f'(0) - 1 = 0.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions f which are univalent in $\mathbb{U}$.
A function $f \in \mathcal{A}$ is the starlike function of order α, $0 \le \alpha < 1$ if f satisfies the inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad z \in \mathbb{U}.$$

For $0 \le \alpha < 1$, we denote by $\mathcal{S}(\alpha)$ the class of starlike functions of order α.
A function $f \in \mathcal{A}$ is the convex function of order $\alpha, 0 \le \alpha < 1$ if f satisfies the inequality

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha, \qquad z \in \mathbb{U}.$$

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For $0 \leq \alpha < 1$, we denote by $\mathcal{K}(\alpha)$ the class of convex functions of order α.

In [7], for $0 < b \leq 1$ Silverman considered the class

$$G_b = \left\{ f \in \mathcal{A} : \left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right| < b \left| \frac{zf'(z)}{f(z)} \right|, \qquad z \in \mathbb{U} \right\}. \tag{1}$$

For $f_i(z)$, $g_i(z) \in \mathcal{A}$ and $\alpha_i > 0$, $\gamma_i > 0$, $(i = 1, ..., n)$, we define the integral operator $G_n(z)$ given by

$$G_n(z) = \int_0^z \prod_{i=1}^{n} \left(\left(\frac{f_i(t)}{t} \right)^{\alpha_i} \left(g_i'(t) \right)^{\gamma_i} \right) dt. \tag{2}$$

Remark 1. *For $n = 1$, $\alpha_1 = 1$, $\gamma_1 = 0$ and $f_1(z) = f(z) \in \mathcal{A}$, we obtain Alexander integral operator introduced in 1915 in [1]*

$$I(z) = \int_0^z \frac{f(t)}{t} dt \qquad z \in \mathbb{U}.$$

Remark 2. *For $n = 1$, $\alpha_1 = \alpha$, $\gamma_1 = 0$ and $f_1(z) = f(z) \in \mathcal{A}$, we obtain the integral operator*

$$I_\alpha(z) = \int_0^z \left(\frac{f(t)}{t} \right)^{\alpha} dt \qquad z \in \mathbb{U}$$

studied in [4], [5] and [6].

Remark 3. *For $\alpha_i > 0$ $(i = 1, ..., n)$, $\gamma_1 = \gamma_2 = ... = \gamma_n = 0$ and $f_i(z) \in \mathcal{A}$, we obtain the integral operator*

$$I_n(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\alpha_i} dt$$

studied in [2], [3].

2 Main results

Theorem 1. *For a natural number $n \geq 1$, let $f_i \in \mathcal{S}(\beta_i)$, $0 \leq \beta_i < 1$ and $g_i \in \mathcal{K}(\lambda_i)$, $0 \leq \lambda_i < 1$, $i = 1, ..., n$. For positive real numbers $\alpha_i, \gamma_i > 0$, $i = 1, ..., n$ satisfying*

$$\sum_{i=1}^{n} \left(\alpha_i \left(1 - \beta_i \right) + \gamma_i \left(1 - \lambda_i \right) \right) < 1,$$

the integral operator $G_n(z)$ given by (2) defines a convex function of order

$$\lambda = 1 + \sum_{i=1}^{n} \left(\alpha_i \left(\beta_i - 1 \right) + \gamma_i \left(\lambda_i - 1 \right) \right).$$

Proof. From (2) we compute the first and second derivatives of $G_n(z)$. We obtain:

$$G'_n(z) = \prod_{i=1}^{n} \left(\left(\frac{f_i(z)}{z} \right)^{\alpha_i} (g'_i(z))^{\gamma_i} \right)$$

and

$$G''_n(z) = \sum_{i=1}^{n} \left[\alpha_i \left(\frac{f_i(z)}{z} \right)^{\alpha_i - 1} \left(\frac{z f'_i(z) - f_i(z)}{z^2} \right) (g'_i(z))^{\gamma_i} \right] \prod_{\substack{k=1 \\ k \neq i}}^{n} \left(\left(\frac{f_k(z)}{z} \right)^{\alpha_k} (g'_k(z))^{\gamma_k} \right)$$

$$+ \sum_{i=1}^{n} \left[\left(\frac{f_i(z)}{z} \right)^{\alpha_i} \gamma_i (g'_i(z))^{\gamma_i - 1} g''_i(z) \right] \prod_{\substack{k=1 \\ k \neq i}}^{n} \left(\left(\frac{f_k(z)}{z} \right)^{\alpha_k} (g'_k(z))^{\gamma_k} \right).$$

After the calculus, we have

$$\frac{zG''_n(z)}{G'_n(z)} = \sum_{i=1}^{n} \left(\alpha_i \left(\frac{z f'_i(z)}{f_i(z)} - 1 \right) + \gamma_i \frac{z g''_i(z)}{g'_i(z)} \right)$$

$$= \sum_{i=1}^{n} \left(\alpha_i \frac{z f'_i(z)}{f_i(z)} - \alpha_i + \gamma_i \frac{z g''_i(z)}{g'_i(z)} \right). \tag{3}$$

We calculate the real part from both terms of the above expression and we obtain

$$\mathrm{Re} \left(\frac{zG''_n(z)}{G'_n(z)} + 1 \right) = \sum_{i=1}^{n} \left(\alpha_i \mathrm{Re} \frac{z f'_i(z)}{f_i(z)} - \alpha_i + \gamma_i \mathrm{Re} \frac{z g''_i(z)}{g'_i(z)} \right) + 1$$

$$= \sum_{i=1}^{n} \left(\alpha_i \mathrm{Re} \frac{z f'_i(z)}{f_i(z)} - \alpha_i + \gamma_i \mathrm{Re} \left(\frac{z g''_i(z)}{g'_i(z)} + 1 \right) - \gamma_i \right) + 1. \tag{4}$$

Taking the real part of the above equality, and using the fact that by hypothesis $f_i \in \mathcal{S}(\beta_i)$ and $g_i \in \mathcal{K}(\lambda_i)$, $i = 1, ..., n$, we obtain:

$$\mathrm{Re} \left(\frac{zG''_n(z)}{G'_n(z)} + 1 \right) > 1 + \sum_{i=1}^{n} (\alpha_i \beta_i - \alpha_i + \gamma_i \lambda_i - \gamma_i)$$

$$= 1 + \sum_{i=1}^{n} (\alpha_i (\beta_i - 1) + \gamma_i (\lambda_i - 1))$$

$$= \lambda,$$

which shows that the integral operator $G_n(z)$ defines a convex function of order λ. □

Setting $n = 1$ in Theorem 1, we have

Corollary 1. *Let* $f_i \in \mathcal{S}(\beta)$, $0 \leq \beta < 1$ *and* $g \in \mathcal{K}(\lambda)$, $0 \leq \lambda < 1$. *For positive real numbers* $\alpha, \gamma > 0$ *satisfying*

$$(\alpha (1-\beta) + \gamma (1-\lambda)) < 1,$$

the integral operator

$$G(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\alpha (g'(t))^\gamma \, dt$$

defines a convex function of order

$$1 + (\alpha (\beta - 1) + \gamma (\lambda - 1)).$$

Theorem 2. *For a natural number* $n \geq 1$, *let* $f_i, g_i \in \mathcal{A}$, *where* $g_i \in G_{b_i}$, $0 < b_i \leq 1$, $i = 1, ..., n$. *For positive real numbers* $M_i > 0$ *and* α_i, $\gamma_i > 0$ *satisfying the conditions*

$$\left|\frac{f_i'(z)}{f_i(z)}\right| \leq M_i \qquad z \in \mathbb{U}, \qquad \left|\frac{z g_i'(z)}{g_i(z)} - 1\right| < 1 \qquad z \in \mathbb{U} \tag{5}$$

and

$$\sum_{i=1}^{n} (\alpha_i (M_i + 1) + \gamma_i (2b_i + 1)) < 1,$$

the integral operator $G_n(z)$ *given by (2) defines a convex function of order*

$$\lambda = 1 - \sum_{i=1}^{n} (\alpha_i (M_i + 1) + \gamma_i (2b_i + 1)).$$

Proof. Following the same steps as in Theorem 1, we obtain

$$\frac{zG_n''(z)}{G_n'(z)} = \sum_{i=1}^{n} \left(\alpha_i \left(\frac{z f_i'(z)}{f_i(z)} - 1\right) + \gamma_i \frac{z g_i''(z)}{g_i'(z)}\right)$$

$$= \sum_{i=1}^{n} \left(\alpha_i \left(\frac{z f_i'(z)}{f_i(z)} - 1\right) + \gamma_i \left(\frac{z g_i''(z)}{g_i'(z)} - \frac{z g_i'(z)}{g_i(z)} + 1\right) + \gamma_i \left(\frac{z g_i'(z)}{g_i(z)} - 1\right)\right).$$

Thus, we have

$$\left|\frac{zG_n''(z)}{G_n'(z)}\right| \leq \sum_{i=1}^{n} \left(\alpha_i \left(\left|\frac{z f_i'(z)}{f_i(z)}\right| + 1\right) + \gamma_i \left|\frac{z g_i''(z)}{g_i'(z)} - \frac{z g_i'(z)}{g_i(z)} + 1\right| + \gamma_i \left|\frac{z g_i'(z)}{g_i(z)} - 1\right|\right).$$

From the hypothesis (5) of Theorem 2, we have

$$\left|\frac{f_i'(z)}{f_i(z)}\right| \leq M_i \qquad z \in \mathbb{U} \qquad \text{and} \qquad \left|\frac{z g_i'(z)}{g_i(z)} - 1\right| < 1 \qquad z \in \mathbb{U}$$

for all $i = 1, 2, ..., n$.
Since $g_i \in G_{b_i}$, $0 < b_i \leq 1$ for $i = 1, 2, ..., n$, from (1) we obtain

$$\begin{aligned}
\left|\frac{zG''_n(z)}{G'_n(z)}\right| &\leq \sum_{i=1}^{n}\left(\alpha_i\left(M_i+1\right)+\gamma_i b_i\left|\frac{zg'_i(z)}{g_i(z)}\right|+\gamma_i\left|\frac{zg'_i(z)}{g_i(z)}-1\right|\right)\\
&\leq \sum_{i=1}^{n}\left(\alpha_i\left(M_i+1\right)+\gamma_i b_i\left(\left|\frac{zg'_i(z)}{g_i(z)}-1\right|+1\right)+\gamma_i\left|\frac{zg'_i(z)}{g_i(z)}-1\right|\right)\\
&\leq \sum_{i=1}^{n}\left(\alpha_i\left(M_i+1\right)+\gamma_i b_i\left|\frac{zg'_i(z)}{g_i(z)}-1\right|+\gamma_i b_i+\gamma_i\left|\frac{zg'_i(z)}{g_i(z)}-1\right|\right)\\
&\leq \sum_{i=1}^{n}\left(\alpha_i\left(M_i+1\right)+\left(\gamma_i b_i+\gamma_i\right)\left|\frac{zg'_i(z)}{g_i(z)}-1\right|+\gamma_i b_i\right)\\
&\leq \sum_{i=1}^{n}\left(\alpha_i\left(M_i+1\right)+\gamma_i\left(2b_i+1\right)\right)\\
&= 1-\lambda.
\end{aligned}$$

So, the integral operator $G_n(z)$ defined by (2) is in $\mathcal{K}(\lambda)$. □

Setting $n = 1$ in Theorem 2, we have

Corollary 2. *Let $f, g \in \mathcal{A}$, where $g \in G_b$, $0 < b \leq 1$. For positive real numbers $M > 0$ and α, $\gamma > 0$ satisfying the conditions*

$$\left|\frac{f'_i(z)}{f_i(z)}\right| \leq M_i \qquad z \in \mathbb{U}, \qquad \left|\frac{zg'_i(z)}{g_i(z)}-1\right| < 1 \qquad z \in \mathbb{U}$$

and

$$\left(\alpha\left(M+1\right)+\gamma\left(2b+1\right)\right) < 1,$$

the integral operator

$$G_(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\alpha}\left(g'(t)\right)^{\gamma} dt$$

defines a convex function of order

$$\lambda = 1-\left(\alpha\left(M+1\right)+\gamma\left(2b+1\right)\right).$$

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