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DARBOUX-STIELTJES CALCULUS ON BANACH SPACES

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Abstract

The purpose of this paper is to extend in the vector case the study of Darboux-Stieltjes integrability as it was initiated by I. Bucur and to give some new results. Among them we note: the symmetry principle, the formula of integration by parts, the extension integrability principle, a convergence theorem. The link with other procedures of integration is given.

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1 Preliminaries and first results

For a given interval $[a, b]$ of $\mathbb{R}$ we denote by $\mathcal{D}[a, b]$ the set of all divisions $d = (a = x_0 \leq x_1 \leq x_2 \leq ... \leq x_n = b)$ of this interval. The norm of this division is denoted by $\nu(d)$, i.e. $\nu(d) = \max\{x_{i+1} - x_i |\ i = 0, 1, \dots, n-1\}$.

By *intermediary system* of d we shall understand a new divison ξ of $[a, b]$, $\xi = (a = \xi_0 \leq \xi_1 \leq \xi_2 \leq ... \leq \xi_n \leq \xi_{n+1} = b)$ where $\xi_1 \in [x_0, x_1],\ \xi_2 \in [x_1, x_2],\ ...\ \xi_n \in [x_{n-1}, x_n]$. The set of all intermediary systems of d will be noted by $\mathcal{I}(d)$. Obviously we have $\xi \in \mathcal{I}(d) \Rightarrow d \in \mathcal{I}(\xi)$ and $\nu(\xi) \leq 2\nu(d)$, $\nu(d) \leq 2\nu(\xi)$.

Let X be a Banach space over $\mathbb{R}$, let $f : [a, b] \to X,\ g : [a, b] \to \mathbb{R}$ be two arbitrary bounded functions. For $d \in \mathcal{D}[a, b],\ \xi \in \mathcal{I}(d)$ as below we denote by $\sigma(f, g, d, \xi)$ the element of X given by

$$\sigma(f, g; d, \xi) = \sum_{i=1}^{n} f(\xi_i)(g(x_i) - g(x_{i-1})).$$

The following *reciprocity formula* may be easily verified

$$\sigma(f, g; d, \xi) = f(b)g(b) - f(a)g(a) - \sigma(g, f; \xi, d).$$

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If $d_1, d_2 \in \mathcal{D}[a,b]$ we say that d_2 *is finner than* d_1 and we write $d_1 \le d_2$ if any element of d_1 belongs to d_2.

Definition 1. *We say that the function* f is Riemann-Stieltjes integrable with respect to the function g *if there exists an element* $I \in X$ *such that for any* $\epsilon > 0$ *there exists* $\eta_\epsilon > 0$ *such that*

$$||\sigma(f,g;d,\xi) - I|| < \epsilon, \ \forall d \in \mathcal{D}[a,b] \ \textit{with} \ \nu(d) \le \eta_\epsilon, \ \forall \xi \in \mathcal{I}(d).$$

The element I *of the Banach space* X *is uniquely determined and it is called the Riemann-Stieltjes integral of* f *with respect to* g.

We write $f \in RS(g)$ *instead of "*f *is Riemann-Stieltjes integrable with respect to* g*" and*

$$I = (RS)\int_a^b f dg = RS\int_a^b f dg.$$

It is well known that the function f is Riemann-Stieltjes integrable with respect to g iff for any sequence $(d_n)_n$ in $\mathcal{D}[a,b]$ with $\lim\limits_{n\to\infty} \nu(d_n) = 0$ and any $\xi^n \in \mathcal{I}(d_n)$, the sequence $(\sigma(f,g;d_n,\xi^n))_n$ is convergent in X. One can see that the above limit does not depend on the sequences $(d_n)_n, (\xi^n)_n, \xi^n \in \mathcal{I}(d_n)$ with $\lim\limits_{n\to\infty} \nu(d_n) = 0$.

Also we have the following

Proposition 1 (Cauchy criterion). *One has* $f \in RS(g)$ *iff for any* $\epsilon > 0$ *there exists* $\eta_\epsilon > 0$ *such that*

$$\big|\big|\sigma(f,g;d',\xi') - \sigma(f,g;d'',\xi'')\big|\big| < \epsilon,$$

for all d', d'' *with* $\nu(d') < \eta_\epsilon$, $\nu(d'') < \eta_\epsilon$, $\xi' \in \mathcal{I}(d')$, $\xi'' \in \mathcal{I}(d'')$.

As in the real case one can show that the set $RS(g)$ is a linear vector space over $\mathbb{R}$ and we have: $\alpha, \beta \in \mathbb{R}$, f_1, $f_2 \in RS(g) \Rightarrow \alpha f_1 + \beta f_2 \in RS(g)$ and

$$(RS)\int_a^b (\alpha f_1 + \beta f_2) dg = \alpha (RS)\int_a^b f_1 dg + \beta (RS)\int_a^b f_2 dg.$$

Using the above reciprocity formula, we have $f \in RS(g) \Rightarrow g \in RS(f)$ and

$$(RS)\int_a^b f dg = f(b)g(b) - f(a)g(a) - (RS)\int_a^b g df$$

and moreover, if $f \in RS(g_1) \cap RS(g_2)$ and $\alpha, \beta \in \mathbb{R}$ we have $f \in RS(\alpha g_1 + \beta g_2)$ and

$$(RS)\int_a^b f d(\alpha g_1 + \beta g_2) = \alpha (RS)\int_a^b f dg_1 + \beta (RS)\int_a^b f dg_2.$$

Definition 2. *The function f is called Darboux-Stieltjes integrable with respect to g if there exists $I \in X$ and for any $\epsilon > 0$ there exists $d_\epsilon \in \mathcal{D}[a,b]$ such that*

$$||\sigma(f, g; d, \xi) - I|| \leq \epsilon, \ \forall \ d \in \mathcal{D}[a,b], d_\epsilon \leq d, \forall \xi \in \mathcal{I}(d).$$

It is not difficult to show that the element $I \in X$ in the above definition is uniquely determined and it will be called the *Darboux-Stieltjes integral of f with respect to g*.

We write $f \in DS(g)$ instead of "the function f is Darboux-Stieltjes integrable with respect to g" and we denote the element I as follows

$$I = (DS) \int_a^b f dg \text{ or } I = \int_a^b f dg.$$

The following assertion generalizes a well known Riemann-Stieltjes integrability criterion using sequences of divisions.

Proposition 2. *The function f is Darboux-Stieltjes integrable with respect to g iff there exists a sequence $(d_n)_n$ in $\mathcal{D}[a,b]$ such that for any sequence $(d'_n)_n$ in $\mathcal{D}[a,b]$, with $d_n \leq d'_n$ (for any $n \in \mathbb{N}$) and any $\xi' \in \mathcal{I}(d'_n)$ (for any $n \in \mathbb{N}$) the sequence $(\sigma(f, g; d'_n, \xi'_n))_n$ converges in the Banach space X.*

Proof. We suppose that $f \in DS(g)$. Just from the definition we deduce that there exists $I \in X$ such that for any $n \in \mathbb{N}^*$, there exists $d_n \in \mathcal{D}[a,b]$ for which we have

$$\left|\left|\sigma(f, g; d'_n, \xi'_n) - I\right|\right| < \frac{1}{n}, \ \forall \ d'_n \in \mathcal{D}[a,b], d_n \leq d'_n, \forall \ \xi'_n \in \mathcal{I}(d'_n).$$

Hence, we deduce

$$\lim_{n\to\infty} \sigma(f, g, d'_n, \xi'_n) = I.$$

Conversely, we suppose the existence of a sequence $(d_n)_n$ in $\mathcal{D}[a,b]$ such that for any sequence $(d'_n)_n$ in $\mathcal{D}[a,b]$, with $d_n \leq d'_n$, for any $n \in \mathbb{N}$ and any $\xi'_n \in \mathcal{I}(d'_n)$ the sequence $\sigma(f, g, d'_n, \xi'_n)$ converges in X. Using a mixing procedure we deduce that the element $\lim\limits_{n\to\infty} \sigma(f, g.d'_n, \xi'_n)$ of X does not depend on the above sequences $(d'_n)_n$ and $(\xi'_n)_n$. We denote by I this limit and we show that for any $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that

$$\left|\left|\sigma(f, g; d', \xi') - I\right|\right| < \epsilon, \ \forall d' \in \mathcal{D}[a,b], d' \geq d_{n_\epsilon}, \forall \xi' \in \mathcal{I}(d').$$

In the contrary case there exists $\epsilon_0 > 0$ such that for any $n \in \mathbb{N}$ there exists $d'_n \in \mathcal{D}[a,b]$, $d_n \leq d'_n$ and $\xi'_n \in \mathcal{I}(d'_n)$ such that

$$\left|\left|\sigma(f, g, d'_n, \xi'_n) - I\right|\right| \geq \epsilon_0.$$

The contradiction we have arrived shows that $f \in DS(g)$. $\square$

The following assertion, the Cauchy criterion, is almost obvious:

Proposition 3. *The function f is Darboux-Stieltjes integrable with respect to g iff for any $\epsilon > 0$ there exists a division d_ϵ of $[a,b]$ such that*

$$\left|\left|\sigma(f,g;d',\xi') - \sigma(f,g;d'',\xi'')\right|\right| < \epsilon$$

for any d', $d'' \in \mathcal{D}[a,b]$, $d_\epsilon \leq d'$, $d_\epsilon \leq d''$ and for any $\xi' \in \mathcal{I}(d')$, $\xi'' \in \mathcal{I}(d'')$.

Remark 1. *It is easy to see that for $X = \mathbb{R}$ and g an increasing function on $[a,b]$ the fact that $f \in DS(g)$ is equivalent with the relation $\underline{\int_a^b} fdg = \overline{\int_a^b} fdg$ where $\underline{\int_a^b} fdg$ $\left(\text{respectively } \overline{\int_a^b} fdg\right)$ means the lower (respectively upper) Darboux-Stieltjes integral of f with respect to g; that is we get the well known classical situation.*

2 Relation between RS and DS integrability

The functions f and g will be as before. If the function f is Riemann-Stieltjes integrable with respect to g then we consider an arbitrary sequence $(d_n)_n$ in $\mathcal{D}[a,b]$ such that $\lim\limits_{n\to\infty} \nu(d_n) = 0$.

If we consider another sequence $(d'_n)_n$ in $\mathcal{D}[a,b]$ with $d_n \leq d'_n$ for any $n \in \mathbb{N}$ then we have $\nu(d'_n) \leq \nu(d_n)$ and therefore $\lim\limits_{n\to\infty} \nu(d'_n) = 0$. In this case we have

$$\lim_{n\to\infty} \sigma(f,g,d'_n,\xi'_n) = (RS)\int_a^b fdg, \ \forall \xi'_n \in \mathcal{I}(d'_n)$$

and therefore, using Proposition 2 we deduce $f \in DS(g)$.

Hence we have the following assertion

Proposition 4. *If $f \in RS(g)$ then $f \in DS(g)$ and*

$$(DS)\int_a^b fdg = (RS)\int_a^b fdg.$$

Remark 2. *The converse of the above proposition is not always true. Indeed, we consider an element $y \in X$, $y \neq 0_X$ and the functions $f : [0,2] \to X$, $g : [0,2] \to \mathbb{R}$ given by $f(x) = \begin{cases} y, & \text{if } 1 \leq x \leq 2 \\ 0_X, & \text{if } 0 \leq x < 1 \end{cases}$, $g(x) = \begin{cases} 1, & \text{if } 1 < x \leq 2 \\ 0, & \text{if } 0 \leq x \leq 1 \end{cases}$.*

Let $d_0 \in \mathcal{D}[0,2]$, $d_0 = (0 = x_0 < x_1 < x_2 = 2)$ be such that $x_1 = 1$ and let $d' \in \mathcal{D}[0,2]$, $d' \geq d_0$ be of the form

$$d' = (0 = x'_0 < x'_1 < x'_2 < \ldots < x'_k < 1 < x'_{k+2} < \ldots x'_n = 2).$$

If we consider $\xi' \in \mathcal{I}(d')$, $\xi' = (0 = \xi'_0 \leq \xi'_1 \leq \xi'_2 \leq \ldots \leq \xi'_k \leq \ldots \leq \xi'_{n+1} = 2)$, *where*

$$0 = \xi'_0 \leq \xi'_1 \leq \xi'_2 \leq \ldots \leq \xi'_k \leq x'_k \leq \xi'_{k+1} \leq 1 \leq \xi'_{k+2} \leq x'_{k+2} \leq \ldots \leq \xi'_{n+1} = 2,$$

we have

$$\sigma(f, g, d', \xi') = \sum_{i=1}^{n} f(\xi'_i)(g(x'_i) - g(x'_{i-1})) = f(\xi'_{k+2})(g(x'_{k+2}) - g(1)) = y,$$

$||\sigma(f, g, d', \xi') - y|| = 0.$

From the definition we deduce $f \in DS(g)$ *and* $(DS)\int_a^b f dg = y$. *On the other hand, if we consider an arbitrary division* d *of* $[0, 2]$,

$$d = (0 = x_0 < x_1 < x_2 < \ldots < x_p < x_{p+1} < \ldots < x_m = 2)$$

such that $x_p < 1 < x_{p+1}$ *and* $\xi' \in \mathcal{I}(d)$, $\xi'' \in \mathcal{I}(d)$,

$$\xi' = (0 = \xi'_0 \leq \xi'_1 \leq \xi'_2 \leq \ldots \leq \xi'_{m+1} = 2),\ \xi'_i \in [x_{i-1}, x_i], i \in \overline{1, m},$$

$$\xi'' = (0 = \xi''_0 \leq \xi''_1 \leq \xi''_2 \leq \ldots \leq \xi''_{m+1} = 2),\ \xi''_i \in [x_{i-1}, x_i],\ i \in \overline{1, m}$$

and $\xi'_{p+1} \in (x_p, 1)$, $\xi''_{p+1} \in (1, x_{p+1})$, *we have*

$$\sigma(f, g, d, \xi') = 0_X,\ \ \sigma(f, g, d, \xi'') = y,\ \ y \neq 0_X.$$

Using now the Cauchy criterion of Riemann-Stieltjes integrability we deduce that f *is not Riemann-Stieltjes integrable with respect to* g.

The non-(RS)-integrability in our previous remark is an immediate consequence of the next result. The interested reader can easily find more examples using our technique. The following statement shows how far is Riemann-Stieltjes integrability from the Darboux-Stieltjes integrability.

Proposition 5. *a) If the functions* f *and* g *have a common point of discontinuity on the left hand side (or on the right hand side) then the function* f *is not Darboux-Stieltjes integrable with respect to* g.

b) If the functions f *and* g *have a common point of discontinuity then the function* f *is not Riemann-Stieltjes integrable with respect to* g.

Proof. a) We suppose that f and g are discontinuous on the left at the point $c \in (a, b]$. In this case there exists $r' > 0$, $r'' > 0$ and two sequences $(x'_n)_n$, $(x''_n)_n$ which increase to c and such that for any $n \in \mathbb{N}$ we have

$$x'_n < x''_n < x'_{n+1} < c,\ \ ||f(x'_n) - f(c)|| > r',\ \ |g(x''_n) - g(c)| > r''.$$

Let now d be an arbitrary division of $[a, b]$,

$$d = (a = x_0 < x_1 < \ldots < x_{k_0} < \ldots < x_n = b),$$

such that $x_{k_0} = c$. For n sufficiently large we have

$$x_{k_0-1} < x''_n < x'_{n+1} < c$$

and we consider the division d' of $[a,b]$ obtained from d adding the point x''_{n_0} with $x_{k_0-1} < x''_{n_0} < c$. We consider now ξ, ξ' in $\mathcal{I}(d')$ which differ between them only by the points $\xi_{k_0} = x'_{n_0+1}$, $\xi'_{k_0} = c$ of the interval $[x''_{n_0}, c]$ of the division d'.

We have

$$\sigma(f,g,d',\xi) - \sigma(f,g,d',\xi') = f(x'_{n_0+1})(g(x''_{n_0}) - g(c)) - f(c)(g(x''_{n_0}) - g(c)),$$

$$||\sigma(f,g,d',\xi) - \sigma(f,g,d',\xi')|| = ||f(x'_{n_0+1}) - f(c)|| \cdot |g(x''_{n_0}) - g(c)| > r' \cdot r''.$$

Now, using Proposition 3 and the fact that the division d of $[a,b]$ was arbitrary we deduce that the function f is not Darboux-Stieltjes integrable with respect to the function g.

An analogous treatment may be done for the case where f and g are discontinuous on the right at a point $c \in [a,b)$.

b) The functions f and g are both discontinuous on the same side of a point $c \in [a,b]$; this is a trivial consequence of the assertion a). So, let $c \in (a,b)$ such that f is discontinuous on the left but it is continuous on the right at the point c whereas the function g is continuous on the left, but it is discontinuous on the right at the point c. In this case there exist $r' > 0$, $r'' > 0$ and there exist two sequences: $(x'_n)_n$ strictly increasing to c and $(x''_n)_n$ strictly decreasing to c such that we have

$$||f(x'_n) - f(c)|| > r', \ |g(x''_n) - g(c)| > r'', \ \forall n \in \mathbb{N}.$$

Let now $d \in \mathcal{D}[a,b]$ be an arbitrary division such that c is not a point of d

$$d = (a = x_0 < x_1 < \ldots < x_{k_0} < x_{k_0+1} < \ldots < x_m = b), \ x_{k_0} < c < x_{k_0+1}.$$

For n_0 sufficiently large we have $x_{k_0} < x'_{n_0} < c < x''_{n_0} < x_{k_0+1}$. We add to the division d the points x'_n, x''_n with $n \geq n_0$ and we note by d_n this new division of $[a,b]$. Further we consider ξ', ξ'' in $\mathcal{I}(d_n)$ which differ between them only by the intermediary point ξ'_n, respectively ξ''_n in the interval $[x'_n, x''_n]$, namely $\xi'_n = x'_n$, $\xi''_n = x''_n$. We shall have

$$\begin{aligned}\sigma(f,g,d_n,\xi'') - \sigma(f,g,d_n,\xi') &= (f(\xi'_n) - f(\xi''_n)) \cdot (g(x'_n) - g(x''_n)) = \\ &= (f(x'_n) - f(x''_n)) \cdot (g(x'_n) - g(x''_n)).\end{aligned}$$

Since f is continuous on the right and g is continuous on the left at the point c and $||f(x'_n) - f(c)|| > r'$, $|g(x''_n) - g(c)| > r''$, for all n, we deduce that

$$||f(x'_n) - f(x''_n)|| > \frac{r'}{2}, |g(x''_n) - g(x'_n)| > \frac{r''}{2}$$

if n is sufficiently large. So we have

$$||\sigma(f,g,d_n,\xi'') - \sigma(f,g,d_n,\xi')|| > \frac{r' \cdot r''}{4}$$

for n sufficiently large. Using the fact that the division d is arbitrary we can deduce that $f \notin RS(g)$ from the Cauchy criterion. $\square$

Proposition 6. *If $f \in DS(g)$ and the function f and g have no common point of discontinuity then we have $f \in RS(g)$.*

Proof. Let us denote $||f|| = \sup\{||f(x)||; x \in [a,b]\}$, $||g|| = \sup\{|g(x)|; x \in [a,b]\}$ and $\epsilon > 0$ be arbitrary. We consider $d_\epsilon \in \mathcal{D}[a,b]$ such that for any $d \in \mathcal{D}[a,b]$, $d_\epsilon \le d$ and any $\xi \in \mathcal{I}(d)$ we have

$$\left|\left|\sigma(f,g,d,\xi) - (DS)\int_a^b f dg\right|\right| < \epsilon.$$

If $d_\epsilon = (a = x_0 < x_1 < x_2 < \ldots < x_k = b)$, then using the hypothesis concerning the continuity we may consider $\eta > 0$ such that for any $i \in \{0,1,2,\ldots,k\}$ we have at least one of the relations

$$|z - x_i| < \eta \Rightarrow ||f(z) - f(x_i)|| \le \frac{\epsilon}{r} \text{ or } |g(z) - g(x_i)| \le \frac{\epsilon}{r},$$

where $r := 4k(||f|| \cdot ||g||)$.

Let now $d_0 \in \mathcal{D}[a,b]$, $d_0 = (a = y_0 < y_1 < \ldots < y_n = b)$ with $\nu(d_0) < \eta$ and let $\xi = (a = \xi_0 < \xi_1 \le \xi_2 \le \ldots \le \xi_n \le \xi_{n+1} = b)$, $\xi \in \mathcal{I}(d_0)$ with $\xi_i \in [y_{i-1}, y_i]$ for all $i \in \{1,2,\ldots\ n\}$.

Suppose that one point x_i of the division d_ϵ belongs to the interval $[y_{j_0}, y_{j_0+1}]$. We choose $\xi'_{j_0} \in [y_{j_0}, x_i]$, $\xi''_{j_0} \in [x_i, y_{j_0+1}]$ and we consider the division $d_{x_i} \in \mathcal{D}[a,b]$ obtained by adding the point x_i to the division d_0. As an intermediary system $\xi^* \in \mathcal{I}(d_{x_i})$ we take the following one

$$\xi^* = \{a = \xi_0 \le \xi_1 \le \ldots \le \xi_{j_0} \le \xi'_{j_0} \le \xi''_{j_0} \le \xi_{j_0+2} \le \xi_{j_0+3} \le \ldots \le \xi_{n+1} = b\}.$$

We shall have

$$\begin{aligned}
&\sigma(f,g,d_0,\xi) - \sigma(f,g,d_{x_i},\xi^*) = \\
&\quad f(\xi_{j_0+1})(g(y_{j_0+1}) - g(y_{j_0})) - f(\xi'_{j_0})(g(x_i) - g(y_{j_0})) - f(\xi''_{j_0})(g(y_{j_0+1}) - g(x_i)) = \\
&\qquad = (f(\xi_{j_0+1}) - f(\xi'_{j_0}))(g(x_i) - g(y_{j_0})) + (f(\xi_{j_0+1}) - f(\xi''_{j_0}))(g(y_{j_0+1}) - g(x_i)).
\end{aligned}$$

$$\begin{aligned}
||\sigma(f,g,d_0,\xi) - \sigma(f,g,d_{x_i},\xi^*)|| &\le \left|\left|f(\xi_{j_0+1}) - f(\xi'_{j_0})\right|\right| \cdot |g(x_i) - g(y_{j_0})| + \\
&+ \left|\left|f(\xi_{j_0+1}) - f(\xi''_{j_0})\right|\right| \cdot |g(y_{j_0+1}) - g(x_i)| \le 4(||f|| \cdot ||g||) \cdot \frac{\epsilon}{r}
\end{aligned}$$

We start with the divisions d and ξ as before and taking $i = 1$ we construct as above the division $d_1 = d_0 \cup \{x_1\}$ and the division $\xi^1 := \xi^*$. We have

$$\left|\left|\sigma(f,g,d_0,\xi) - \sigma(f,g,d_1,\xi^1)\right|\right| \le 4(||f|| \cdot ||g||) \cdot \frac{\epsilon}{r}.$$

Then starting with the divisions d_1, ξ^1 we construct in a similar manner $d_2 = d_1 \cup \{x_2\}$, $\xi^2 = (\xi^1)^* \in \mathcal{I}(d_2)$. We have

$$\left|\left|\sigma(f,g,d_1,\xi^1) - \sigma(f,g,d_2,\xi^2)\right|\right| \le 4(||f|| \cdot ||g||) \cdot \frac{\epsilon}{r}.$$

We continue this procedure $(k-1)$-times and we construct the divisions (d_1,ξ^1), (d_2,ξ^2), (d_3,ξ^3), …, (d_{k-1},ξ^{k-1}) such that $d_{i+1}=d_i\cup\{x_{i+1}\}$, $\xi^{i+1}=(\xi^i)^*$.

By construction we have

$$\left|\left|\sigma(f,g,d_i,\xi^i)-\sigma(f,g,d_{i+1},\xi^{i+1})\right|\right|\le 4(||f||\cdot||g||)\cdot\frac{\epsilon}{r},\ i+1\le k-1$$

and therefore, applying this k times and taking into account the fact that $r=4k(||f||\cdot||g||)$, we get

$$\left|\left|\sigma(f,g,d_0,\xi)-\sigma(f,g,d_{k-1},\xi^{k-1})\right|\right|\le k\cdot 4||f||\cdot||g||\cdot\frac{\epsilon}{r}=\epsilon.$$

But $d_\epsilon\le d_{k-1}$ and therefore we have

$$\left|\left|\sigma(f,g,d_{k-1},\xi^{k-1})-(DS)\int_a^b fdg\right|\right|\le\epsilon$$

From the last two inequalities follows

$$\left|\left|\sigma(f,g,d_0,\xi)-(DS)\int_a^b fdg\right|\right|\le 2\epsilon$$

for any $d_0\in\mathcal{D}[a,b]$ with $\nu(d_0)<\eta$ and any $\xi\in\mathcal{I}(d_0)$. □

Corollary 1. *If we have $f\in DS(g)$ and one of the functions f or g is continuous on $[a,b]$, then $f\in RS(g)$.*

Remark 3. *The Proposition 6 and the Corollary 1 were previously considered in the scalar case for g increasing ([3], [4]).*

The concept of Darboux-Stieltjes integrability is much more related with the concept of Lebesgue (or Bochner) integrability than the Riemann-Stieltjes concept is.

Let g be increasing and continuous on the left and let μ_g be the measure on $([a,b],\mathcal{B})$ were $\mathcal{B}$ is the set of all Borel subsets of $[a,b]$, for which we have

$$\mu([c,d))=g(d)-g(c),\ \forall c,d\in\mathbb{R},\ a\le c<d\le b.$$

If $f:[a,b]\to\mathbb{R}$ is a bounded function then, proceeding as in [4] and [5] we can prove the following results:

Proposition 7. *If the function f is Darboux-Stieltjes integrable with respect to g then the function f is Bochner integrable and we have*

$$(DS)\int_a^b fdg=\int_a^b fd\mu_g.$$

Proposition 8. *If $(f_n)_n$ is a sequence of uniformely bounded real functions on $[a,b]$ such that $f_n \in DS(g)$, for all n and this sequence is pointwise convergent to a function f such that $f \in DS(g)$, then we have*

$$\lim_{n\to\infty}(DS)\int\limits_a^b f_n dg = DS\int\limits_a^b f dg.$$

3 Hereditary properties and the formula of integration by parts

It is well known that if a bounded real function f on the interval $[a,b]$ is Riemann Stieltjes integrable with respect to the function g defined on the same interval, then, for any $c,d \in [a,b],\ c < d$, the restriction of f to $[c,d]$ is Riemann-Stieltjes integrable with respect to the restriction of g to $[c,d]$. Generally the converse assertion is not true i.e. the Riemann-Stieltjes integrability of f with respect to g on the intervals $[a,c]$ and $[c,d]$ does not imply the RS-integrability of the function f with respect to g on the whole interval $[a,b]$. From this point of view the Darboux-Stieltjes integral is more convenient.

Proposition 9. *If $f : [a,b] \to X$ and $g : [a,b] \to \mathbb{R}$ are bounded functions then we have:*

a) If f is Darboux-Stieltjes integrable with respect to g on $[a,b]$ then f is Darboux-Stieltjes integrable with respect to g on any subinterval $[c,d]$ of $[a,b]$ i.e. the restriction of f to $[c,d]$ is Darboux-Stieltjes integrable with respect to the restriction of g to $[c,d]$. Moreover, we have

$$(DS)\int\limits_a^b f dg = (DS)\int\limits_a^c f dg + (DS)\int\limits_c^d f dg + (DS)\int\limits_d^b f dg.$$

b) If c is a point in $[a,b]$ and the function f is Darboux-Stieltjes integrable with respect to g on the interval $[a,c]$ and $[c,b]$, then f is Darboux-Stieltjes integrable with respect to g on $[a,b]$.

Proof. For any divisions $d' \in \mathcal{D}[a,c]$, $d'' \in \mathcal{D}[c,b]$ we denote by $d' \vee d''$ the division of $[a,b]$ given by

$$d' \vee d'' = (a = x_0 < x_1 < x_2 < \ldots < x_n = c = y_0 < y_1 < y_2 \ldots < y_n = b),$$

where $d' = (a = x_0 < x_1 < x_2 < \ldots < x_n = c)$, $d'' = (c = y_0 < y_1 < y_2 \ldots < y_n = b)$.

We use an analogous notation $\xi' \vee \xi''$ for $\xi' \in \mathcal{I}(d')$, $\xi'' \in \mathcal{I}(d'')$.

Obviously we have

$$\sigma(f,g,d' \vee d'', \xi' \vee \xi'') = \sigma(f,g,d',\xi') + \sigma(f,g,d'',\xi'').$$

a) The proof follows using Cauchy criterion of Darboux-Stieltjes integrability.

b) Let $({d'_n}^0)_n$, $({d''_n}^0)_n$ be two sequences in $\mathcal{D}[a,c]$, respectively $\mathcal{D}[c,b]$ such that for any sequences $(d'_n)_n \in \mathcal{D}[a,c]$, ${d'_n}^0 \leq d'_n$, $(d''_n)_n \in \mathcal{D}[c,b]$, ${d''_n}^0 \leq d''_n$ and for any $\xi'_n \in \mathcal{I}(d'_n)$, respectively any $\xi''_n \in \mathcal{I}(d''_n)$, we have

$$\lim_{n\to\infty} \sigma(f,g,d'_n,\xi'_n) = \int_a^c f dg, \ \lim_{n\to\infty} \sigma(f,g,d''_n,\xi''_n) = \int_c^b f dg.$$

Let now $(d_n^0)_n$ be a sequence in $\mathcal{D}[a,b]$ such that $d_n^0 = {d'_n}^0 \vee {d''_n}^0$, $\forall n \in \mathbb{N}$ and let $(d_n)_n$ be a sequence in $\mathcal{D}[a,b]$ such that $d_n^0 \leq d_n$ for any n. If we choose $\xi_n \in \mathcal{I}(d_n)$ and we denote $d'_n = d_n \cap [a,c]$, $\xi'_n = \xi_n \cap [a,c]$, $d''_n = d_n \cap [c,b]$, $\xi''_n = \xi_n \cap [c,b]$ we have ${d'_n}^0 \leq d'_n$, $\xi'_n \in \mathcal{I}(d'_n)$, ${d''_n}^0 \leq d''_n$, $\xi'' \in \mathcal{I}(d''_n)$ and therefore

$$\lim_{n\to\infty} \sigma(f,g,d'_n,\xi'_n) = \int_a^c f dg, \ \lim_{n\to\infty} \sigma(f,g,d''_n,\xi''_n) = \int_c^b f dg.$$

It is obvious that $d'_n \vee d''_n = d_n$, $\xi'_n \vee \xi''_n = \xi_n$. We have

$$\sigma(f,g,d_n,\xi_n) = \sigma(f,g,d'_n,\xi'_n) + \sigma(f,g,d''_n,\xi''_n),\ \forall n \in \mathbb{N},$$

$$\lim_{n\to\infty} \sigma(f,g,d_n,\xi_n) = \int_a^c f dg + \int_c^b f dg.$$

Hence using Proposition 2, the function f is Darboux-Stieltjes integrable with respect to g. $\square$

Proposition 10. *(Symmetry principle) If the function f is Darboux-Stieltjes integrable with respect to g then the function g is Darboux-Stieltjes integrable with respect to f and we have*

$$\int_a^b g df = f \cdot g \Big|_a^b - \int_a^b f dg = f(b)g(b) - f(a)g(a) - \int_a^b f dg.$$

(Integration by parts).

Proof. For $\epsilon > 0$ we consider $d_\epsilon \in \mathcal{D}[a,b]$, $d_\epsilon = (a = y_0 < y_1 < y_2 \cdots < y_k = b)$ such that for any $d \in \mathcal{D}[a,b]$, $d_\epsilon \leq d$ and any $\xi \in \mathcal{I}(d)$ we have

$$\left|\left|\sigma(f,g,d,\xi) - \int_a^b f dg\right|\right| < \epsilon.$$

Using the hypothesis and Proposition 5-b) we deduce that for any y_i, $i \in \{1,2,\ldots,k\}$ at least one of the functions f and g is continuous on the left at the point y_i. Hence we may choose $\eta > 0$ such that, for any $z \in [a,b]$, $z \in [y_i - \eta, y_i]$ we have

$$||f(y_i) - f(z)|| < \frac{\epsilon}{M} \text{ or } |g(y_i) - g(z)| < \frac{\epsilon}{M},$$

where $M := k(||f||+1)(||g||+1)$.

Let now d'_ϵ be a division of $[a,b]$ such that $\nu(d'_\epsilon) < \eta$ and such that $d_\epsilon \leq d'_\epsilon$. We take an arbitrary division d of $[a,b]$ such that $d'_\epsilon \leq d$ and we consider an arbitrary $\xi \in \mathcal{I}(d)$. We have $d = (a = x_0 = y_0 < x_1 < x_2 < \ldots < x_{j_1} < y_1 < x_{j_1+1} < \ldots < x_{j_2} < y_2 < x_{j_2+1} < \ldots < x_{j_k} < y_k = b)$, $\xi = (a = \xi_0 \leq \xi_1 \leq \ldots \leq \xi_{j_k+k} = b)$, $\xi \in \mathcal{I}(d)$ and we modify ξ replacing the element ξ_{j_p} in the interval $[x_{j_p}, y_p]$ by the element y_p, for all $p = 1, 2, \ldots, k$. We obtain a new intermediary division ξ' of d and we have

$$\left|\left|g\left(\xi_{j_p}\right)\left(f\left(y_p\right) - f\left(x_{j_p}\right)\right) - g\left(y_p\right)\left(f\left(y_p\right) - f\left(x_{j_p}\right)\right)\right|\right| = \\ \left|\left|f\left(y_p\right) - f\left(x_{j_p}\right)\right|\right| \cdot \left|g\left(y_p\right) - g\left(\xi_{j_p}\right)\right| \leq 2(||f||+1)(||g||+1)\cdot\frac{\epsilon}{M}.$$

We deduce the relation

$$\left|\left|\sigma(g,f,d,\xi) - \sigma(g,f,d,\xi')\right|\right| \leq \sum_{p=1}^{k} \left|\left|f\left(y_p\right) - f\left(x_{j_p}\right)\right|\right| \cdot \left|g\left(y_p\right) - g\left(\xi_{j_p}\right)\right|,$$

$$\left|\left|\sigma(g,f,d,\xi) - \sigma(g,f,d,\xi')\right|\right| \leq k(||f||+1)(||g||+1)\cdot\frac{\epsilon}{M} = \epsilon.$$

We remark that $d_\epsilon \leq \xi'$ and therefore we have

$$\left|\left|\sigma(f,g,\xi',d) - \int_a^b f dg\right|\right| \leq \epsilon.$$

On the other hand, using the reciprocity formula, we get

$$\left|\left|\sigma(g,f,d,\xi) - \left(f\cdot g\Big|_a^b - \int_a^b f dg\right)\right|\right| \leq \left|\left|\sigma(g,f,d,\xi) - \sigma(g,f,d,\xi')\right|\right| + \\ + \left|\left|\sigma(g,f,d,\xi') - \left(f\cdot g\Big|_a^b - \int_a^b f dg\right)\right|\right| \leq \epsilon + \left|\left|\int_a^b f dg - \sigma(f,g,\xi',d)\right|\right| \leq 2\epsilon.$$

Hence the function g is Darboux-Stieltjes integrable with respect to f and we have the following rule

$$\int_a^b g df = f\cdot g\Big|_a^b - \int_a^b f dg.$$

$\square$

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