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ON AN EXTENSION OF THE HILBERT INTEGRAL INEQUALITY

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Abstract

In this paper it is shown that an extension of the Hilbert integral inequality can be built by introducing a parameter λ $(\lambda > -1)$. Constant factor expressed by Γ-function is proved to be the best possible. As applications, some equivalent forms are given.

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1 Introduction

Let f and g be two real functions. If $0 \leq \int_0^\infty f^2(x)dx < +\infty$ and $0 \leq \int_0^\infty g^2(x)dx < +\infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)\,g(y)}{x+y}dxdy \leq \pi \left\{ \int_0^\infty f^2(x)\,dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)\,dx \right\}^{\frac{1}{2}} \tag{1.1}$$

where the constant factor π is the best possible. Equality in (1.1) holds if and only if $f(x) = 0$ or $g(x) = 0$. This is the famous Hilbert integral inequality, (see [1],[2]). Recently, various improvements and extensions of (1.1) are listed in paper [3]

$\left(\ln \frac{x}{y}\right)^0 = 1$,when $x = y$. The purpose of the present paper is to establish the Hilbert-type integral inequality of the form

$$\int_0^\infty \int_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda f(x)\,g(y)}{x+y}dxdy \leq C \left\{ \int_0^\infty f^2(x)\,dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)\,dx \right\}^{\frac{1}{2}} \tag{1.2}$$

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where $\lambda > -1$. We will give the constant factor C, and will prove the constant factor C in (1.2) to be the best possible, and then give some results, study some equivalent forms of them. Evidently, the inequality (1.2) is an extension of (1.1). The new inequality established is significant in theory and applications.

For convenience, we introduce the following function and signs:

$\Gamma-$ function is defined by $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ $(Re\ z\ > 0)$. The following formula is given in the paper [10] (see pp. 226, formula 1053): $\int_0^\infty x^n e^{-ax}dx = \frac{\Gamma(n+1)}{a^{n+1}}$ $(a > 0,\ n > -1))$.

Let E_n be Euler's numbers, viz. $E_0 = 1,\ E_1 = 1,\ \ E_2 = 5,\ \ E_3 = 61,\ \ \ E_4 = 1385$, etc.

We will frequently use these signs throughout paper.

2 Lemmas

In order to prove our main results, we need the following lemmas.

Lemma 2.1. *Let a be a positive number and $\lambda > -1$. Then*

$$\int_0^\infty x^\lambda e^{-ax}dx = \frac{\Gamma(\lambda+1)}{a^{\lambda+1}}, \tag{2.3}$$

where $\Gamma(z)$ is Γ-function.

Lemma 2.2. *Let a be a positive number. Then*

$$\int_0^\infty \frac{x}{\cosh ax}dx = \frac{2G}{a^2}, \tag{2.4}$$

where G is Catalan constant, i.e. $G = 0.915965594\cdots$.

Proof. Let $\lambda > -1$. Expanding the hyperbolic secant function $\frac{1}{\cosh ax}$, and then using Lemma 2.1 we have

$$\begin{aligned}
\int_0^\infty \frac{x^\lambda}{\cosh ax}dx &= 2\int_0^\infty \frac{x^\lambda e^{-ax}}{1+e^{-2ax}}dx = 2\int_0^\infty x^\lambda e^{-ax}\sum_{k=1}^\infty (-1)^{k+1} e^{-2(k-1)ax}dx \\
&= 2\sum_{k=1}^\infty (-1)^{k+1}\int_0^\infty x^\lambda e^{-(2k-1)ax}dx = \frac{2\Gamma(\lambda+1)}{a^{\lambda+1}}\sum_{k=1}^\infty \frac{(-1)^{k+1}}{(2k-1)^{\lambda+1}} \\
&= \frac{2\Gamma(\lambda+1)}{a^{\lambda+1}}G(\lambda),
\end{aligned} \tag{2.5}$$

where the function $G(\lambda)$ is defined by

$$G(\lambda) = \sum_{k=1}^\infty \frac{(-1)^{k+1}}{(2k-1)^{\lambda+1}}. \tag{2.6}$$

Let $\lambda = 1$.Then $\Gamma(\lambda+1) = 1$. In accordance with the definition of the Catalan constant (see [10], pp. 503.),

$$G(1) = G = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^2} = 0.915965594 \cdots .$$

□

Lemma 2.3. *Let $\lambda > -1$. Then*

$$\int_0^1 t^{-\frac{1}{2}} \left(\ln \frac{1}{t}\right)^{\lambda} \frac{1}{1+t} dt = 2^{\lambda+1}\Gamma(\lambda+1)G(\lambda), \tag{2.7}$$

where the function $G(\lambda)$ is defined by (2.6).

Proof. $x = \ln \frac{1}{t}$, it is easy to deduce that

$$\int_0^1 t^{-\frac{1}{2}} \left(\ln \frac{1}{t}\right)^{\lambda} \frac{1}{1+t} dt = \int_0^{\infty} \frac{x^{\lambda} e^{-\frac{1}{2}x}}{1+e^{-x}} dx = \int_0^{\infty} \frac{x^{\lambda}}{e^{\frac{1}{2}x} + e^{-\frac{1}{2}x}} dx = \frac{1}{2} \int_0^{\infty} \frac{x^{\lambda}}{\cosh \frac{1}{2}x} dx.$$

□

By using (2.5), the equality (2.7) follows.

Lemma 2.4.

$$\int_0^{\infty} t^{-\frac{1}{2}} \left|\ln \frac{1}{t}\right|^{\lambda} \frac{1}{1+t} dt = 2^{\lambda+2}\Gamma(\lambda+1)G(\lambda), \tag{2.8}$$

where $G(\lambda)$ is defined by (2.6).

Proof.

$$\begin{aligned}
\int_0^{\infty} t^{-\frac{1}{2}} \left|\ln \frac{1}{t}\right|^{\lambda} \frac{1}{1+t} dt &= \int_0^{1} t^{-\frac{1}{2}} \left|\ln \frac{1}{t}\right|^{\lambda} \frac{1}{1+t} dt + \int_1^{\infty} t^{-\frac{1}{2}} \left|\ln \frac{1}{t}\right|^{\lambda} \frac{1}{1+t} dt \\
&= \int_0^{1} t^{-\frac{1}{2}} \left|\ln \frac{1}{t}\right|^{\lambda} \frac{1}{1+t} dt + \int_0^{1} v^{-\frac{1}{2}} \left|\ln v\right|^{\lambda} \frac{1}{1+v} dv \\
&= \int_0^{1} t^{-\frac{1}{2}} \left(\ln \frac{1}{t}\right)^{\lambda} \frac{1}{1+t} dt + \int_0^{1} v^{-\frac{1}{2}} \left(\ln \frac{1}{v}\right)^{\lambda} \frac{1}{1+v} dv \\
&= 2\int_0^{1} t^{-\frac{1}{2}} \left(\ln \frac{1}{t}\right)^{\lambda} \frac{1}{1+t} dt.
\end{aligned}$$

By Lemma 2.3, the equality (2.8) follows. □

3 Theorems and their proofs

In this section, we will prove our assertions by using the above Lemmas.

Theorem 3.1. *Let f and g be two real functions and $\lambda > -1$. If $0 \le \int_0^\infty f^2(x)dx < +\infty$ and $0 \le \int_0^\infty g^2(x)dx < +\infty$, then*

$$\int_0^\infty \int_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda f(x)\, g(y)}{x + y} dxdy \le C \left\{ \int_0^\infty f^2(x)\, dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)\, dx \right\}^{\frac{1}{2}}, \tag{3.9}$$

where the constant factor C is defined by

$$C = 2^{\lambda+2}\Gamma(\lambda+1)G(\lambda), \tag{3.10}$$

the function $G(\lambda)$ is defined by (2.6) *and $\Gamma(z)$ is Γ-function. Constant factor C in* (3.9) *is the best possible. Equality in* (3.9) *holds if and only if $f(x) = 0$ or $g(x) = 0$.*

Proof. We can apply the Cauchy inequality to estimate the left-hand side of (3.9) as follows.

$$\int_0^\infty \int_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda f(x)g(y)}{x + y} dxdy \le \left(\int_0^\infty \omega(x) f^2(x) dx \right)^{\frac{1}{2}} \left(\int_0^\infty \omega(x) g^2(x) dx \right)^{\frac{1}{2}}, \tag{3.11}$$

where $\omega(x) = \int\limits_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda}{x+ y} \left(\frac{x}{y}\right)^{\frac{1}{2}} dy.$

By proper substitution of variable, and then by Lemma 2.4, it is easy to deduce that

$$\begin{aligned} \omega(x) &= \int_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda}{x\left(1 + \left(\frac{y}{x}\right)\right)} \left(\frac{x}{y}\right)^{\frac{1}{2}} dy \\ &= \int_0^\infty t^{-\frac{1}{2}} \left|\ln t\right|^\lambda \frac{1}{1 + t} dt = C, \end{aligned} \tag{3.12}$$

where the constant factor C is defined by (2.8).

It follows from (3.11) and (3.12) that

$$\int_0^\infty \int_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda f(x)\, g(y)}{x + y} dxdy \le C \left\{ \int_0^\infty f^2(x)\, dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(x)\, dx \right\}^{\frac{1}{2}}, \tag{3.13}$$

If (3.13) takes the form of the equality, then there is a pair of non-zero constants c_1 and c_2 such that

$$c_1 \frac{\left|\ln \frac{x}{y}\right|^\lambda}{x+y} f^2(x) \left(\frac{x}{y}\right)^{\frac{1}{2}} = c_2 \frac{\left|\ln \frac{x}{y}\right|^\lambda}{x+y} g^2(y) \left(\frac{y}{x}\right)^{\frac{1}{2}} \quad a.e. \text{ on } \quad (0, +\infty) \times (0, +\infty).$$

Then we have

$$c_1 x\, f^2(x) = c_2 y\, g^2(y) = C_0. \text{ (constant)} \quad \text{a.e. on } (0, +\infty) \times (0, +\infty).$$

Without losing the generality, we suppose that $c_1 \neq 0$, then

$$\int_0^\infty f^2(x)\, dx = \frac{C_0}{c_1} \int_0^\infty x^{-1} dx.$$

This contradicts $0 < \int_0^\infty f^2(x)\, dx < +\infty$. It is obvious that the equality in (3.13) holds if and only if $f(x) = 0$, or $g(x) = 0$. It follows that the inequality (3.9) is valid.

It remains only to show that C in (3.9) is the best possible. Let $0 < \varepsilon < 1$.

$$\tilde{f}(x) = \begin{cases} 0 & x \in (0,\ 1) \\ x^{-\frac{1+\varepsilon}{2}} & x \in [1,\ \infty) \end{cases} \quad \text{and} \quad \tilde{g}(y) = \begin{cases} 0 & y \in (0,\ 1) \\ y^{-\frac{1+\varepsilon}{2}} & y \in [1,\ \infty) \end{cases}.$$

It is easy to deduce that

$$\int_0^{+\infty} \tilde{f}^2(x)\, dx = \int_0^{+\infty} \tilde{g}^2(y)\, dy = \frac{1}{\varepsilon}.$$

If C in (3.9) is not the best possible, then there exists $C^* > 0$, such that $C^* < C$ and

$$\begin{aligned} H(\lambda) &= \int_0^\infty \int_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda \tilde{f}(x)\, \tilde{g}(y)}{x + y} dx dy \leq C^* \left(\int_0^\infty \tilde{f}^2(x)\, dx \right)^{\frac{1}{2}} \left(\int_0^\infty \tilde{g}^2(y)\, dy \right)^{\frac{1}{2}}. \\ &= C^* \left(\int_1^\infty \tilde{f}^2(x)\, dx \right)^{\frac{1}{2}} \left(\int_1^\infty \tilde{g}^2(y)\, dy \right)^{\frac{1}{2}} = \frac{C^*}{\varepsilon}. \end{aligned} \tag{3.14}$$

On the other hand, we have

$$
\begin{aligned}
H(\lambda) &= \int_0^\infty\int_0^\infty \frac{\left|\ln\frac{x}{y}\right|^{\lambda}\tilde{f}(x)\,\tilde{g}(y)}{x+y}dxdy = \int_1^\infty\int_1^\infty \frac{\left\{x^{-\frac{1+\varepsilon}{2}}\right\}\left\{\left|\ln\frac{x}{y}\right|^{\lambda}y^{-\frac{1+\varepsilon}{2}}\right\}}{x+y}dxdy\\
&= \int_1^\infty\left\{\int_1^\infty \frac{\left|\ln\frac{x}{y}\right|^{\lambda}y^{-\frac{1+\varepsilon}{2}}}{x\left(1+\frac{y}{x}\right)}dy\right\}\left\{x^{-\frac{1+\varepsilon}{2}}\right\}dx\\
&= \int_1^\infty\left\{\int_{1/x}^\infty \frac{\left|\ln\frac{1}{t}\right|^{\lambda}t^{-\frac{1+\varepsilon}{2}}}{1+t}dt\right\}\left\{x^{-1-\varepsilon}\right\}dx\\
&= \int_1^\infty\left\{\int_{1/x}^1 \frac{\left|\ln t\right|^{\lambda}t^{-\frac{1+\varepsilon}{2}}}{1+t}dt\right\}\left\{x^{-1-\varepsilon}\right\}dx + \int_1^\infty\left\{\int_1^\infty \frac{\left|\ln t\right|^{\lambda}t^{-\frac{1+\varepsilon}{2}}}{1+t}dt\right\}\left\{x^{-1-\varepsilon}\right\}dx\\
&= \int_0^1\left\{\int_{1/t}^\infty x^{-1-\varepsilon}dx\right\}\frac{\left|\ln t\right|^{\lambda}t^{-\frac{1+\varepsilon}{2}}}{1+t}dt + \int_1^\infty\left\{\int_1^\infty \frac{\left|\ln t\right|^{\lambda}t^{-\frac{1+\varepsilon}{2}}}{1+t}dt\right\}\left\{x^{-1-\varepsilon}\right\}dx\\
&= \frac{1}{\varepsilon}\int_0^1 \frac{\left|\ln t\right|^{\lambda}t^{-\frac{1-\varepsilon}{2}}}{1+t}dt + \frac{1}{\varepsilon}\int_1^\infty \frac{\left|\ln t\right|^{\lambda}t^{-\frac{1+\varepsilon}{2}}}{1+t}dt. \qquad (3.15)
\end{aligned}
$$

When ε is sufficiently small, we obtain from (3.15) that

$$
\begin{aligned}
H(\lambda) &= \frac{1}{\varepsilon}\left(\int_0^1 \frac{\left|\ln t\right|^{\lambda}t^{-\frac{1}{2}}}{1+t}dt + \circ_1(1)\right) + \frac{1}{\varepsilon}\left(\int_1^\infty \frac{\left|\ln t\right|^{\lambda}t^{-\frac{1}{2}}}{1+t}dt + \circ_2(1)\right)\\
&= \frac{1}{\varepsilon}\left(\int_0^\infty \frac{\left|\ln t\right|^{\lambda}t^{-\frac{1}{2}}}{1+t}dt + \circ(1)\right)(\varepsilon\to 0).
\end{aligned}
$$

By(3.12), we have

$$H(\lambda) = \frac{1}{\varepsilon}\left(C+\circ(1)\right). \quad (\varepsilon\to 0) \qquad (3.16)$$

Evidently, the (3.16) is in contradiction with (3.14). Therefore, the constant factor C in (3.9) is the best possible. Thus the proof of Theorem is completed. □

Based on Theorem 3.1, we have the following

Theorem 3.2. *If* $0 \le \int_0^\infty f^2(x)dx < +\infty$ *and* $0 \le \int_0^\infty g^2(x)dx < +\infty$, *then*

$$\int_0^\infty\int_0^\infty \frac{\left|\ln\frac{x}{y}\right| f(x)\,g(y)}{x+y}dxdy \le 8G\left\{\int_0^\infty f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty g^2(x)\,dx\right\}^{\frac{1}{2}}. \qquad (3.17)$$

where G is the Catalan constant. Constant factor $8G$ in (3.17) is the best possible. Equality in (3.17) holds if and only if $f(x) = 0$, or $g(x) = 0$.

Theorem 3.3. *Let $\lambda = 2n$ ($n \in N_0$, where N_0 the set of non-integers). If $0 \leq \int_0^\infty f^2(x)dx < +\infty$ and $0 \leq \int_0^\infty g^2(x)dx < +\infty$, then*

$$\int_0^\infty\int_0^\infty \frac{\left(\ln\frac{x}{y}\right)^{2n} f(x)\,g(y)}{x+\,y}dxdy \;\leq\; \left(\pi^{2n+1}E_n\right)\left\{\int_0^\infty f^2(x)\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty g^2(x)\,dx\right\}^{\frac{1}{2}}, \tag{3.18}$$

where E_n are Euler's numbers. And the constant factor $\pi^{2n+1}E_n$ in (3.18) is the best possible. Equality in (3.18) holds if and only if $f(x) = 0$, or $g(x) = 0$.

Proof. We need only to show that the constant factor in (3.18). When $\lambda = 2n$, it is known from (3.10) that

$$C = 2^{\lambda+2}\Gamma(\lambda+1)G(\lambda) =^{2n+2}\Gamma(2n+1)G(2n) = 2^{2n+2}\,(2n)!\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^{2n+1}}.$$

According to the paper [11] (pp. 231.), we have

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^{2n+1}} = \frac{\pi^{2n+1}}{2^{2n+2}(2n)!}E_n.$$

where E_n are Euler's numbers Notice that $\sum\limits_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)} = \frac{\pi}{4}$, hence we obtain $C = \pi^{2n+1}E_n$.

□

4 Some applications

As applications, we will establish some new inequalities.

Theorem 4.1. *Let f be a real function and $\lambda > -1$. If $0 \leq \int_0^\infty f^2(x)dx < +\infty$, then*

$$\int_0^\infty\left\{\int_0^\infty\frac{\left|\ln\frac{x}{y}\right|^{\lambda}}{x\;+\;y}f(x)\,dx\right\}^2 dy \leq \; C^2\int_0^\infty f^2(x)\,dx, \tag{4.19}$$

where C is defined by (3.10) and the constant factor C^2 in (4.19) is the best possible. Inequality (4.19) is equivalent to (3.9). Equality in (4.19) holds if and only if $f(x) = 0$.

Proof. Setting a real function $g(y)$ as

$$g(y) = \int_0^\infty\frac{\left|\ln\frac{x}{y}\right|^{\lambda}}{x\;+\;y}f(x)\,dx, \quad y \in (0,\,+\infty)$$

By using (3.9), we have

$$\int_0^\infty \left\{ \int_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda}{x\ +\ y} f\left(x\right) dx \right\}^2 dy = \int_0^\infty \int_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda}{x\ \ +\ y} f\left(x\right) g(y) dx dy$$

$$\le C \left\{ \int_0^\infty f^2(x) dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(y) dy \right\}^{\frac{1}{2}}$$

$$= C \left\{ \int_0^\infty f^2(x) dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty \left(\int_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda}{x\ \ +\ y} f(x) dx \right)^2 dy \right\}^{\frac{1}{2}}. \tag{4.20}$$

It follows from (4.20) that the inequality (4.19) is valid after some simplifications.

On the other hand, assume that the inequality (4.19) keeps valid, by applying the Cauchy inequality and (4.19), we have

$$\int_0^\infty \int_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda}{x\ \ +\ y} f\left(x\right) g\left(y\right) dx dy = \int_0^\infty \left\{ \int_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda}{x\ \ +\ y} f\left(x\right) dx \right\} g\left(y\right) dy$$

$$\le \left\{ \int_0^\infty \left(\int_0^\infty \frac{\left|\ln \frac{x}{y}\right|^\lambda}{x\ \ +\ y} f\left(x\right) dx \right)^2 dy \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2\left(y\right) dy \right\}^{\frac{1}{2}}$$

$$\le \left\{ C^2 \int_0^\infty f^2\left(x\right) dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2\left(y\right) dy \right\}^{\frac{1}{2}}$$

$$= C \left\{ \int_0^\infty f^2\left(x\right) dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2\left(y\right) dy \right\}^{\frac{1}{2}}. \tag{4.21}$$

Therefore the inequality (4.19) is equivalent to (3.9) .

If the constant factor C^2 in (4.19) is not the best possible, then it is known from (4.21) that the constant factor C in (3.9) is also not the best possible. This is a contradiction. It is obvious that the equality in (4.19) holds if and only if $f(x) = 0$. Theorem is proved. □

Theorem 4.2. *Let f be a real function. If $0 \le \int_0^\infty f^2(x)dx < +\infty$, then*

$$\int_0^\infty \left\{ \int_0^\infty \frac{\left|\ln \frac{x}{y}\right|}{x\ +\ y} f\left(x\right) dx \right\}^2 dy \le\ 64G^2 \int_0^\infty f^2\left(x\right) dx. \tag{4.22}$$

where G is the Catalan constant and the constant factor $64G^2$ in (4.22) is the best possible. Inequality (4.22) is equivalent to (3.17) . Equality in (4.22) holds if and only if $f(x) = 0$.

Theorem 4.3. *Let f be a real function and $n \in N_0$. If $0 < \int_0^\infty f^2(x)dx < +\infty$, then*

$$\int_0^\infty \left\{ \int_0^\infty \frac{\left|\ln \frac{x}{y}\right|^{2n}}{x\ +\ y} f\left(x\right) dx \right\}^2 dy \leq \ \left(\pi^{2n+1} E_n\right)^2 \int_0^\infty f^2\left(x\right) dx, \tag{4.23}$$

where E_n are Euler's numbersConstant factor $\left(\pi^{2n+1}E_n\right)^2$ in (4.23) *is the best possible. Inequality* (4.23) *is equivalent to* (3.18) . *Equality in* (4.23) *holds if and only if $f(x) = 0$.*

The proofs of Theorems 4.2 and 4.3 are similar to one of Theorem 4.1. Hence they are omitted.

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