

QUASICONFORMAL MAPPINGS ON CERTAIN CLASSES OF DOMAINS IN METRIC SPACES

Anca ANDREI¹

Abstract

First, we give some results about certain properties of domains and balls in a geodesic metric space. We shall prove that a domain in a geodesic metric space is arc connected and any ball in a geodesic metric space is arc connected. Moreover, if the metric space is GC-geodesic then we shall prove that any ball in a geodesic metric space is locally arc connected on the boundary. Next, our framework are the Ahlfors Q -regular metric measure spaces, with $1 \leq Q < \infty$. The Ahlfors Q -regular spaces are a natural setting for the theory of quasiconformal mappings since in these spaces the three definitions of quasiconformality in Euclidean spaces of dimension at least two, can be formulated, but they are not equivalent. We consider in this note the notion of geometric quasiconformality. In the Euclidean space, Gehring [3] has introduced the definition of global quasiconformal collared domain on the boundary. We give an analogous definition for this domain in a Q -regular metric measure space. We shall prove that if a domain D in a Q -regular GC-geodesic metric space is globally quasiconformally on the boundary, then D is locally arc connected on the boundary. We shall prove that any bounded Q -Loewner domain in a Q -regular geodesic metric space is arc strictly quasiconformally accessible (Definition 10[1]) on the boundary. In the end of this note we give several boundary extension theorems for quasiconformal mappings.

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1 Introduction

Throughout the paper we shall consider only metric spaces.

We first recall some definitions and make some remarks of a general character which will be use in this paper.

Let X be a metric space. The space X is said to be **path connected** (or **arc connected**) if for any two points x and y in X there exists a continuous function γ from the unit interval $[0, 1]$ to X with $\gamma(0) = x$ and $\gamma(1) = y$. This function is called

¹National Computer Science College " Spiru Haret", *Suceava*, University of Braşov, Romania, e-mail: anca56@hotmail.com

a path from x to y . We say that x and y are the endpoints of γ and that γ *joins* (or *connects*) the points x and y . An **arc domain** D in X is an open arc connected set in X . By a **continuum** we mean a compact connected set which contains at least two points.

A path is **rectifiable** if its length is a finite number. A metric space is said to be **rectifiably connected** if any two points can be joined by a rectifiable path. By a **curve** in a metric space X we mean either a path γ or its image. We usually abuse notation and call γ both the path and its image.

Note that every path connected metric spaces is connected and the image of a path is always a path connected compact space.

2 Path connectedness and locally path connectedness properties in metric spaces

Let (X, d) be a metric space. We denote by $B(x, r) = \{y \in X, d(x, y) < r\}$ the open ball of center x in X and radius $r < \text{diam}X$ and its closure $\overline{B}(x, r)$. The closed ball in X centered at the point x and with radius r is the set $B[x, r] = \{y \in X, d(x, y) \leq r\}$. It is known that $\overline{B}(x, r) \subset B[x, r]$ and $B(x, r) \subset \text{Int}B[x, r]$. Also, we know that open balls of a metric space are open sets and closed balls are closed sets. In this context, we recall the next theorem.

Remark 2.1. *Suppose that (X, d) is a metric space, $x \in X$ and $r > 0$. Then:*

- (i) $\partial B(x, r) \subset \{y, d(x, y) = r\}$;
- (ii) $\partial B[x, r] \subset \{y, d(x, y) = r\}$. (Theorem 5.1.7[6])

At the beginning, we give the following result.

Theorem 2.1. *If X is a rectifiably connected and locally path connected space and D is a domain in X , then D is an arc domain.*

Proof. For each point x in D let C_x denote the path component of D to which x belongs. Fix $x = a$ and we prove that $C_a = D$. Suppose that there exists at least $y \in D \setminus C_a$. Since D is open and X is locally path connected (by Theorem 5.18[2]), it follows that C_a is open. Let $C = \bigcup_{y \in D \setminus C_a} C_y$, where C_y is path component of y . Then, the sets C and C_a are disjoint, non-empty open subsets of D , with $C \cup C_a = D$. Thus, we conclude that D is disconnected, which contradicts the fact that D is connected. Consequently, $D = C_a$ and hence D must be arc connected. \square

Definition 2.1. *A **geodesic path** (or, simply, a **geodesic**) in a metric space X is a path γ which connects two points in X and the length of γ is equal to the distance between the points. A metric space is called **geodesic space** if every pair of distinct points can be connected by a geodesic. (Definitions 2.2.1, 2.4.1[7])*

We remember the following Lemma:

Lemma 2.1. *Let E be a connected subset of a topological space X . If $A \subset X$ and neither $E \cap A$ nor $E \cap (X \setminus A)$ is empty, then $E \cap \partial A \neq \emptyset$. ([9],11.26).*

Next, we shall give some results about domains and balls in geodesic metric spaces.

Theorem 2.2. *If X is a geodesic metric space then $\overline{B}(x, r) = B[x, r]$, where $x \in X$ and $r < \text{diam}X$.*

Proof. Let y be a point in X such that $d(x, y) = r$. For any neighbourhood V of y , there exists $0 < \varepsilon < \frac{r}{2}$ such that $B[y, \varepsilon] \subset V$. Since X is a geodesic space, there exists a curve γ which connects x and y and $l(\gamma) = d(x, y) = r$. On the other hand, γ is connected, y lies in the ball $B(y, \varepsilon)$ and x is not in the ball $B(y, \varepsilon)$. Therefore, it follows that there exists $z \in \gamma \cap \partial B(y, \varepsilon)$ and hence $z \in V$. Using Theorem 12.9.5 [6], we have $d(x, z) \leq l(\gamma|[x, z]) \leq r - d(z, y) < r$, where $\gamma|[x, z]$ is the restriction of γ which has endpoints x and z . Therefore, we obtain that $z \in B(x, r)$. Consequently, $z \in V \cap B(x, r)$ and hence $y \in \overline{B}(x, r)$. Such, we proved that $\overline{B}(x, r) = B[x, r]$. \square

Definition 2.2. *We say that a metric space X has property (P) if the closure of any open ball $B(x, r)$ in X is the closed ball $B[x, r]$.*

Remark 2.2. *A ball in a connected metric space does not need to be connected, but it is known that in a compact metric space X with property (P), any open or closed ball is connected. Therefore, if (X, d) is a compact geodesic metric space, by Theorem 2, X has property (P) and hence any ball is connected.*

Proposition 2.1. *If (X, d) is a geodesic space then any ball $B(a, r)$ with center a in X and radius $r < \text{diam}X$ is arc connected. Moreover, any geodesic which connects two points in $B(a, r)$ has length $< 2r$ and it lies in $B(a, 2r)$*

Proof. Let x and y be two points in the ball $B(a, r)$. Since X is geodesic, then there exist a geodesic γ_1 which connects a and x , and a geodesic γ_2 which connects a and y , with $l(\gamma_1) = d(a, x)$ and $l(\gamma_2) = d(a, y)$. We prove that γ_1 lies in $B(a, r)$. Suppose that there exists $z \in \gamma_1$ such that $z \notin B(a, r)$. Since γ_1 is arc connected, by Lemma 1, it follows that there is $w \in \partial B(a, r)$ and hence $w \notin B(a, r)$. On the other hand, $d(a, w) \leq l(\gamma_1|[a, w]) < l(\gamma_1|[a, x]) = d(a, x) < r$, and hence $w \in B(a, r)$, which is a contradiction. Consequently, γ_1 and γ_2 lie in $B(a, r)$. The curve $\gamma = \gamma_1 \cup \gamma_2$ lies in $B(a, r)$ and connects x and y . Therefore, $B(a, r)$ is arc connected. Let γ' be a geodesic which connects x and y and let z' be a point of γ' .

$$l(\gamma') = d(x, y) \leq d(x, a) + d(y, a) < 2r. \quad (*)$$

We reason by contradiction. Suppose that there exists a point $z' \in \gamma'$ such that z' is not contained in $B(a, 2r)$ and hence $d(a, z') \geq 2r$. Then we would have

$$d(x, z') \geq d(a, z') - d(a, x) \geq 2r - d(a, x) > r$$

$$d(y, z') \geq d(a, z') - d(a, y) \geq 2r - d(a, y) > r.$$

Summing the last relations, we obtain $l(\gamma') = d(x, y) > 2r$ which contradicts the relation (*). Thus, we proved that γ' lies in $B(a, 2r)$. \square

Corollary 2.1. *If (X, d) is a geodesic metric space then it is locally path connected, and consequently, locally connected.*

Proof. According to Proposition 1, for any $x \in X$, the ball $B(x, r)$ with $0 < r < \text{diam}X$, is arc connected. Therefore, X is locally path connected. \square

Corollary 2.2. *Let (X, d) be a geodesic metric space. If D is a domain in X then D is arc connected.*

Proof. Since X is geodesic, it follows that X is rectifiable connected and by Corollary 1, X is locally path connected. We obtain that D is connected, locally path connected open set. Therefore, D is arc connected. \square

Remark 2.3. *If (X, d) is a geodesic space then any ball $B[a, r]$ with center a in X and radius $r < \text{diam}X$ is arc connected.*

Proof. Let x and y be two points in the ball $B[a, r]$. We shall study three cases.

Case 1. Suppose that x and y are two points in $B(a, r)$. By Proposition 1, there exists a path which connects x and y in $B(a, r)$.

Case 2. Suppose that $x \in B(a, r)$ and $y \in \partial B(a, r)$. By the proof of Proposition 1, there exists a geodesic γ_1 in $B(a, r)$ which connects a and x . Since X is a geodesic space, there exists a geodesic γ_2 in X which connects a and y . Let z be a point of γ_2 such that $z \neq y$. We have:

$$d(a, z) \leq l(\gamma_2)[a, z] \leq l(\gamma_2) - d(z, y) < r$$

and hence, $z \in B(a, r)$. Therefore, γ_2 lies in $B[a, r]$. The curve $\gamma = \gamma_1 \cup \gamma_2$ lies in $B[a, r]$ and connects x and y .

Case 3. Suppose that x and y are two boundary points of $B(a, r)$. According with case 2, there exist a geodesic γ_1 which connects a and x in $B[a, r]$ and a geodesic γ_2 which connects a and y in $B[a, r]$. The curve $\gamma = \gamma_1 \cup \gamma_2$ connects x and y in $B[a, r]$. By these three cases, it follows that $B[a, r]$ is arc connected. \square

Remark 2.4. *By Proposition 1 and Remark 3 it follows that, if (X, d) is a geodesic space then any ball in X is connected.*

We shall introduce the following definitions:

Definition 2.3. *We say that a ball $B(a, r)$ in a geodesic metric space X is a **GC-ball** if for every points x and y in the ball $B[a, r]$, then any geodesic which connects x and y lies in $B[a, r]$.*

Definition 2.4. *We say that a metric space X is **GC-geodesic space** if the following condition are satisfied:*

- (i) X is a geodesic metric space;
- (ii) any ball $B(a, r)$ with center a in X and radius $r < \text{diam}X$ is a GC-ball.

Proposition 2.2. *Let (X, d) be a metric space which is GC-geodesic and let $B(a, r_1), B(b, r_2)$ be two balls in X with $a, b \in X, a \neq b$, and $r_1, r_2 < \text{diam}X$. If $B(a, r_1) \cap B(b, r_2) \neq \emptyset$ then $\overline{B}(a, r_1) \cap \overline{B}(b, r_2)$ is arc connected.*

Proof. We denote $\overline{B}(a, r_1) \cap \overline{B}(b, r_2) = E$ and let x, y be two distinct points in E . Note that there exist two distinct points in E because $B(a, r_1) \cap B(b, r_2)$ is an open set. Since $\overline{B}(b, r_2)$ is arc connected it follows that there exists a rectifiable curve γ in $\overline{B}(b, r_2)$, $\gamma(0) = x$ and $\gamma(1) = y$. If $\gamma \subset E$ the proof is complete. Now, we assume the contrary, that is not in E . Using the fact that γ is connected and compact we have $\gamma \cap \partial B(a, r_1) \neq \emptyset$. Let us denote $\gamma \cap \partial B(a, r_1) = \{z_1, z_2, \dots, z_n, \dots\}$ such that $\gamma(t_i) = z_i$, $0 \leq t_1 < t_2 < \dots < t_n < \dots \leq 1$. If $\gamma_{z_i z_{i+1}} \subset B(b, r_2) \setminus E$ then replace $\gamma_{z_i z_{i+1}}$ with an geodesic path $\gamma'_{z_i z_{i+1}}$ that connects z_i with z_{i+1} . Since $B(a, r_1)$ and $B(b, r_2)$ are GC-balls, it follows that $\gamma'_{z_i z_{i+1}}$ lies in E . Thus, we obtain a rectifiable curve in E which connects x and y . Note that we denoted by $\gamma_{z_i z_{i+1}}$ the curve with endpoints z_i and z_{i+1} . \square

Proposition 2.3. *Let (X, d) be a metric space which is GC-geodesic and let $B(a, r_1), B(b, r_2)$ be two balls in X with $a, b \in X, a \neq b$, and $r_1, r_2 < \text{diam}X$. If $B(a, r_1) \cap B(b, r_2) \neq \emptyset$ then $B(a, r_1) \cap B(b, r_2)$ is arc connected.*

Proof. We denote $B(a, r_1) \cap B(b, r_2) = E$ and let x, y be two distinct points in E . For x there exist the balls $B(x, \varepsilon_1) \subset B(b, r_2)$, $B(x, \delta_1) \subset B(a, r_1)$ and for y there exist the balls $B(y, \varepsilon_2) \subset B(b, r_2)$, $B(y, \delta_2) \subset B(a, r_1)$. Let us denote $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and $\delta = \min\{\delta_1, \delta_2\}$. Therefore $B(b, r_2 - \varepsilon) \subset B(b, r_2)$, $x, y \in B(b, r_2 - \varepsilon)$ and $B(a, r_1 - \delta) \subset B(a, r_1)$, $x, y \in B(a, r_1 - \delta)$. By Proposition 2, it follows that $\overline{B}(a, r_1 - \delta) \cap \overline{B}(b, r_2 - \varepsilon) = F$ is arc connected and $F \subset E$. Since $x, y \in F$ and F is arc connected there exists a rectifiable curve γ joining x and y in F . Consequently γ connects x and y in E . Therefore, E is arc connected. \square

Next, we recall the definition of a domain locally connected at a boundary point and we shall prove that any ball in a GC-geodesic metric space is locally arc connected on the boundary.

Definition 2.5. *Let (X, d) be a metric space and D a domain in X . D is called **locally (arc) connected** at a boundary point b if for every neighbourhood V of b , there exists a neighbourhood U of b , $U \subset V$, such that $U \cap D$ is (arc) connected. If D is locally (arc) connected at every boundary point then we say that D has property on the boundary.*

Proposition 2.4. *Let (X, d) be a GC-geodesic metric space. Then every ball $B(x, R)$ in X with center $x \in X$ and radius $R < \text{diam}X$ is locally arc connected on the boundary.*

Proof. We consider b a boundary point of $B(x, R)$ and U a neighbourhood of b . Then there exists a ball $B(b, r) \subset U$ with $r < R$. Using Proposition 3, it follows that $B(b, r) \cap B(x, R)$ is arc connected. Consequently, $B(x, R)$ is locally arc connected at b and since b was arbitrarily chosen we obtain then $B(x, R)$ is locally arc connected on the boundary. \square

3 Boundary accessibility properties of some domains in metric spaces

The most important tool of the quasiconformal theory is the conformal modulus. To define the modulus of a family of curves in a metric space we require an additional structure.

Let (X, d, μ) be a rectifiably connected metric space equipped with a locally finite Borel regular measure μ . Next, we assume that all metric spaces are rectifiably connected and all measures are locally finite and Borel regular with dense support.

We recall some definitions which will be use in this section.

Definition 3.1. Let Γ be a family of curves in a metric measure space (X, d, μ) . The (*conformal*) p -*modulus* of Γ , $1 \leq p < \infty$, is the value

$$M_p(\Gamma) = \inf \int_X \rho^p d\mu ,$$

where the infimum is taken over all Borel functions $\rho : X \rightarrow [0, \infty]$ which satisfy the inequality $\int_{\gamma} \rho ds \geq 1$, for each locally rectifiable curve $\gamma \in \Gamma$. Here, $\int_{\gamma} \rho ds$ denotes the line integral of ρ along γ , which is defined using the arc length parametrization of γ in the usual manner. By definition, the modulus of the family of all curves in X that are not locally rectifiable is zero.

Definition 3.2. A metric measure space (X, d, μ) is said to be Q -*Ahlfors regular*, $Q > 0$, if there is a constant $c \geq 1$ so that

$$\frac{r^Q}{c} \leq \mu(B_r) \leq cr^Q$$

for all balls B_r in X of radius $r < \text{diam}X$. ([5] , p.15)

Note that if above relation holds then X has Hausdorff dimension Q ([5] , p.15). It is known that one can replace μ in the above definition by the Q -dimensional Hausdorff measure (see, for instance, [8] , Lemma C. 3).

If E, F, D are subsets of X with $E \subset \overline{D}$, $F \subset \overline{D}$, we denote by $\Delta(E, F; D)$ the family of all curves which join E and F in D .

Next, we consider two Q - (Ahlfors) regular metric measure spaces (X, d, μ) and (Y, d', μ') with $1 \leq Q < \infty$, and two domains $D \subset X$, $D' \subset Y$. Similar to the geometric definition in R^n , $n \geq 2$, by Väisälä ([10],13.1) we have:

Definition 3.3. A homeomorphism $f : D \rightarrow D'$ is called K - (*geometrically*) *quasi-conformal*, $K \in [1, \infty)$ if

$$\frac{M_Q(\Gamma)}{K} \leq M_Q(f(\Gamma)) \leq KM_Q(\Gamma)$$

for every family Γ of curves in D . We also say that a homeomorphism $f : D \rightarrow D'$ is (geometrically) quasiconformal if f is K - (geometrically) quasiconformal for some $K \in [1, \infty)$, i.e., if the distortion of moduli of curve families under the mapping f is bounded.

To simplify notation, we write quasiconformal mapping instead of geometrically quasiconformal mapping.

In the Euclidean n -space, Gehring [3] has introduced the definition of global quasiconformal collared domain on the boundary in the following way:

“A domain D is globally quasiconformally collared on the boundary if there exist a neighbourhood U of ∂D and a homeomorphism $g : U \cap \bar{D} \rightarrow \{x \in \mathbb{R}^n, a < |x| \leq 1\}$, $a \geq 0$, such that $g|_{U \cap D}$ is quasiconformal.”

In our framework, we give an analogous definition:

Definition 3.4. A domain D in a metric measure space X is called **globally quasiconformally collared on the boundary** if there exist two positive constants ε, δ such that $0 \leq \varepsilon < \delta < \text{diam}X$, an arbitrarily small neighbourhood U of boundary of D and a homeomorphism $g : U \cap \bar{D} \rightarrow \{x \in X, \varepsilon < d(a, x) \leq \delta\}$, $a \in X$, such that the restriction $g|_{U \cap D}$ is geometrically quasiconformal.

In the sequel, we shall prove the following result.

Theorem 3.1. Let (X, d, μ) be a Q -Ahlfors regular and GC-geodesic metric space and D a domain in X . If D is globally quasiconformally collared on the boundary then D is locally arc connected on the boundary.

Proof. Proof. Since D is globally quasiconformally collared on the boundary then there exist $0 \leq \varepsilon < \delta < \text{diam}X$, an arbitrarily small neighbourhood U of boundary of D and a homeomorphism $g : U \cap \bar{D} \rightarrow \{x \in X, \varepsilon < d(a, x) \leq \delta\}$, $a \in X$, such that the restriction $g|_{U \cap D}$ is geometrically quasiconformal. We denote $C_{\varepsilon, \delta}(a) = \{x \in X, \varepsilon < d(a, x) < \delta\}$ and $S_\delta(a) = \{x \in X, d(a, x) = \delta\}$. Let b be a boundary point of D and a neighbourhood V of b . The set $U_1 = V \cap U$ is a neighbourhood of b . Since $g^{-1} : C_{\varepsilon, \delta}(a) \cup S_\delta(a) \rightarrow U \cap \bar{D}$ is continuous at $g(b)$, it follows that there exists $0 < r < \delta - \varepsilon$ such that $g^{-1}(B(g(b), r) \cap (C_{\varepsilon, \delta}(a) \cup S_\delta(a))) \subset U_1 \cap \bar{D}$. We consider the set $W = g^{-1}(B(g(b), r) \cap (C_{\varepsilon, \delta}(a) \cup S_\delta(a))) \cup (U_1 \setminus \bar{D})$. Obviously, W is a neighbourhood of b and $W \subset (U_1 \cap \bar{D}) \cup (U_1 \setminus \bar{D}) \subset U_1 \subset V$. On the other hand,

$$W \cap D = g^{-1}(B(g(b), r) \cap C_{\varepsilon, \delta}(a)) = g^{-1}(B(g(b), r) \cap B(a, \delta)),$$

since $C_{\varepsilon, \delta}(a) \cap \bar{B}(a, r) = \emptyset$ and $C_{\varepsilon, \delta}(a) \cup \bar{B}(a, r) = B(a, \delta)$. By Proposition 3, the set $B(g(b), r) \cap B(a, \delta)$ is arc connected and hence $W \cap D$ is arc connected. \square

Let D be a domain in a Q -Ahlfors regular metric space X . Next, we recall the following definitions.

Definition 3.5. D is called Q -**Loewner domain**, $Q \geq 1$, if there exists a decreasing function $\varphi : (0, \infty) \rightarrow (0, \infty)$ such that

$$M_Q(\Delta(E, F; D)) \geq \varphi(t)$$

whenever E and F are disjoint nondegenerate continua in D satisfying

$$\text{dist}(E, F) \leq t \cdot \min\{\text{diam}E, \text{diam}F\}.$$

The (arc) quasiconformal accessibility in the Euclidean space can be formulated in the same way in our framework.

Definition 3.6. We say that D is **(arc) quasiconformally accessible** at $b \in \partial D$ if for every neighbourhood U of b there exist a continuum $F \subset D$ and a positive number $\delta > 0$ such that

$$M_Q(\Delta(E, F; D)) \geq \delta$$

for all (arc) connected sets $E \subset D$ with $b \in \bar{E}$ and $E \cap \partial U \neq \emptyset$.

Definition 3.7. We say that D is **(arc) strictly quasiconformally accessible** at $b \in \partial D$ if for every neighbourhood U of b there exist a neighbourhood V of b , $V \subset U$, a continuum F in D and a real number $\delta > 0$ such that $M_Q(\Delta(E, F; D)) \geq \delta$ for every $E \subset D$ (arc) connected with $E \cap \partial V \neq \emptyset$ and $E \cap \partial U \neq \emptyset$. (Definition 10.[1])

In the sequel, we shall prove the following result.

Theorem 3.2. Let (X, d, μ) be a Q -Ahlfors regular, rectifiably connected and locally path connected metric space, $Q \geq 1$. Then any bounded Q -Loewner domain D in X is arc strictly quasiconformally accessible on the boundary.

Proof. Since D is Q -Loewner domain, it follows that if there exists a decreasing function $\varphi : (0, \infty) \rightarrow (0, \infty)$ such that

(*)

$$M_Q(\Delta(E, F; D)) \geq \varphi(t)$$

whenever E and F are disjoint non-degenerate continua in D satisfying

$$\text{dist}(E, F) \leq t \cdot \min\{\text{diam}E, \text{diam}F\}.$$

Let x be a boundary point of D and let U be a neighbourhood of x . Hence, there exists a ball $B(x, r)$ of center x and radius $r < \text{diam}D$ such that $B(x, r) \subset U$. We denote $S(x, r) = \{y \in X, d(x, y) = r\}$. We can take two points $a \in S(x, \frac{2r}{3}) \cap D$ and $b \in S(x, \frac{5r}{6}) \cap D$. By Theorem 1, we have that D is an arc domain and hence there exists a curve γ in D which connected a and b . We can pick a subcurve γ' of γ which connects $S(x, \frac{2r}{3})$ and $S(x, \frac{5r}{6})$ in $B[x, \frac{5r}{6}] \setminus B(x, \frac{2r}{3})$. Thus, $l(\gamma') \geq \frac{r}{6}$. We set $F = \gamma'$ which is a continuum in D , and $V = B(x, \frac{r}{3}) \subset U$. Every arc connected set E in D which intersects ∂U and ∂V will also intersect $S(x, \frac{2r}{3})$ and $S(x, \frac{5r}{6})$. Since E is arc

connected set, we can choose a curve γ_1 in E which connects $S(x, \frac{r}{2})$ and $S(x, \frac{r}{3})$ and lies in $B[x, \frac{r}{2}] \setminus B(x, \frac{r}{3})$. Set $E' = \gamma_1$. Note that E' is a continuum and $E' \cap F = \emptyset$. On the other hand, $\min\{diam E', diam F\} \geq \frac{r}{6}$ that implies $\frac{dist(E', F)}{\min\{diam E', diam F\}} \leq \frac{6dist(E', F)}{r} \leq \frac{6diam D}{r}$. Using (*), we obtain

$$M_Q(\Delta(E, F; D)) \geq M_Q(\Delta(E', F; D)) \geq \varphi\left(\frac{6diam D}{r}\right) > 0.$$

We denote $\delta = \varphi\left(\frac{6diam D}{r}\right)$ and hence

$$M_Q(\Delta(E, F; D)) \geq \delta,$$

whenever $E \subset D$ is arc connected set with $E \cap \partial U \neq \emptyset$ and $E \cap \partial V \neq \emptyset$. Thus, we get the desired conclusion. \square

Remark 3.1. *Note that (arc) strictly quasiconformal accessibility implies (arc) quasiconformal accessibility, it follows that in above theorem, D is arc quasiconformally accessible on the boundary.*

Corollary 3.1. *If (X, d, μ) is a Q - Ahlfors regular and geodesic metric space, $Q \geq 1$, then any bounded Q -Loewner domain D in X is arc strictly quasiconformally accessible on the boundary.*

Proof. The proof follows from Theorem 4, since every geodesic metric space is rectifiably connected and locally path connected. \square

4 Continuous boundary extension of quasiconformal mapping between domains in metric spaces

In this section we give several boundary extension theorems for quasiconformal mappings.

Let us consider two Q - Ahlfors regular and geodesic metric measure spaces (X, d, μ) and (Y, d', μ') , with $1 \leq Q < \infty$, and two domains $D \subset X$, $D' \subset Y$.

We recall the following two results which will be use for to prove our theorems of boundary extension.

Lemma 4.1. *If D is locally (arc) connected on the boundary, $\overline{D'}$ is compact and D' is (arc) strictly quasiconformally accessible on the boundary, then every quasiconformal mapping $f : D \rightarrow D'$ has a continuous extension $f^* : \overline{D} \rightarrow \overline{D'}$. (Corollary 5[1])*

Lemma 4.2. *Suppose that $f : D \rightarrow D'$ is quasiconformal mapping and $\overline{D}, \overline{D'}$ are compact sets. If D, D' are locally (arc) connected on the boundary and (arc) strictly quasiconformally accessible on the boundary, then f can be extended to a homeomorphism $f^* : \overline{D} \rightarrow \overline{D'}$. (Corollary 7[1])*

Theorem 4.1. *Suppose that D is locally arc connected on the boundary, D' is a Q - Loewner domain and $\overline{D'}$ is compact. Then every quasiconformal mapping $f : D \rightarrow D'$ has a continuous extension $f^* : \overline{D} \rightarrow \overline{D'}$.*

Proof. Since D' is a bounded Q - Loewner domain, by Corollary 3 it follows that D' is arc strictly quasiconformally accessible on the boundary. Using Lemma 2, we get the desired conclusion. \square

Corollary 4.1. *Suppose that X is a GC-geodesic spaces, D' is a Q - Loewner domain and $\overline{D'}$ is compact. Then every quasiconformal mapping $f : B \rightarrow D'$ has a continuous extension $f^* : \overline{B} \rightarrow \overline{D'}$, where B is a ball in X .*

Proof. By Proposition 4, we have that B is locally arc connected on the boundary and using theorem 5 we obtain the desired conclusion. \square

Theorem 4.2. *Suppose that X is a GC-geodesic space, D is globally quasiconformally collared on the boundary, $\overline{D'}$ is compact and D' is arc strictly quasiconformally accessible on the boundary. Then every quasiconformal mapping $f : D \rightarrow D'$ has a continuous extension $f^* : \overline{D} \rightarrow \overline{D'}$.*

Proof. The proof follows by Theorem 3 and Lemma 2. \square

Theorem 4.3. *Suppose that X is a GC-geodesic space, D is globally quasiconformally collared on the boundary, D' is a Q - Loewner domain and $\overline{D'}$ is compact. Then every quasiconformal mapping $f : D \rightarrow D'$ has a continuous extension $f^* : \overline{D} \rightarrow \overline{D'}$.*

Proof. Since D is globally quasiconformally collared on the boundary, by Theorem 3 we get D is locally arc connected on the boundary. On the other hand, D' is a Q - Loewner domain and using Corollary 3, it follows that D' is arc strictly quasiconformally accessible on the boundary. By Lemma 2, we obtain the desired conclusion. \square

Theorem 4.4. *Suppose that X and Y are two GC-geodesic spaces, $f : D \rightarrow D'$ is a quasiconformal mapping and $\overline{D}, \overline{D'}$ are compact sets. If D, D' are globally quasiconformally collared on the boundary and Q - Loewner domains, then f can be extended to a homeomorphism $f^* : \overline{D} \rightarrow \overline{D'}$.*

Proof. The proof follows by Theorem 3, Theorem 4 and Lemma 2. \square

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