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ON PROJECTIVE COMPLEX RANDERS CHANGES

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Abstract

In this paper we study the relation between complex Randers changes and projective changes of complex Finsler metrics. We consider complex Randers changes of a generalized Berwald complex Finsler metric and we determine the necessary and sufficient conditions for the generalized Berwald property to be preserved by these changes. Using this theory, a recursive sequence of projectively related complex Berwald metrics is pointed out.

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1 Introduction

The problem of projective changes between two real Finsler metrics is quite old in geometry and it has been studied by many geometers, [8, 16, 21, 13, 19, 9, 15]. Its origin is formulated in Hilbert's Fourth Problem: determine the metrics on an open subset in $\mathbf{R}^n$, whose geodesics are straight lines. Two Finsler metrics, on a common underlying manifold, are called projectively related if they have the same geodesics as point sets.

The notion of Randers change has been proposed by M. Matsumoto in [16]. Further substantial contributions on this topic are due to C. Shibata [22], M. Hashiguchi, Y. Ichijyō [13], H. S. Park, I. Y. Lee [19], Bácsó, Z. Kovacs [9], etc.

The main themes from projective real Finsler geometry have recently been developed in complex Finsler geometry, [5, 6, 7]. Two complex Finsler metrics F and $\tilde{F}$, on a common underlying manifold M, are called projectively related if any complex geodesic curve, in [1]' s sense, of the first metric is also a complex geodesic curve for the second one and conversely. As we proved in [5], this means that between the spray coefficients G^i and $\tilde{G}^i$ there is a so-called *projective change* $\tilde{G}^i = G^i + B^i + P\eta^i$, where P is a smooth function on $T'M$ with complex values and $B^i := \frac{1}{2}(\tilde{\theta}^{*i} - \theta^{*i})$.

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Complex Randers metrics $\alpha + |\beta|$, where α is a purely Hermitian metric and β is a $(1,0)$ - form, both on the base manifold, are remarkable in complex Finsler geometry, and they represent a situation in which Hermitian geometry properly interferes with complex Finsler geometry, [3]. A generalization of a complex Randers metric is given by $\tilde{F} = F + |\beta|$, where (M, F) is a complex Finsler space, which compared to projective changes lead us to *complex Randers changes,* which constitute the subject of the present paper.

We consider a complex Randers change of a generalized Berwald complex Finsler metric. The complex Finsler metric obtained by a complex Randers change is also a generalized Berwald one if and only if the $(1,0)$ - form β satisfies a special regularity condition, (see Lemma 4.1 and Corollary 4.1). The necessary and sufficient conditions for a complex Randers change to be a projective change are given in Lemma 4.2. By requiring the complex Finsler metric F to be a complex Berwald one, [6], the complex Randers change $\tilde{F} = F + |\beta|$ is a projective change if and only if $\tilde{F}$ is a complex Berwald metric, (see Theorem 4.1). Moreover, by means of the obtained results we construct a recursive sequence of complex Berwald metrics which are projectively related, (Corollary 4.2).

The paper is organized as follows. In Section 2, we recall some preliminary properties of $n-$dimensional complex Finsler spaces, needed for our aforementioned study. In Section 3 we make a survey of projectively related complex Finsler metrics. Section 3 contains the proofs of the above mentioned theorems and some interesting examples.

2 Preliminaries

Let M be an n-dimensional complex manifold and $z = (z^k)_{k=\overline{1,n}}$ be the complex coordinates in a local chart. The complexified $T_C M$ of the real tangent bundle $T_R M$, splits into the sum of the holomorphic tangent bundle $T'M$ and its conjugate $T''M$. The bundle $T'M$ is itself a complex manifold and the local coordinates in a local chart will be denoted by $u = (z^k, \eta^k)_{k=\overline{1,n}}$. These are transformed into $(z'^k, \eta'^k)_{k=\overline{1,n}}$ by the rules $z'^k = z'^k(z)$ and $\eta'^k = \frac{\partial z'^k}{\partial z^l}\eta^l$.

A *complex Finsler space* is a pair (M, F), where $F : T'M \to \mathbb{R}^+$ is a continuous function satisfying the following conditions:

i) $L := F^2$ is smooth on $\widetilde{T'M} := T'M \backslash \{0\}$;

ii) $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;

iii) $F(z, \lambda\eta) = |\lambda| F(z, \eta)$ for $\forall \lambda \in \mathbb{C}$;

iv) the Hermitian matrix $\big(g_{i\bar{j}}(z, \eta)\big)$ is positive definite, where $g_{i\bar{j}} := \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ is the fundamental metric tensor. Equivalently, this means that the indicatrix of the space is strongly pseudo-convex.

Consequently, from *iii*) we have $\frac{\partial L}{\partial \eta^k}\eta^k = \frac{\partial L}{\partial \bar{\eta}^k}\bar{\eta}^k = L$, $\frac{\partial g_{i\bar{j}}}{\partial \eta^k}\eta^k = \frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k}\bar{\eta}^k = 0$ and $L = g_{i\bar{j}}\eta^i\bar{\eta}^j$.

Consider the sections of the complexified tangent bundle of $T'M$. Then by $VT'M \subset T'(T'M)$ we denote the vertical bundle, locally spanned by $\{\frac{\partial}{\partial \eta^k}\}$, and by $VT''M$, its conjugate. The idea of complex nonlinear connection, briefly (*c.n.c.*), is an instrument in the 'linearization' of the geometry of the manifold $T'M$. A (*c.n.c.*) is a supplementary complex subbundle to $VT'M$ in $T'(T'M)$, i.e. $T'(T'M) = HT'M \oplus VT'M$. The horizontal distribution $H_uT'M$ is locally spanned by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z,\eta)$ are the coefficients of the (*c.n.c.*). The pair $\{\delta_k := \frac{\delta}{\delta z^k}, \dot{\partial}_k := \frac{\partial}{\partial \eta^k}\}$ will be called the adapted frame of the (*c.n.c.*), which obey the change rules $\delta_k = \frac{\partial z'^j}{\partial z^k}\delta'_j$ and $\dot{\partial}_k = \frac{\partial z'^j}{\partial z^k}\dot{\partial}'_j$. By conjugation everywhere we obtain an adapted frame $\{\delta_{\bar{k}}, \dot{\partial}_{\bar{k}}\}$ on $T''_u(T'M)$. The dual adapted frames are $\{dz^k, \delta\eta^k := d\eta^k + N_j^k dz^j\}$ and $\{d\bar{z}^k, \delta\bar{\eta}^k\}$.

Let $S \in T'(T'M)$ be a complex spray. Locally, it can be expressed as follows

$$S = \eta^k \frac{\partial}{\partial z^k} - 2G^k(z,\eta)\frac{\partial}{\partial \eta^k} \tag{2.1}$$

where G^k are the spray coefficients, [17], which are $(2,0)$ - homogeneous with respect to η, i.e. $(\dot{\partial}_k G^i)\eta^k = 2G^i$ and $(\dot{\partial}_{\bar{k}} G^i)\bar{\eta}^k = 0$.

Between the notions of complex spray and (*c.n.c.*) there exists an interdependence, each one of them determining the other, (for more details see [17]).

A (*c.n.c.*) related only to the fundamental function of the complex Finsler space (M,F) is the so-called Chern-Finsler (*c.n.c.*), (see [1]), with the local coefficients $N_j^i := g^{\overline{m}i}\frac{\partial g_{l\overline{m}}}{\partial z^j}\eta^l$. Subsequently, δ_k is the adapted frame with respect to the Chern-Finsler (*c.n.c.*). A Hermitian connection D, of $(1,0)-$ type, which satisfies in addition $D_{JX}Y = JD_XY$, for all horizontal vectors X, where J is the natural complex structure of the manifold, is the Chern-Finsler connection, [1]. It is locally given by the following coefficients (see [17]):

$$L^i_{jk} := g^{\bar{l}i}\delta_k g_{j\bar{l}} = \dot{\partial}_j N^i_k \;\; ; \; C^i_{jk} := g^{\bar{l}i}\dot{\partial}_k g_{j\bar{l}}. \tag{2.2}$$

In [1]'s terminology, the complex Finsler space (M,F) is *Kähler* iff $T^i_{jk}\eta^j = 0$ and *weakly Kähler* iff $g_{i\bar{l}}T^i_{jk}\eta^j\bar{\eta}^l = 0$, where $T^i_{jk} := L^i_{jk} - L^i_{kj}$. We notice that in the particular case of complex Finsler metrics which come from Hermitian metrics on M, called *purely Hermitian metrics* in [17], (i.e. $g_{i\bar{j}} = g_{i\bar{j}}(z)$), these two notions of Kähler are the same. On the other hand, as in Aikou's work [2], a complex Finsler space which is Kähler and $L^i_{jk} = L^i_{jk}(z)$ is called a *complex Berwald* space.

In [17] is proved that the Chern-Finsler (*c.n.c.*) does not generally come from a complex spray. But, its local coefficients N^i_j always determine a complex spray with coefficients $G^i = \frac{1}{2}N^i_j\eta^j$. Further, G^i induce a (*c.n.c.*) denoted by $\overset{c}{N^i_j} := \dot{\partial}_j G^i$ which is called *canonical* in [17], and is proved that it coincides with Chern-Finsler (*c.n.c.*) if and only if the complex Finsler metric is Kähler. With respect to the canonical (*c.n.c.*), we consider the frame $\{\overset{c}{\delta}_k, \dot{\partial}_k\}$, where $\overset{c}{\delta}_k := \frac{\partial}{\partial z^k} - \overset{c}{N^j_k}\dot{\partial}_j$, and its dual coframe $\{dz^k, \overset{c}{\delta}\eta^k\}$, where $\overset{c}{\delta}\eta^k := d\eta^k + \overset{c}{N^k_j} dz^j$. Moreover, we associate to the canonical (*c.n.c.*) a complex linear connection of Berwald type $B\Gamma$ with its connection form

$$\omega^i_j(z,\eta) = G^i_{jk}dz^k + G^i_{j\bar{k}}d\bar{z}^k, \tag{2.3}$$

where $G^i_{jk} := \dot{\partial}_k \overset{c}{N^i_j} = G^i_{kj}$ and $G^i_{j\bar{k}} := \dot{\partial}_{\bar{k}} \overset{c}{N^i_j}$.

Note that the spray coefficients obey the relations $2G^i = N^i_j\eta^j = \overset{c}{N^i_j}\eta^j = G^i_{jk}\eta^j\eta^k = L^i_{jk}\eta^j\eta^k$. We denote by $G^i_{jkh} := \dot{\partial}_h G^i_{jk}$, $G^i_{jk\bar{h}} = \dot{\partial}_{\bar{h}} G^i_{jk}$ and $G^i_{j\bar{k}h} := \dot{\partial}_h G^i_{j\bar{k}}$ the hv-, $\bar{h}\bar{v}$- and $h\bar{v}$- curvature tensors respectively; their properties are pointed out in [4].

An extension of complex Berwald spaces, directly related to the $B\Gamma$ connection, are *generalized Berwald* spaces, studied by us in [4]. They have the coefficients G^i_{jk} depending only on the position z, equivalently with $\dot{\partial}_{\bar{h}} G^i = 0$, i.e. $B\Gamma$ is of $(1,0)$ - type. Since in the Kähler case $G^i_{jk} = L^i_{jk}$, any complex Berwald space is generalized Berwald. Conversely, in [6] we proved that any generalized Berwald space, which is weakly Kähler, is a complex Berwald space.

3 Projectively related complex Finsler metrics

In Abate-Patrizio's sense, (see [1] p. 101), the equations of a complex geodesic curve $z = z(s)$ of (M, F), with s a real parameter, can be expressed as follows

$$\frac{d^2 z^i}{ds^2} + 2G^i(z(s), \frac{dz}{ds}) = \theta^{*i}(z(s), \frac{dz}{ds})\ ;\ i = \overline{1,n}, \tag{3.1}$$

where by $z^i(s)$, $i = \overline{1,n}$, we denote the coordinates along of curve $z = z(s)$ and $\theta^{*k} := 2g^{\bar{j}k}\, \overset{c}{\delta_{\bar{j}}} L$. Note that θ^{*i} vanishes identically if and only if the space is weakly Kähler.

Let $\tilde{F}$ be another complex Finsler metric on the underlying manifold M. Corresponding to the metric $\tilde{F}$, we have the spray coefficients $\tilde{G}^i$ and the functions $\tilde{\theta}^{*i}$. The complex Finsler metrics F and $\tilde{F}$ on the manifold M, are called *projectively related* if they have the same complex geodesics as point sets. This means that for any complex geodesic curve $z = z(s)$ of (M, F) there is a transformation of its parameter s, $\tilde{s} = \tilde{s}(s)$, with $\frac{d\tilde{s}}{ds} > 0$, such that $z = z(\tilde{s}(s))$ is a geodesic of $(M, \tilde{F})$, and conversely.

Theorem 3.1. *([5]). Let F and $\tilde{F}$ be two complex Finsler metrics on the manifold M. Then F and $\tilde{F}$ are projectively related if and only if there exists a smooth function P on $T'M$ with complex values, such that*

$$\tilde{G}^i = G^i + B^i + P\eta^i;\ i = \overline{1,n}, \tag{3.2}$$

where $B^i := \frac{1}{2}(\tilde{\theta}^{*i} - \theta^{*i})$.

The relations (3.2) between the spray coefficients $\tilde{G}^i$ and G^i of the projectively related complex Finsler metrics F and $\tilde{F}$ is called a *projective change.* The projective change (3.2) gives rise to various projective invariants, for more details see [6].

Theorem 3.2. *([5]). Let F and $\tilde{F}$ be complex Finsler metrics on the manifold M. Then, F and $\tilde{F}$ are projectively related if and only if*

$$\dot{\partial}_{\bar{r}}(\delta_k \tilde{F})\eta^k + 2(\dot{\partial}_{\bar{r}} G^l)(\dot{\partial}_l \tilde{F}) = \frac{1}{\tilde{F}}(\delta_k \tilde{F})\eta^k(\dot{\partial}_{\bar{r}} \tilde{F})\ ;$$

$$B^r = -\frac{1}{\tilde{F}}\theta^{*l}(\dot{\partial}_l\tilde{F})\eta^r \ ; \ P = \frac{1}{\tilde{F}}[(\delta_k\tilde{F})\eta^k + \theta^{*i}(\dot{\partial}_i\tilde{F})]. \tag{3.3}$$

Moreover, the projective change is $\tilde{G}^i = G^i + \frac{1}{\tilde{F}}(\delta_k\tilde{F})\eta^k\eta^i$.

Note that, the weakly Kähler property is preserved by projective changes. Moreover, if the metric F is generalized Berwald, then $\tilde{F}$ is also generalized Berwald.

4 Complex Randers changes

We consider a complex Finsler metric $F(z,\eta) = \sqrt{g_{i\bar{j}}(z,\eta)\eta^i\bar{\eta}^j}$ and a differential $(1,0)-$form $\beta(z,\eta) := b_i(z)\eta^i$, both on the manifold M.

Definition 4.1. *A change of complex Finsler metrics $F(z,\eta) \to \tilde{F}(z,\eta)$ is called a complex Randers change of F if*

$$\tilde{F}(z,\eta) = F(z,\eta) + |\beta|. \tag{4.1}$$

In particular, if F is a purely Heimitian metric, i.e. $F(z,\eta) = \sqrt{g_{i\bar{j}}(z)\eta^i\bar{\eta}^j}$, then $\tilde{F}(z,\eta)$, becomes a complex Randers metric, (see Theorem 2.1, [3]). Taking into account that in the paper [5] we have an exhaustive study of the projectiveness of the complex Randers metric, our next investigation is focused on the complex Randers change with F a non purely Hermitian metric.

It is a technical computation to give the expressions of the geometric objects of the space $(M,\tilde{F})$, obtained by the complex Randers change (4.1). Certainly, they involve some trivial calculus which leads to

$$\begin{aligned} \tilde{g}_{i\bar{j}} &= \frac{\tilde{F}}{F}g_{i\bar{j}} - \frac{\tilde{F}}{2F^3}\eta_i\eta_{\bar{j}} + \frac{\tilde{F}}{2|\beta|}b_ib_{\bar{j}} + \frac{1}{2\tilde{L}}\tilde{\eta}_i\tilde{\eta}_{\bar{j}}; \\ \tilde{g}^{\bar{j}i} &= \frac{F}{\tilde{F}}g^{\bar{j}i} + \frac{|\beta|(F||b||^2+|\beta|)}{\tilde{L}\gamma}\eta^i\bar{\eta}^j - \frac{F^3}{\tilde{F}\gamma}b^i\bar{b}^j - \frac{F}{\tilde{F}\gamma}(\bar{\beta}\eta^i\bar{b}^j + \beta b^i\bar{\eta}^j); \\ \tilde{N}^i_j &= N^i_j + \frac{1}{\gamma}(\eta_{\bar{r}}\frac{\partial\bar{b}^r}{\partial z^j} - \frac{\beta^2}{|\beta|^2}\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)\xi^i + \frac{\beta}{2|\beta|}k^{\bar{r}i}\frac{\partial b_{\bar{r}}}{\partial z^j}, \end{aligned} \tag{4.2}$$

where $k^{\bar{r}i} := 2Fg^{\bar{r}i} + \frac{2(F||b||^2+2|\beta|)}{\gamma}\eta^i\bar{\eta}^r - \frac{2F^3}{\gamma}b^i\bar{b}^r - \frac{2F}{\gamma}(\bar{\beta}\eta^i\bar{b}^r + \beta b^i\bar{\eta}^r)$, $\gamma := \tilde{F}^2 + F^2(||b||^2-1)$, $\xi^i := \bar{\beta}\eta^i + F^2b^i$, $N^k_j := g^{\bar{m}k}\frac{\partial g_{l\bar{m}}}{\partial z^j}\eta^l$, with the settings

$$\begin{aligned} \eta_i \ &: = 2F(\dot{\partial}_iF) \ ; \ \tilde{\eta}_i := 2\tilde{F}(\dot{\partial}_i\tilde{F}) = \frac{\tilde{F}}{F}\eta_i + \frac{\tilde{F}\bar{\beta}}{|\beta|}b_i; \\ \dot{\partial}_i|\beta| \ &= \ \frac{\bar{\beta}}{2|\beta|}b_i \ ; \ b^i := g^{\bar{j}i}b_{\bar{j}} \ \ ; \ \ ||b||^2 := g^{\bar{j}i}b_ib_{\bar{j}} \ \ ; \ \ b^{\bar{i}} := \bar{b}^i. \end{aligned} \tag{4.3}$$

Therefore, the spray coefficients are

$$\tilde{G}^i = G^i + \frac{1}{2\gamma}(\eta_{\bar{r}}\frac{\partial\bar{b}^r}{\partial z^j} - \frac{\beta^2}{|\beta|^2}\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)\xi^i\eta^j + \frac{\beta}{4|\beta|}k^{\bar{r}i}\frac{\partial b_{\bar{r}}}{\partial z^j}\eta^j. \tag{4.4}$$

Next, for complex Randers changes of a generalized Berwald metric we can prove the following.

Lemma 4.1. *Let (M,F) be a connected generalized Berwald space and let $\tilde{F}(z,\eta)=F(z,\eta)+|\beta|$ be a complex Randers change. Then, $(M,\tilde{F})$ is a generalized Berwald space if and only if $(\bar{\beta}\eta_{\bar{r}}\frac{\partial \bar{b}^r}{\partial z^j}+\beta\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)\eta^j=0$. Moreover, given any of them, $\tilde{G}^i=G^i$.*

Proof. First, we prove the direct implication. If $(M,\tilde{F})$ is generalized Berwald space, then $2\tilde{G}^i=\tilde{G}^i_{jk}(z)\ \eta^j\eta^k$, which means that $\tilde{G}^i$ is quadratic in η. Also, G^i is quadratic in η. Thus, using (4.4) we have

$$F|\beta|\{-\beta[(F^2||b||^2+|\beta|^2)g^{\bar{r}i}+||b||^2\bar{\eta}^r\eta^i-F^2\bar{b}^r b^i-\bar{\beta}\eta^i\bar{b}^r-\beta b^i\bar{\eta}^r]\frac{\partial b_{\bar{r}}}{\partial z^j}\eta^j$$
$$+4|\beta|^2(\tilde{G}^i-G^i)\}+|\beta|^2[2(F^2||b||^2+|\beta|^2)(\tilde{G}^i-G^i)-2F^2\beta g^{\bar{r}i}\frac{\partial b_{\bar{r}}}{\partial z^j}\eta^j$$
$$-(\bar{\beta}\eta_{\bar{r}}\frac{\partial \bar{b}^r}{\partial z^j}+\beta\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)\eta^j\eta^i-\frac{F^2\beta}{|\beta|^2}(\bar{\beta}\eta_{\bar{r}}\frac{\partial \bar{b}^r}{\partial z^j}-\beta\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)\eta^j b^i]=0,$$

which contains an irrational part and a rational part. Thus, we deduce

$$\begin{aligned}
&\beta[(F^2||b||^2+|\beta|^2)g^{\bar{r}i}+||b||^2\bar{\eta}^r\eta^i-F^2\bar{b}^r b^i-\bar{\beta}\eta^i\bar{b}^r-\beta b^i\bar{\eta}^r]\frac{\partial b_{\bar{r}}}{\partial z^j}\eta^j\\
=\ &4|\beta|^2(\tilde{G}^i-G^i)\ \text{ and}\\
&(\bar{\beta}\eta_{\bar{r}}\frac{\partial \bar{b}^r}{\partial z^j}+\beta\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)\eta^j\eta^i+\frac{F^2\beta}{|\beta|^2}(\bar{\beta}\eta_{\bar{r}}\frac{\partial \bar{b}^r}{\partial z^j}-\beta\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)\eta^j b^i+2F^2\beta g^{\bar{r}i}\frac{\partial b_{\bar{r}}}{\partial z^j}\eta^j\\
=\ &2(\alpha^2||b||^2+|\beta|^2)(\tilde{G}^i-G^i).
\end{aligned}$$

Contractions with b_i and η_i yield

$$(\tilde{G}^i-G^i)b_i=0; \tag{4.5}$$

$$\begin{aligned}
4|\beta|^2(G^i-\tilde{G}^i)\eta_i+2\beta F^2(||b||^2\bar{\eta}^r-\bar{\beta}\bar{b}^r)\frac{\partial b_{\bar{r}}}{\partial z^j}\eta^j &= 0;\\
\bar{\beta}(F^2||b||^2+|\beta|^2)\eta_{\bar{r}}\frac{\partial \bar{b}^r}{\partial z^j}\eta^j-\beta(\alpha^2||b||^2-|\beta|^2)\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r\eta^j+2\alpha^2|\beta|^2\bar{b}^r\frac{\partial b_{\bar{r}}}{\partial z^j}\eta^j &= 0;\\
(\alpha^2||b||^2+|\beta|^2)(G^i-\tilde{G}^i)\eta_i+\alpha^2(\bar{\beta}\eta_{\bar{r}}\frac{\partial \bar{b}^r}{\partial z^j}+\beta\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)\eta^j &= 0.
\end{aligned}$$

Adding the second and the third relations from (4.5), we obtain $4|\beta|^2(G^i-\tilde{G}^i)\eta_i+(F^2||b||^2+|\beta|^2)(\bar{\beta}\eta_{\bar{r}}\frac{\partial \bar{b}^r}{\partial z^j}+\beta\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)\eta^j=0$.

This together with the fourth equation from (4.5) implies $(G^i-\tilde{G}^i)\eta_i=0$ and $(\bar{\beta}\eta_{\bar{r}}\frac{\partial \bar{b}^r}{\partial z^j}+\beta\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)\eta^j=0$.

Conversely, if $(\bar{\beta}\eta_{\bar{r}}\frac{\partial \bar{b}^r}{\partial z^j}+\beta\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)\eta^j=0$, its differentiation with respect to $\bar{\eta}^m$ and the fact that G^i are holomorphic in η, gives $(l_{\bar{r}}\frac{\partial \bar{b}^r}{\partial z^j}b_{\bar{m}}+\beta\frac{\partial b_{\bar{m}}}{\partial z^j})\eta^j=0$. The last two relations imply

$$g^{\bar{m}i}\frac{\partial b_{\bar{m}}}{\partial z^j}\eta^j=\frac{\beta}{|\beta|^2}\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r b^i\eta^j\ \text{ and }\ \bar{b}^m\frac{\partial b_{\bar{m}}}{\partial z^j}\eta^j=||b||^2\frac{\beta}{|\beta|^2}\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r\eta^j,$$

which substituted into (4.4) imply $\tilde{G}^i=G^i$ and so, $\tilde{G}^i$ are holomorphic in η, i.e., $\tilde{F}$ is generalized Berwald. □

Subsequently, our aim is to determine the necessary and sufficient conditions in which the complex Randers change (4.1) is a projective change, that is, to establish when the complex Finsler metrics F and $\tilde{F}$ from (4.1) are projectively related. A simple computation shows that

$$(\delta_k\tilde{F})\eta^k = (\delta_k|\beta|)\eta^k = \frac{1}{2|\beta|}(\bar{\beta}\eta_{\bar{r}}\frac{\partial\bar{b}^r}{\partial z^k} + \beta\frac{\partial b_{\bar{r}}}{\partial z^k}\bar{\eta}^r)\eta^k, \tag{4.6}$$

because $(\delta_k F)\eta^k = 0$ and

$$\theta^{*i}(\dot{\partial}_i\tilde{F}) = \frac{\bar{\beta}}{|\beta|}(\overset{c}{\delta}_{\bar{m}} F)b^{\bar{m}}. \tag{4.7}$$

Thanks to Lemma 4.1 we have proven,

Corollary 4.1. *Let (M,F) be a connected generalized Berwald space and let $\tilde{F}(z,\eta) = F(z,\eta)+|\beta|$ be a complex Randers change. Then, $(M,\tilde{F})$ is a generalized Berwald space if and only if $(\delta_k|\beta|)\eta^k = 0$.*

Lemma 4.2. *Let (M,F) be a connected generalized Berwald space. Then, the complex Randers change (4.1) is a projective change if and only if*

$$(\delta_k|\beta|)\eta^k = 0 \text{ and } B^i = -P\eta^i,$$

for any $i=\overline{1,n}$, where $P = \frac{\bar{\beta}}{|\beta|}(\overset{c}{\delta}_{\bar{m}} F)b^{\bar{m}}$. Moreover, given any of them, the projective change is $\tilde{G}^i = G^i$.

Proof. Since F is generalized Berwald and the metrics F and $\tilde{F}$ are projectively related, then $\tilde{F}$ is also generalized Berwald. So that, by (4.6), (4.7) and Corollary 4.1, the conditions (3.3) are reduced to $B^i = -P\eta^i$, for any $i=\overline{1,n}$, where $P = \frac{\bar{\beta}}{|\beta|}(\overset{c}{\delta}_{\bar{m}} F)b^{\bar{m}}$.

Conversely, since $(\delta_k|\beta|)\eta^k = 0$, then the first condition from (3.3) is identically satisfied and by (4.7), $B^i = -\frac{1}{\tilde{F}}\theta^{*l}(\dot{\partial}_l\tilde{F})\eta^i$ and $P = \frac{1}{\tilde{F}}\theta^{*i}(\dot{\partial}_i\tilde{F})$. All these conditions imply that the metrics F and $\tilde{F}$ are projectively related. □

Theorem 4.1. *Let (M,F) be a connected complex Berwald space. Then, the complex Randers change (4.1) is a projective change if and only if $\tilde{F}$ is a complex Berwald metric.*

Proof. Suppose that the complex Randers change (4.1) is a projective change. Hence, it preserves the weakly Kähler and generalized Berwald properties of the metric F. Thus, $\tilde{F}$ is a complex Berwald metric.

Conversely, since F and $\tilde{F}$, related by (4.1), are complex Berwald metrics, the conditions (3.3) are identically satisfied. Thus, the metrics F and $\tilde{F}$ are projectively related. □

An example. Let $\Delta = \left\{(z,w)\in\mathbf{C}^2,\ |w|<|z|<1\right\}$ be the Hartogs triangle with the Kähler-purely Hermitian metric

$$a_{i\bar{j}} = \frac{\partial^2}{\partial z^i\partial\bar{z}^j}(\log\frac{1}{(1-|z|^2)\,(|z|^2-|w|^2)});\ \alpha^2(z,w;\eta,\theta) = a_{i\bar{j}}\eta^i\bar{\eta}^j, \tag{4.8}$$

where z, w, η, θ are the local coordinates z^1, z^2, η^1, η^2, respectively, and $|z^i|^2 := z^i\bar{z}^i$, $z^i \in \{z, w\}$, $\eta^i \in \{\eta, \theta\}$. We choose

$$b_z = \frac{w}{|z|^2 - |w|^2};\ b_w = -\frac{z}{|z|^2 - |w|^2}. \tag{4.9}$$

With these tools we construct $\alpha(z, w, \eta, \theta) := \sqrt{a_{i\bar{j}}(z, w)\eta^i\bar{\eta}^j}$ and $\beta(z, \eta) = b_i(z, w)\eta^i$ and from here, we obtain the complex Randers metric $F = \alpha + |\beta|$. After some calculations it folows that the spray coefficients of the metric F are

$$\begin{aligned} G^z &= \overset{a}{G^z} = \frac{\bar{z}\eta^2}{1 - |z|^2}\ ; \\ G^w &= \overset{a}{G^w} = \frac{\bar{z}w\eta^2}{z}\left(\frac{1}{1 - |z|^2} + \frac{1}{|z|^2 - |w|^2}\right) - \frac{(|z|^2 + |w|^2)\eta\theta}{z\,(|z|^2 - |w|^2)} + \frac{\overline{w}\theta^2}{|z|^2 - |w|^2}, \end{aligned} \tag{4.10}$$

where $\overset{a}{G^z}$ and $\overset{a}{G^w}$ are the spray coefficients corresponding to the metric α. From (4.10) and (4.9) we deduce that F is a complex Berwald metric, and so, by Theorem 4.1, α and F are projectively related.

Given the complex Berwald metric $F = \alpha + |\beta|$, (with (4.8) and (4.9)), we consider the Randers change $\tilde{F} = F + |\beta|$, where β is same as in (4.9). Since F is a complex Berwald metric and $(\delta_k|\beta|)\eta^k = 0$, we obtain $\tilde{G}^z = G^z$ and $\tilde{G}^w = G^w$, which alows us to conclude that $\tilde{F}$ is also a complex Berwald metric. Applying Theorem 4.1, the considered Randers change is projective.

We complete our considerations with the following statement.

Corollary 4.2. *Let* (M, F_0) *be a connected complex Berwald space. Then,*

$$F_m = F_{m-1} + |\beta|,\ m \in \mathbf{N}, \tag{4.11}$$

is a recursive sequence of complex Berwald metrics, on the complex manifold M*, if and only if (4.11) is a projective Randers change for any* $m \in \mathbf{N}$*.*

The proof is obtained inductively, by using Theorem 4.1.

The recursive sequence can be rewritten as $F_m = F_0 + m|\beta|$, $m \in \mathbf{N}$, and by this, we can generate some examples of complex Berwald metrics. Indeed, choosing the tools $F_0 = \alpha$ and β from (4.8) and (4.9), we produce a lot of concrete examples of complex Berwald metrics on Hartogs triangle.

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