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ON THE DURRMEYER-KANTOROVICH TYPE OPERATOR

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Abstract

The purpose of this article is to give a Kantorovich generalization of Durrmeyer operators. We obtain convergence properties of our operators in the continous function space and Lebesque spaces.

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1 Introduction

For any given $n \in \mathbb{N}$ and $f \in C[0,1]$ Durrmeyer operators are defined by:

$$D_n(f,x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)\,f(t)\,dt,\ \ x \in [0,1], \tag{1}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Durrmeyer operators were introduced by J.L.Durrmeyer in 1967 [6] and studied intensively by M.M. Derriennic in 1981[4].

The Kantorovich operators are Bernstein operators modified given by [10]:

$$K_n(f,x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt,\ \ n \in \mathbb{N},\ \ f \in L_1[0,1],\ \ x \in [0,1]. \tag{2}$$

Similar Kantorovich type operators are obtain and studied by modification of other operators [1], [3], [5], [7], [8], [9], [11], [13], [14], [15], [16], [17].

In our paper we study new aspects of Durrmeyer-Kantorovich operators.

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2 Definition. Basic results.

In this article, we study the Durrmeyer-Kantorovich operators.

Definition 1. For any $n \in \mathbb{N}$ we define the operator $\widetilde{K}_n : L_p[0,1] \to C[0,1]$, given by

$$\widetilde{K}_n(f,x) = (n+3)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n+2,k+1}(t)\, f(t)\, dt, \tag{3}$$

for $f \in L_p[0,1]$, $x \in [0,1]$. We will refer to them as Durrmeyer-Kantorovich operators.

Lemma 1. Durrmeyer-Kantorovich operators are linear and positive, for $n \in \mathbb{N}$.

Proof. It is clear. The assertions follow from definition. □

Further note $e_k(t) = t^k,\ t \in [0,1],\ k \in \mathbb{N}$.

Lemma 2. For $x \in [0,1]$ and $n \in \mathbb{N}$ the Durrmeyer-Kantorovich operators (3) has the following property:

$$\widetilde{K}_n(e_{r+1},x) = \frac{nx+r+2}{n+r+4}\widetilde{K}_n(e_r,x) + \frac{x(1-x)}{n+r+4}\left(\widetilde{K}_n(e_r,x)\right)'. \tag{4}$$

Proof. Using the definition (3) it follows directly:

$$\widetilde{K}_n(e_r,x) = (n+3)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n+2,k+1}(t)\, t^r dt =$$

$$= (n+3)\sum_{k=0}^{n} p_{n,k}(x)\binom{n+2}{k+1} B(k+r+2, n-k+2).$$

Whence

$$\left(\widetilde{K}_n(e_r,x)\right)' = (n+3)\sum_{k=0}^{n}\frac{k-nx}{x(1-x)} p_{n,k}(x)\binom{n+2}{k+1} B(k+r+2,n-k+2) =$$

$$= \frac{(n+3)}{x(1-x)}\sum_{k=0}^{n} p_{n,k}(x)\binom{n+2}{k+1}\left[(k-nx)\, B(k+r+2,n-k+2)\right] =$$

$$= \frac{(n+3)}{x(1-x)}\sum_{k=0}^{n} p_{n,k}(x)\binom{n+2}{k+1}\left[(n+r+4)\, B(k+r+3,n-k+2) - \right.$$

$$\left. -(nx+r+2)\, B(k+r+2,n-k+2)\right] =$$

$$= \frac{1}{x(1-x)}\left[(n+r+4)\,\widetilde{K}_n(e_{r+1},x) - (nx+r+2)\,\widetilde{K}_n(e_r,x)\right].$$

From which we obtain the lemma immediately. □

Corollary 1. For $x \in [0,1]$ and $n \in \mathbb{N}$ the Durrmeyer-Kantorovich operators (3) have the following properties:

$$\widetilde{K}_n(e_0,x) = 1, \tag{5}$$

$$\widetilde{K}_n(e_1,x) = \frac{nx+2}{n+4}, \tag{6}$$

$$\widetilde{K}_n(e_2,x) = \frac{n(n-1)x^2+6nx+6}{(n+4)(n+5)}. \tag{7}$$

Proof. Using the definition (3) it follows directly

$$\widetilde{K}_n(e_0,x) = (n+3)\sum_{k=0}^{n} p_{n,k}(x) \int_0^1 \binom{n+2}{k+1} t^{k+1}(1-t)^{n+1-k}\,dt =$$

$$= (n+3)\sum_{k=0}^{n} p_{n,k}(x) \binom{n+2}{k+1} \beta(k+2, n+2-k) =$$

$$= (n+3)\sum_{k=0}^{n} p_{n,k}(x) \frac{(n+2)!}{(k+1)!\,(n+1-k)!} \cdot \frac{(k+1)!\,(n+1-k)!}{(n+3)!} = \sum_{k=0}^{n} p_{n,k}(x) = 1.$$

Relations (6) and (7) are obtained immediately using (5) and Lemma 2. □

Further we use the following notation:
(i) D is the differentiation operator,

$$D(f,x) = f'(x), \ f \in C_1[0,1], x \in [0,1],$$

(ii) I is the antiderivative operator,

$$I(f,x) = \int_0^x f(t)\,dt, \ f \in C[0,1], \ x \in [0,1].$$

Lemma 3. Let $n \in \mathbb{N}$. We have
i) $(D \circ I)(f) = f$, for all $f \in C[0,1]$,
ii) $(I \circ D)(f) = f$, for all $f \in C_1[0,1]$, such that $f(0) = 0$.

Proof. It is clear. □

Lemma 4. The operators $\widetilde{K}_n(f,x)$, as defined in (3), verify:
i) $\widetilde{K}_n((t-x),x) = \frac{2-4x}{n+4}$,
ii)$\widetilde{K}_n\left((t-x)^2,x\right) = \frac{(2n-20)x(1-x)+6}{(n+4)(n+5)}$.

Proof. Taking into account the linearity of $\widetilde{K}_n$ and Corollary 1, we have
i) $\widetilde{K}_n((t-x),x) = \widetilde{K}_n(e_1,x) - x\widetilde{K}_n(e_0,x) = \frac{nx+2}{n+4} - x = \frac{2-4x}{n+4}$.
ii) $\widetilde{K}_n\left((t-x)^2,x\right) = \widetilde{K}_n(e_2,x) - 2x\widetilde{K}_n(e_1,x) + x^2\widetilde{K}_n(e_0,x) = \frac{n(n-1)x^2+6nx+6}{(n+4)(n+5)} - 2x\frac{nx+2}{n+4} + x^2 = \frac{(2n-20)x(1-x)+6}{(n+4)(n+5)}$. □

Theorem 1. For any $n \in \mathbb{N}$ we have:

$$\widetilde{K}_n(f,x) = \frac{n+3}{n+1}(D \circ D_{n+1} \circ I)(f,x). \tag{8}$$

Proof. We have

$$(D_{n+1} \circ I)(f,x) = D_{n+1}(I(f,x)) = D_{n+1}\left(\int_0^x f(u)\,du\right) =$$

$$= (n+2)\sum_{k=0}^{n+1} p_{n+1,k}(x)\int_0^1 p_{n+1,k}(t)\int_0^t f(u)\,dudt =$$

$$= (n+2)\sum_{k=0}^{n+1} p_{n+1,k}(x)\int_0^1 f(u)\int_u^1 p_{n+1,k}(t)\,dtdu$$

In the last equality we have changed the order of integration, $\begin{cases} t \in [0,1] \\ u \in [0,t] \end{cases} \iff \begin{cases} u \in [0,1] \\ t \in [u,1] \end{cases}$ .Taking into account that

$$(p_{n+1,k}(x))' = (n+1)(p_{n,k-1}(x) - p_{n,k}(x))$$

and considering $p_{n,k}(x) = 0$, then $k < 0$ or $k > n$, we obtain

$$(D \circ D_{n+1} \circ I)(f,x) = D(D_{n+1} \circ I)(f,x) =$$

$$= (n+2)\sum_{k=0}^{n+1}(p_{n+1,k}(x))'\int_0^1 f(u)\int_u^1 p_{n+1,k}(t)\,dtdu =$$

$$= (n+2)\sum_{k=0}^{n+1}(n+1)(p_{n,k-1}(x) - p_{n,k}(x))\int_0^1 f(u)\int_u^1 p_{n+1,k}(t)\,dtdu =$$

$$= (n+1)(n+2)\sum_{k=0}^{n+1} p_{n,k}(x)\int_0^1 f(u)\int_u^1 (p_{n+1,k+1}(t) - p_{n+1,k}(t))\,dtdu =$$

$$= (n+1)(n+2)\sum_{k=0}^{n+1} p_{n,k}(x)\int_0^1 f(u)\int_u^1 -\frac{1}{n+2}(p_{n+2,k+1}(t))'\,dtdu =$$

$$= (n+1)\sum_{k=0}^{n+1} p_{n,k}(x)\int_0^1 f(u)\,p_{n+2,k+1}(u)\,du,$$

from where we obtain the relation (8).Thus, the proof is completed. □

3 Convergence properties

Remind now, a theorem obtained by Shisha and Mond [13].

Theorem A. Let L be a linear positive operator such that $L : C[a,b] \to C[a,b]$

i) If $f \in C[a,b]$ and $x \in [a,b]$, then we have

$$\begin{aligned} |L(f,x) - f(x)| &\le |f(x)| \cdot |L(e_0,x) - 1| + \\ &+ \left\{ L(e_0,x) + \frac{1}{\delta}\sqrt{L(e_0,x)\, L\left((t-x)^2, x\right)} \right\} \cdot \omega(f;\delta). \end{aligned}$$

ii) If $f' \in C[a,b]$ and $x \in [a,b]$, then we have

$$\begin{aligned} |L(f,x) - f(x)| &\le |f(x)| \cdot |L(e_0,x) - 1| + \\ &+ \left|f'(x)\right| \cdot |L((t-x),x)| + \sqrt{L\left((t-x)^2, x\right)} \times \\ &\times \left\{ \sqrt{L(e_0,x)} + \frac{1}{\delta}\sqrt{L\left((t-x)^2, x\right)} \right\} \cdot \omega\left(f';\delta\right). \end{aligned}$$

Further, we will give a theorem on the degree of approximation of a continuous function f by the sequence of $\widetilde{K}_n(f,x)$. To this end, we will use the modulus of continuity of function f given by

$$\omega(f;\delta) = \sup\{|f(x) - f(y)| : x, y \in [0,1], |x-y| \le \delta\} \tag{9}$$

for any positive number δ [2].

Theorem 2. For any $f \in C[0,1]$ and each $x \in [0,1]$ Durrmeyer-Kantorovich type operators (3) have the properties:

i)

$$\left|\widetilde{K}_n(f,x) - f(x)\right| \le 2\omega\left(f; \sqrt{\frac{(2n-20)\,x(1-x)+6}{(n+4)(n+5)}}\right). \tag{10}$$

ii)

$$\left\|\widetilde{K}_n(f) - f\right\| \le 2\omega\left(f; \frac{1}{2}\sqrt{\frac{4+2n}{(n+4)(n+5)}}\right). \tag{11}$$

Proof. i) Considering Theorem A, we can write

$$\begin{aligned} \left|\widetilde{K}_n(f,x) - f(x)\right| &\le |f(x)| \cdot \left|\widetilde{K}_n(e_0,x) - 1\right| + \\ &+ \left\{ \widetilde{K}_n(e_0,x) + \frac{1}{\delta}\sqrt{\widetilde{K}_n(e_0,x)\, \widetilde{K}_n\left((t-x)^2, x\right)} \right\} \cdot \omega(f;\delta) \end{aligned}$$

Using Corollary 1 and Lemma 4, it follows

$$\left|\widetilde{K}_n(f,x)-f(x)\right| \leq \left(1+\frac{1}{\delta}\sqrt{\frac{(2n-20)\,x(1-x)+6}{(n+4)(n+5)}}\right)\cdot\omega(f;\delta) \tag{12}$$

Choosing

$$\delta=\sqrt{\frac{(2n-20)\,x(1-x)+6}{(n+4)(n+5)}}$$

we have

$$\left|\widetilde{K}_n(f,x)-f(x)\right| \leq 2\omega\left(f;\sqrt{\frac{(2n-20)\,x(1-x)+6}{(n+4)(n+5)}}\right).$$

ii)Taking into account that for $x\in[0,1]$ we have $x(1-x)\leq\frac{1}{4}$, from (10) it results

$$\left\|\widetilde{K}_n(f)-f\right\| \leq \left|\widetilde{K}_n(f,x)-f(x)\right| \leq \left(1+\frac{1}{\delta}\sqrt{\frac{4+2n}{4(n+4)(n+5)}}\right)\cdot\omega(f;\delta).$$

Choosing

$$\delta=\frac{1}{2}\sqrt{\frac{2n+4}{(n+4)(n+5)}}$$

we obtain (11).

The theorem is proved. □

Theorem 3. For any $f\in C^1[0,1]$ and each $x\in[0,1]$ Durrmeyer-Kantorovich type operators (3) verify

i)

$$\begin{aligned}\left|\widetilde{K}_n(f,x)-f(x)\right| &\leq \left|f'(x)\right|\cdot\left|\frac{2-4x}{n+4}\right|+2\sqrt{\frac{(2n-20)\,x(1-x)+6}{(n+4)(n+5)}}\\ &\cdot\omega\left(f';\sqrt{\frac{(2n-20)\,x(1-x)+6}{(n+4)(n+5)}}\right).\end{aligned} \tag{13}$$

ii)

$$\begin{aligned}\left\|\widetilde{K}_n(f)-f\right\| &\leq \frac{2}{n+4}\left\|f'\right\|+\sqrt{\frac{2n+4}{(n+4)(n+5)}}\\ &\cdot\omega\left(f';\frac{1}{2}\sqrt{\frac{2n+4}{(n+4)(n+5)}}\right).\end{aligned} \tag{14}$$

Proof. i) Considering Theorem A and using Corollary 1 and Lemma 4 the proof is completed, with $\delta = \sqrt{\frac{(2n-20)x(1-x)+6}{(n+4)(n+5)}}$.

ii) Considering Theorem A, we can write

$$\begin{aligned}\left|\widetilde{K}_n(f,x) - f(x)\right| \leq & \ |f(x)| \cdot \left|\widetilde{K}_n(e_0,x) - 1\right| + \\ & + \left|f'(x)\right| \cdot \left|\widetilde{K}_n((t-x),x)\right| + \sqrt{\widetilde{K}_n\left((t-x)^2,x\right)} \times \\ & \times \left\{\sqrt{\widetilde{K}_n(e_0,x)} + \frac{1}{\delta}\sqrt{\widetilde{K}_n\left((t-x)^2,x\right)}\right\} \cdot \omega\left(f';\delta\right).\end{aligned}$$

Using Corollary 1 and Lemma 4, it follows

$$\begin{aligned}\left|\widetilde{K}_n(f,x) - f(x)\right| \leq & \ \left|f'(x)\right| \cdot \left|\frac{2-4x}{n+4}\right| + \sqrt{\frac{(2n-20)x(1-x)+6}{(n+4)(n+5)}} \times \\ & \times \left\{1 + \frac{1}{\delta}\sqrt{\frac{(2n-20)x(1-x)+6}{(n+4)(n+5)}}\right\} \cdot \omega\left(f';\delta\right).\end{aligned}$$

Taking into account that for $x \in [0,1]$ we have $x(1-x) \leq \frac{1}{4}$ and $\left|\frac{2-4x}{n+4}\right| \leq \frac{2}{n+4}$, we obtain

$$\begin{aligned}\left\|\widetilde{K}_n(f) - f(x)\right\| \leq & \ \left|\widetilde{K}_n(f,x) - f(x)\right| \leq \frac{2}{n+4}\left|f'(x)\right| + \frac{1}{2}\sqrt{\frac{2n+4}{(n+4)(n+5)}} \times \\ & \times \left\{1 + \frac{1}{\delta} \cdot \frac{1}{2}\sqrt{\frac{2n+4}{(n+4)(n+5)}}\right\} \cdot \omega\left(f';\delta\right).\end{aligned}$$

Choosing $\delta = \frac{1}{2}\sqrt{\frac{2n+4}{(n+4)(n+5)}}$, we have (14). □

Corollary 2. Let $f \in C[0,1]$. Then

$$\lim_{n\to\infty}\left\|\widetilde{K}_n(f) - f\right\|_{C[0,1]} = 0. \tag{15}$$

Theorem 4. Let $f \in L_p[0,1], for 1 \leq p < \infty$. Then

$$\lim_{n\to\infty}\left\|\widetilde{K}_n(f) - f\right\|_{L_p[0,1]} = 0. \tag{16}$$

Proof. From the Luzin theorem for a given $\epsilon > 0$, there is $g \in C[0,1]$ such that

$$\|f - g\|_{L_p[0,1]} < \epsilon$$

On the other hand, by using theorem 2 for a given $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$\left\|\widetilde{K}_n(g,x) - g(x)\right\|_{C[0,1]} < \epsilon, \text{ for } n \in \mathbb{N},\ n > n_0.$$

We have

$$\left\|\widetilde{K}_n(f,x)-f(x)\right\|_{L_p[0,1]} \leq \left\|\widetilde{K}_n(f,x)-\widetilde{K}_n(g,x)\right\|_{L_p[0,1]}+ \tag{17}$$
$$+\left\|\widetilde{K}_n(g,x)-g(x)\right\|_{C[0,1]}+\|f-g\|_{L_p[0,1]}.$$

Now, we show that there is a $C>0$ such that $\left\|\widetilde{K}_n\right\|_{L_p[0,1]}\leq C$, for any $n\in\mathbb{N}$. For this purpose, we have

$$\begin{aligned}\left|\widetilde{K}_n(f,x)\right|^p &= \left|(n+3)\sum_{k=0}^{n}p_{n,k}(x)\int_0^1 p_{n+2,k+1}(t)\,f(t)\,dt\right|^p\leq\\ &\leq \left\{(n+3)\sum_{k=0}^{n}p_{n,k}(x)\int_0^1 p_{n+2,k+1}(t)\,|f(t)|\,dt\right\}^p\leq\\ &\leq \sum_{k=0}^{n}p_{n,k}(x)\left\{(n+3)\int_0^1 p_{n+2,k+1}(t)\,|f(t)|\,dt\right\}^p\leq\\ &\leq \sum_{k=0}^{n}p_{n,k}(x)\left[(n+3)\int_0^1 p_{n+2,k+1}(t)\,|f(t)|^p\,dt\right].\end{aligned}$$

I used Jensen's inequality in these relations. Now we have

$$\begin{aligned}\int_0^1\left|\widetilde{K}_n(f,x)\right|^p dx &\leq \int_0^1\sum_{k=0}^{n}p_{n,k}(x)\left[(n+3)\int_0^1 p_{n+2,k+1}(t)\,|f(t)|^p\,dt\right]dx\leq\\ &\leq \sum_{k=0}^{n}(n+3)\int_0^1 p_{n,k}(x)\,dx\int_0^1 p_{n+2,k+1}(t)\,|f(t)|^p\,dt.\end{aligned}$$

Since $\int\limits_0^1 p_{n,k}(x)\,dx=\frac{1}{n+1}$, it follows

$$\int_0^1\left|\widetilde{K}_n(f,x)\right|^p dx\leq\frac{n+3}{n+1}\sum_{k=0}^{n}\int_0^1 p_{n+2,k+1}(t)\,|f(t)|^p\,dt\leq\frac{n+3}{n+1}\int_0^1|f(t)|^p\,dt.$$

Hence

$$\left\|\widetilde{K}_n(f,x)\right\|_{L_p[0,1]}\leq\sqrt[p]{\frac{n+3}{n+1}}\,\|f\|_{L_p[0,1]}\leq\sqrt{3}\,\|f\|_{L_p[0,1]}.$$

From this fact it results $\left\|\widetilde{K}_n\right\|_{L_p[0,1]}\leq\sqrt{3}$.

As

$$\left\|\widetilde{K}_n(f,x)-\widetilde{K}_n(g,x)\right\|_{L_p[0,1]}=\left\|\widetilde{K}_n(f-g,x)\right\|_{L_p[0,1]}\leq\left\|\widetilde{K}_n\right\|_{L_p[0,1]}\|f-g\|_{L_p[0,1]},$$

we obtain, taking into account (17),

$$\begin{aligned}\left\|\widetilde{K}_n\left(f,x\right)-f\left(x\right)\right\|_{L_p[0,1]} &\leq \sqrt{3}\left\|f-g\right\|_{L_p[0,1]}+\epsilon+\left\|f-g\right\|_{L_p[0,1]}\leq \\ &\leq \left(\sqrt{3}+2\right)\epsilon.\end{aligned}$$

With the help of this expression, we find (16). □

References

[1] Adell, J.A. and J. de la Cal, *Bernstein-Durrmeyer operators* , Comput. Math. Appl. **30** (1995), 1-14.

[2] Altomare, F. and Campiti, M., *Korovkin-Type Approximation Theory and Its Applications,* De Gruyter Studies in Mathematics, vol. **17**, Walter de Gruyter and Co., Berlin.

[3] Barbosu, D., *Kantorovich-Stancu type operators,* J. Inequal.Pure Appl. Math. **5** (2004) no. 3, article 53.

[4] Derriennic, M. M., *Sur l'approximation des fonctions integrables sur* $[0,1]$ *par des polynomes de Bernstein modifies,* J. Approx. Theory **31** (1981), 325-343.

[5] Ditzian, Z. and Zhou, X., *Kantorovich-Bernstein polynomials,* Constr. Approx. **6** (1990), 421-435.

[6] Durrmeyer, J. L., *Une Formule d'inversion de la transformee de Laplace: Application a la theorie des moments,* Faculte des Sciences de l'Universite de Paris (1967).

[7] Gonska H., Heilmannb M., Raşa,I., *Kantorovich Operators of Order k,* Numerical Functional Analysis and Optimization **32** (2011), 717-738.

[8] Heilmann, M, Rasa, I., *K-th order Kantorovich type modification of the operators* U_n^ρ, to appear in J. Applied Functional Analysis, **9** (2014), no. 3-4, 320-334.

[9] Kacso, D., *Quantitative statements for the Bernstein-Durrmeyer operators with Jacobi-weights*, Mathematical Analysis and Approximation Theory (2002), Burg Verlag, 135-144.

[10] Kantorovich, L. V., *Sur certains developpements suivant les polynomes de la forme de S. Bernstein*, I, II, C. R. Acad. URSS (1930), 563-568, 595-600.

[11] Knoop,H.B. and Pottinger P., *Ein Satz vom Korovkin-Typ fur* C^k*-Raume,* Math. Z. **148** (1976), 23-32 .

[12] Li,C., Shi,N. and Huo,X.,*Some approximate properties for a kind of generalized Bernstein-Kantorovich operators,* (Chinese), J. Fujian Norm. Univ., Nat. Sci. **24** (2008), no.4, 1-4

[13] Shisha, *O.* and Mond, B., *The degree of convergence of linear positive operators*, Proc. Nat. Acad. Sci. U.S.A., **60** (1968), 1196-1200.

[14] Sucu, S., Ibikli, E., *Approximation by means of Kantorovich-Stancu type operators,* Numerical Functional Analysis and Optimization, **34** (5), 2013, 557-575.

[15] Totik, V., *Approximation in L_1 by Kantorovich polynomials,* Acta Sci. Math. **46**, Szeged, (1983), 211-222.

[16] Totik, V., *Problems and solutions concerning Kantorovich operators*, J. Approx. Theory **37** (1983), 51-68.

[17] Zenke, W. and Junfang, *A generalization of the Bernstein operators,* (Chinese), J. Baoji Coll. Arts Sci., Nat. Sci. **20** (2000), no. 4, 248-250.