Bulletin of the *Transilvania* University of Braşov • Vol 6(55), No. 2 - 2013
Series III: Mathematics, Informatics, Physics, 27-32

A NOTE ON GENERALIZED BENSTEIN-KANTOROVICH OPERATORS

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Abstract

We study the convergence property of a sequence of the Bernstein-Kantorovich type operators attached to the differential operator $D_b(f) = f' + bf$, $b \in \mathbb{R}$. As consequence we obtain a shape preserving property for the Bernstein operators. In this way we generalize and correct some results given in [2].

2000 *Mathematics Subject Classification:* 41A36, 41A10
Key words: Bernstein-Kantorovich type operators, shape preserving property

1 Introduction

The Bernstein operators of order $n \in \mathbf{N}$ are given by

$$B_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right),\ x \in [0,1],\ f : [0,1] \to \mathbf{R}, \tag{1}$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ 0 \le k \le n. \tag{2}$$

The Kantorovich modification of the Bernstein operators is given by

$$K_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt,\ x \in [0,1],\ f \in C[0,1]. \tag{3}$$

Operators K_n can be given also by means of the following equality

$$K_n = D \circ B_{n+1} \circ I, \tag{4}$$

where D is the differentiation operator: $D(f) = f'$ and I is antiderivative operator: $I(f)(x) = \int_0^x f(t)dt$.

A generalization of Kantorovich operators, can be given in the following form

$$K_n^b = D_b \circ B_{n+1} \circ I_b,\ n \in \mathbf{N},\ b \in \mathbf{R}, \tag{5}$$

where the operators D_b, and I_b are given by:

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- $D_b(f) = f' + bf$, for $f \in C^1[0,1]$,
- $I_b(f)(x) = e^{-bx} \int_0^x e^{bt} f(t)dt$, for $f \in C[0,1]$ and $x \in [0,1]$.

Note that we have $(D_b \circ I_b)(f) = f$, for each $f \in C[0,1]$. Also, integrating by parts we obtain $(I_b \circ D_b)(f) = f$, for each $f \in C^1[0,1]$, such that $f(0) = 0$.

These operators were already considered, for the particular case $b > 0$ in [2], but formula (4) given there as definition of these operators has an error and does not represent the operators K_n^b given in (5). Because operators K_n^b are not positive, see Remark 2.1 in the next section, the study of the convergence property of the sequence of these operators cannot be obtained by a simple use of the Korovkin theorem.

2 Main results

Let $b \in \mathbf{R}$ be fixed. We start with a more explicit representation of operators K_n^b:

Lemma 2.1. *For $f \in C[0,1]$, $n \in \mathbf{N}$, $x \in [0,1]$, we have*

$$K_n^b(f)(x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\Big(I_b(f)\Big(\frac{k+1}{n+1}\Big) - I_b(f)\Big(\frac{k}{n+1}\Big)\Big) \tag{6}$$
$$+bB_{n+1}(I_b(f))(x). \tag{7}$$

Proof. From relation (5) we deduce successively:

$$\begin{aligned}
K_n^b(f)(x) &= D(B_{n+1}(I_b(f)))(x) + bB_{n+1}(I_b(f))(x) \\
&= \sum_{k=0}^{n+1} (p_{n+1,k}(x))' I_b(f)\Big(\frac{k}{n+1}\Big) + bB_{n+1}(I_b(f))(x) \\
&= \sum_{k=0}^{n+1} (n+1)(p_{n,k-1}(x) - p_{n,k}(x)) I_b\Big(\frac{k}{n+1}\Big) + bB_{n+1}(I_b(f))(x) \\
&= (n+1)\sum_{k=0}^{n} p_{n,k}(x)\Big(I_b(f)\Big(\frac{k+1}{n+1}\Big) - I_b(f)\Big(\frac{k}{n+1}\Big)\Big) \\
&\quad +bB_{n+1}(I_b(f))(x).
\end{aligned}$$

Here we considered that $p_{n,-1}(x) = 0$ and $p_{n,n+1}(x) = 0$. □

The main result is the following

Theorem 2.1. *For any $f \in C[0,1]$ we have*

$$\lim_{n\to\infty} K_n^b(f) = f, \text{ uniformly.} \tag{8}$$

Proof. Let $f \in C[0,1]$ be fixed. For an arbitrary number $\varepsilon > 0$ there is $n_\varepsilon \in \mathbf{N}$, such that $|f(u) - f(v)| < \varepsilon$, for all $u, v \in [0,1]$, such that $|v - u| < \frac{1}{n}$, if $n \in \mathbf{N}$, $n \geq n_\varepsilon$. Let fix a such number n. For simplicity, we denote: $F = I_b(f)$, i.e.

$$F(x) = e^{-bx} \int_0^x e^{bt} f(t)dt, \ x \in [0,1].$$

We can write

$$F\Big(\frac{k+1}{n+1}\Big) = e^{-\frac{k+1}{n+1}b} \int_{\frac{k}{n}}^{\frac{k+1}{n+1}} e^{bt} f(t)dt + \Big(e^{-\frac{k+1}{n+1}b} - e^{-\frac{k}{n}b}\Big) \int_0^{\frac{k}{n}} e^{bt} f(t)dt + F\Big(\frac{k}{n}\Big)$$

and

$$F\Big(\frac{k}{n+1}\Big) = -e^{-\frac{k}{n+1}b} \int_{\frac{k}{n+1}}^{\frac{k}{n}} e^{bt} f(t)dt + \Big(e^{-\frac{k}{n+1}b} - e^{-\frac{k}{n}b}\Big) \int_0^{\frac{k}{n}} e^{bt} f(t)dt + F\Big(\frac{k}{n}\Big).$$

From these we deduce

$$\begin{aligned} F\Big(\frac{k+1}{n+1}\Big) - F\Big(\frac{k}{n+1}\Big) &= \Big(e^{-\frac{k+1}{n+1}b} - e^{-\frac{k}{n+1}b}\Big) \int_0^{\frac{k}{n}} e^{bt} f(t)dt \\ &+ e^{-\frac{k+1}{n+1}b} \int_{\frac{k}{n}}^{\frac{k+1}{n+1}} e^{bt} f(t)dt + e^{-\frac{k}{n+1}b} \int_{\frac{k}{n+1}}^{\frac{k}{n}} e^{bt} f(t)dt. \quad (9) \end{aligned}$$

Several times we shall use the following inequality:

$$\Big|e^{-tb} - 1\Big| \leq e^{\frac{|b|}{n+1}} - 1, \ \text{if } |t| \leq \frac{1}{n+1}. \quad (10)$$

Indeed, if $tb \leq 0$, then $\big|e^{-tb} - 1\big| = e^{bt} - 1 \leq e^{\frac{|b|}{n+1}} - 1$ and if $tb \geq 0$, then $\big|e^{-tb} - 1\big| = 1 - e^{-tb} \leq 1 - e^{-\frac{|b|}{n+1}} = e^{-\frac{|b|}{n+1}}\Big(e^{\frac{|b|}{n+1}} - 1\Big) \leq e^{\frac{|b|}{n+1}} - 1$.

In what follows we estimate the terms which appear for an index $0 \leq k \leq n$ on the right side of (9). First, let us notice that we can write

$$e^{-\frac{k+1}{n+1}b} - e^{-\frac{k}{n+1}b} = e^{-\frac{k}{n}b}\beta_n + \gamma_{n,k},$$

where

$$\beta_n = e^{-\frac{b}{n+1}} - 1 \quad (11)$$

$$\gamma_{n,k} = \Big(e^{-\frac{k}{n+1}b} - e^{-\frac{k}{n}b}\Big)\Big(e^{-\frac{b}{n+1}} - 1\Big). \quad (12)$$

Consequently,

$$\Big(e^{-\frac{k+1}{n+1}b} - e^{-\frac{k}{n+1}b}\Big) \int_0^{\frac{k}{n}} e^{bt} f(t)dt = \beta_n F\Big(\frac{k}{n}\Big) + \gamma_{n,k} \int_0^{\frac{k}{n}} e^{bt} f(t)dt. \quad (13)$$

Note that

$$\lim_{n\to\infty} (n+1)\beta_n = -b. \quad (14)$$

Using (10 we have

$$|\gamma_{n,k}| = e^{-\frac{k}{n}b}\left|e^{\frac{k}{n(n+1)}b} - 1\right| \cdot \left|e^{-\frac{b}{n+1}} - 1\right| \le e^{|b|}\left(e^{\frac{|b|}{n+1}} - 1\right)^2.$$

Also,

$$\left|\int_0^{\frac{k}{n}} e^{bt} f(t)dt\right| \le \|f\| \int_0^{\frac{k}{n}} e^{|b|}dt \le e^{|b|}\|f\|.$$

It follows that

$$\left|\gamma_{n,k} \int_0^{\frac{k}{n}} e^{bt} f(t)dt\right| \le e^{2|b|}\left(e^{\frac{|b|}{n+1}} - 1\right)^2 \|f\|. \tag{15}$$

Next, the second term on the right side of (9) can be rewritten in the form:

$$\begin{aligned} & e^{-\frac{k+1}{n+1}b} \int_{\frac{k}{n}}^{\frac{k+1}{n+1}} e^{bt} f(t)dt = \left(\frac{k+1}{n+1} - \frac{k}{n}\right) f\left(\frac{k}{n}\right) \\ & + \int_{\frac{k}{n}}^{\frac{k+1}{n+1}} \left(f(t) - f\left(\frac{k}{n}\right)\right) e^{\left(t - \frac{k+1}{n+1}\right)b} dt + \int_{\frac{k}{n}}^{\frac{k+1}{n+1}} f\left(\frac{k}{n}\right)\left(e^{\left(t-\frac{k+1}{n+1}\right)b} - 1\right)dt \\ & = \left(\frac{k+1}{n+1} - \frac{k}{n}\right) f\left(\frac{k}{n}\right) + J_{n,k}^1 + J_{n,k}^2 \text{ say.} \end{aligned} \tag{16}$$

We obtain

$$|J_{n,k}^1| \le \varepsilon \int_{\frac{k}{n}}^{\frac{k+1}{n+1}} e^{|b|}dt \le \frac{\varepsilon}{n+1} e^{|b|}. \tag{17}$$

Also, using (10) we obtain

$$|J_{n,k}^2| \le \|f\| \int_{\frac{k}{n}}^{\frac{k+1}{n+1}} \left|e^{\left(t-\frac{k+1}{n+1}\right)b} - 1\right| dt \le \frac{\|f\|}{n+1}\left(e^{\frac{|b|}{n+1}} - 1\right). \tag{18}$$

The third term on the right side of (9) can be rewritten in the form:

$$\begin{aligned} & e^{-\frac{k}{n+1}b} \int_{\frac{k}{n+1}}^{\frac{k}{n}} e^{bt} f(t)dt = \left(\frac{k}{n} - \frac{k}{n+1}\right) f\left(\frac{k}{n}\right) \\ & + \int_{\frac{k}{n+1}}^{\frac{k}{n}} \left(f(t) - f\left(\frac{k}{n}\right)\right) e^{\left(t - \frac{k}{n+1}\right)b} dt + \int_{\frac{k}{n+1}}^{\frac{k}{n}} f\left(\frac{k}{n}\right)\left(e^{\left(t-\frac{k}{n+1}\right)b} - 1\right)dt \\ & = \left(\frac{k}{n} - \frac{k}{n+1}\right) f\left(\frac{k}{n}\right) + J_{n,k}^3 + J_{n,k}^4 \text{ say.} \end{aligned} \tag{19}$$

We obtain

$$|J_{n,k}^3| \le \varepsilon \int_{\frac{k}{n+1}}^{\frac{k}{n}} e^{|b|}dt \le \frac{\varepsilon}{n+1} e^{|b|}. \tag{20}$$

Also, using (10) we obtain

$$|J_{n,k}^4| \le \|f\| \int_{\frac{k}{n+1}}^{\frac{k}{n}} \left|e^{\left(t-\frac{k}{n+1}\right)b} - 1\right| dt \le \frac{\|f\|}{n+1}\left(e^{\frac{|b|}{n+1}} - 1\right). \tag{21}$$

Let us denote

$$R_n(f)(x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\Big(\gamma_{n,k}\int_0^{\frac{k}{n}} e^{bt}f(t)dt + J^1_{n,k} + J^2_{n,k} + J^3_{n,k} + J^4_{n,k}\Big). \quad (22)$$

Starting from relation (6) and taking into account relations (9), (13), (16), (19) and (22) we arrive to

$$K_n^b(f)(x) \quad = \quad B_n(f)(x) + (n+1)\beta_n B_n(F)(x) + bB_{n+1}(F)(x) + R_n(f)(x). \quad (23)$$

Using the convergence property of the Bernstein operators, and taking into account relation (14) we obtain the following uniform limits:

$$\lim_{n\to\infty} B_n(f)(x) = f(x),$$
$$\lim_{n\to\infty} ((n+1)\beta_n B_n(F)(x) + bB_{n+1}(F)(x)) = 0.$$

Then there is $n_\varepsilon^1 \in \mathbf{N}$, $n_\varepsilon^1 \geq n_\varepsilon$, such that:

$$\|B_n(f) + (n+1)\beta_n B_n(F) + bB_{n+1}(F) - f\| < \varepsilon, \text{ for } n \geq n_\varepsilon^1. \quad (24)$$

From relations (22), (15), (17), (18), (20) and (21) we get

$$\|R_n\| \leq (n+1)e^{2|b|}\Big(e^{\frac{|b|}{n+1}} - 1\Big)^2\|f\| + 2e^{|b|}\varepsilon + 2\|f\|\Big(e^{\frac{|b|}{n+1}} - 1\Big) \quad (25)$$

Since

$$\lim_{n\to\infty}\Big[(n+1)e^{2|b|}\Big(e^{\frac{|b|}{n+1}} - 1\Big)^2\|f\| + 2\|f\|\Big(e^{\frac{|b|}{n+1}} - 1\Big)\Big] = 0$$

there is an integer $n_\varepsilon^2 \geq n_\varepsilon^1$ such that

$$(n+1)e^{2|b|}\Big(e^{\frac{|b|}{n+1}} - 1\Big)^2\|f\| + 2\|f\|\Big(e^{\frac{|b|}{n+1}} - 1\Big) < \varepsilon, \text{ for } n \geq n_\varepsilon^2. \quad (26)$$

Finally, from relations (23), (24), (25) and (26) we obtain

$$\|K_n^b(f) - f\| \leq \varepsilon\Big(2 + 2e^{|b|}\Big), \text{ for } n \geq n_\varepsilon^2. \quad (27)$$

Since $\varepsilon > 0$ was chosen arbitrarily we obtain (8). □

We have the following immediate

Corollary 2.1. *For any $f \in C[0,1]$, $f > 0$, there is $n_0 \in \mathbf{N}$ such that $K_n^b(f) > 0$, for $n \geq n_0$.*

Define the class of functions

$$\mathcal{D}_b = \{f \in C^1[0,1] \mid D_b(f) > 0,\ f(0) = 0\}. \quad (28)$$

Theorem 2.2. *For any $f \in \mathcal{D}_b$ there is $n_1 \in \mathbf{N}$ for which we have $B_n(f) \in \mathcal{D}_b$, for $n \geq n_1$.*

Proof. Let $f \in \mathcal{D}_b$. Since $f(0) = 0$ it follows that $f = I_b(D_b(f))$. We can write $D_b \circ B_n(f) = (D_b \circ B_n \circ I_b \circ D_b)(f) = K^b_{n-1}(D_b(f))$. Since $D_b(f) > 0$, from Corollary 2.1 there is $n_1 \in \mathbf{N}$, such that $K^b_{n-1}(D_b(f)) > 0$, for $n \geq n_1$. That means that $D_b(B_n(f)) > 0$. Finally, we have $B_n(f)(0) = f(0) = 0$. Consequently, $B_n(f) \in \mathcal{D}_b$, for $n \geq n_1$. □

Remark 2.1. Operators K^b_n are not positive as the following example shows. Let $f : [0, 1] \to \mathbf{R}$, given by $f(t) = 1$, for $t \in [1/3, 2/3]$ and $f(t) = 0$, for $t \in [0, 1/3) \cup (2/3, 1]$. We choose $b = 1$ and $n = 2$. It follows that

$$\begin{aligned} K^1_2(f)(1) &= 3p_{22}(1)(I_1(f)(1/3) - I_1(f)(2/3)) + p_{33}(1)f(1) \\ &= 3\Big[e^{-1}\int_0^1 e^t f(t)dt - e^{-2/3}\int_0^{2/3} e^t f(t)dt\Big] \\ &= 3\Big[e^{-1}\int_{1/3}^{2/3} e^t dt - e^{-2/3}\int_{1/3}^{2/3} e^t dt\Big] \\ &= 3(e^{2/3} - e^{1/3})(e^{-1} - e^{-2/3}) \\ &< 0. \end{aligned}$$

Finally we chose a function $g_\varepsilon \in C[0, 1]$, $g_\varepsilon \geq 0$, of the form $g_\varepsilon(t) = 1$, for $t \in [1/3, 2/3]$, $g_\varepsilon(t) = 0$, $t \in [0, 1/3 - \varepsilon] \cup [2/3 + \varepsilon, 1]$, and linear on each of the intervals $[1/3 - \varepsilon, 1/3]$ and $[2/3, 2/3 + \varepsilon]$, with $0 < \varepsilon < 1/3$. Then $\lim_{\varepsilon \to +0} K^1_2(g_\varepsilon)(1) = K^1_2(f)(1)$. Consequently, for sufficiently small $\varepsilon > 0$ we have $K^1_2(g_\varepsilon)(1) < 1$.

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