Bulletin of the *Transilvania* University of Braşov • Vol 6(55), No. 2 - 2013
Series III: Mathematics, Informatics, Physics, 17-22

A SIMPLE PROOF OF THE GAUSSIAN LOWER BOUND FOR THE NEUMANN HEAT KERNEL OF CONVEX DOMAINS

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Abstract

We present a simple proof of the gaussian lower bound for the Neumann heat kernel of convex domains. The proof is probabilistic in spirit and relies on a geometric property of the extended mirror coupling of reflecting Brownian motions introduced in [7].

2000 *Mathematics Subject Classification:* 35K08, 60J65, 60J35.
Key words: Neumann heat kernel, Gaussian lower bound, reflecting Brownian motion, mirror coupling of reflecting Brownian motions.

1 Introduction

The Neumann heat kernel (the fundamental solution of the heat equation with Neumann boundary conditions) is an important object of study in Physics, and it has many connections in other areas of mathematics, such as Analysis, Probability, and Geometry (for a survey on heat kernels see for example [8]). There are many questions related to the Neumann heat kernels in the literature, among which are the long time behavior, the evolution of the hot or cold spots, the upper and lower bounds, etc.

It is known that the Neumann heat kernel $p_D(t,x,y)$ of a smooth convex domain $D\subset\mathbb{R}^n$ satisfies the two-sided estimate

$$\frac{c}{V\left(x,\sqrt{t}\right)}e^{-C\|x-y\|^2/t}\leq p\left(t,x,y\right)\leq\frac{C}{V\left(x,\sqrt{t}\right)}e^{-c\|x-y\|^2/t},\tag{1}$$

where $c,C>0$ are positive constants, and $V(x,r)$ is the volume of the trace of the Euclidean ball centered at x and of radius $r>0$ in D. This can be proved using the

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volume doubling property and the Poincaré inequality (see [5], p. 10, or pp. 49 – 72 for a proof in the case of inner uniform domains).

The purpose of the present article is to provide a simple proof of the gaussian lower bound for the Neumann heat kernel in the case of convex domains in $\mathbb{R}^n$ (Theorem 1). The proof is probabilistic in spirit, and it is based on the interpretation of the Neumann heat kernel as the density of the reflecting Brownian motion, and on a certain geometric property of the extended mirror coupling of reflecting Brownian motions introduced by the first author in [7]. Aside from its simplicity, the proof also gives explicit values of the constants c and C appearing in (1), that is, it identifies the lower bound in (1) with the heat kernel $p(t,x,y)$ in $\mathbb{R}^n$.

The paper is structured as follows: in Section 2 we introduce the notation and we briefly review the results about the extended mirror coupling of reflecting Brownian motions introduced in [7], needed in the sequel.

In Section 3, we first prove the lower bound for the Neumann heat kernel in the case of polygonal domains in $\mathbb{R}^n$ $(n \geq 1)$, and then we extend the proof to the case of arbitrary smooth convex domains. Next, by means of an example we show that the hypothesis on the convexity of the domain in Theorem 1 plays an important role, and it cannot be dispensed off (the inequality (8) in Theorem 1 does not hold for the non-convex domain in Example 2). We conclude with a remark showing that the gaussian lower bound for the Neumann heat kernel can be viewed as a limiting case of Chavel's conjecture on the domain monotonicity of the Neumann heat kernel.

2 Preliminaries

Let $D \subset \mathbb{R}^n$ be a domain with smooth boundary ∂D, and let $\nu : \partial D \to \mathbb{R}^n$ denote the inward unit normal vector field on the boundary ∂D. A classical solution of the heat equation in D with Neumann boundary conditions on ∂D (thermal insulated boundary) is a function $u \in C^\infty\left((0,\infty) \times \overline{D}\right)$ which satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\Delta u(t,x), \qquad (t,x) \in (0,\infty) \times D \\ \dfrac{\partial u}{\partial \nu}(t,x) = 0, \qquad (t,x) \in (0,\infty) \times \partial D \end{cases} \tag{2}$$

In the above we have used $\frac{1}{2}\Delta$ instead of the usual Laplace operator $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ for the convenience of the probabilistic arguments involved. The same results hold for the unscaled Laplace operator (Δ instead of $\frac{1}{2}\Delta$) by scaling the time variable by a factor of 2.

The Neumann heat kernel $p_D(t,x,y)$ of D is defined as the fundamental solution of (2), that is, for $x \in \overline{D}$ arbitrarily fixed, $p_D(t,x,y) : (0,\infty) \times \overline{D} \to \mathbb{R}$ satisfies (2) and $\lim_{t \searrow 0} p_D(t,x,y) = \delta_x(y)$, where δ_x is the Dirac point-mass at $x \in \overline{D}$. Moreover, for any $u_0 \in L^2(D)$,

$$u(t,x) = \int_D p_D(t,x,y)\, u_0(y)\, dy, \qquad x \in \overline{D}, \tag{3}$$

is the unique solution of (2) with initial temperature distribution $u(0,x) = u_0(x)$, $x \in D$.

The probabilistic interpretation of the Neumann heat kernel for D is as the transition density of the reflecting Brownian motion in D. This is a stochastic process $(X_t)_{t\geq 0}$ defined on $\overline{D}$, which behaves like ordinary Brownian motion when inside D, and it has instantaneous reflection in the direction of inner normal ν when on the boundary ∂D of D. Explicitly, $(X_t)_{t\geq 0}$ can be defined as the strong solution of the following stochastic differential equation

$$X_t = X_0 + B_t + \frac{1}{2}\int_0^t \nu(X_s)\,dL_s^X, \qquad t \geq 0,$$

where $(B_t)_{t\geq 0}$ is a n-dimensional (free) Brownian motion and $\left(L_t^X\right)_{t\geq 0}$ is the local time of X on the boundary (the continuous nondecreasing process which increases only when $X \in D$, i.e. $\int_0^\infty 1_D(X_s)\,dL_s = 0$).

In [7], the authors extended the mirror coupling of reflecting Brownian motions (introduced in [1]) to the case when the two reflecting Brownian motions live in different domains. Explicitly, they showed that given a n-dimensional Brownian motion $(W_t)_{t\geq 0}$ starting at 0, and smooth domains $D_{1,2} \subset \mathbb{R}^n$ (piecewise C^2 domains in $\mathbb{R}^2$ with a finite number of convex corners, or C^2 domains in $\mathbb{R}^n$, $n \geq 3$, will suffice) with $\overline{D_2} \subset D_1$ and D_2 convex domain (or more generally $D_1 \cap D_2$ convex domain), there exists a strong solution of the following system of stochastic differential equations

$$X_t = x + W_t + \int_0^t \nu_{D_1}(X_s)\,dL_s^X \tag{4}$$

$$Y_t = y + Z_t + \int_0^t \nu_{D_2}(Y_s)\,dL_s^Y \tag{5}$$

$$Z_t = \int_0^t G(Y_s - X_s)\,dW_s \tag{6}$$

where $G : \mathbb{R}^n \to \mathcal{M}_{n\times n}$ is defined by:

$$G(z) = \begin{cases} H\left(\frac{z}{\|z\|}\right), & \text{if } z \neq 0 \\ I, & \text{if } z = 0 \end{cases}, \tag{7}$$

and for a unitary vector $m \in \mathbb{R}^n$, $H(m)v = v - 2(mv)m$ is the mirror image of $v \in \mathbb{R}^n$ with respect to the hyperplane through the origin perpendicular to m (m' denotes the transpose of the vector m, vectors being considered as column vectors).

Geometrically, the equations (4) – (7) above show that X and Y are reflecting Brownian motions in D_1, respectively D_2, and that the increments of the corresponding driving Brownian motions W and Z are mirror images with respect to the hyperplane of symmetry between X and Y.

The pair $(X_t, Y_t)_{t\geq 0}$ constructed above is called a *mirror coupling* of reflecting Brownian motions in $D_1 \times D_2$ starting at $(x, y) \in \overline{D_1} \times \overline{D_2}$.

In particular, the above construction can be carried out in the case when D_1 is a convex polygonal domain (a domain bounded by hyperplanes in $\mathbb{R}^n$) and $D_2 = \mathbb{R}^n$

(see Section 3.2 in [7]). In this case, it can be shown that the extended mirror coupling has the following property: when the processes X_t and Y_t are decoupled (i.e. $X_t \neq Y_t$), the hyperplane of symmetry between X_t and Y_t, called the *mirror of the coupling*, lies outside of the domain D_1 (see Proposition 3.10 in [7]). This property of the extended mirror coupling is the key ingredient in the proof of lower bound for the Neumann heat kernel, presented in the next section.

3 Main results

Theorem 1. *Let $D \subset \mathbb{R}^n$ $(n \geq 1)$ be a smooth convex domain and let $p_D(t,x,y)$ denote the Neumann heat for D. Then for all $t > 0$ and $x, y \in D$ we have*

$$p_D(t,x,y) \geq p(t,x,y) = \frac{1}{(2\pi t)^{n/2}} e^{-\frac{\|x-y\|^2}{2t}}. \tag{8}$$

Proof. Consider first the case when $D \subset \mathbb{R}^n$ is a convex polygonal domain (a convex domain bounded by hyperplanes in $\mathbb{R}^n$).

Let $(X_t, Y_t)_{t\geq 0}$ be a mirror coupling of (reflecting) Brownian motions in $D \times \mathbb{R}^n$ starting at $(x,x) \in D \times \mathbb{R}^n$.

By the results in [7] (Proposition 3.10), when the processes X_t and Y_t are not coupled, the mirror of the coupling (the hyperplane of symmetry between X_t and Y_t) lies outside of D. It follows that for all times $t > 0$, either the processes X_t and Y_t are coupled, or $X_t \in D$ and $Y_t \in \mathbb{R}^n - D$. In turn, this shows that for any $y \in D$ an $r > 0$ sufficiently small we have

$$P^x(X_t \in B(y,r)) \geq P^x(Y_t \in B(y,r)),$$

or equivalent

$$\int_{B(y,r)} p_D(t,x,z)\,dz \geq \int_{B(y,r)} p(t,x,z)\,dz.$$

Dividing by the volume $|B(y,r)|$ of the ball $B(y,r) \subset \mathbb{R}^n$ and passing to the limit with $r \searrow 0$ we obtain

$$\begin{aligned} p_D(t,x,y) &= \lim_{r\searrow 0} \frac{1}{|B(y,r)|} \int_{B(y,r)} p_D(t,x,z)\,dz \\ &\geq \lim_{r\searrow 0} \frac{1}{|B(y,r)|} \int_{B(y,r)} p_D(t,x,z)\,dz \\ &= p(t,x,y), \end{aligned}$$

by the continuity of the heat kernel in the space variable, thus concluding the proof of the theorem in the case of polygonal domains.

In order to prove the claim in the general case, consider an increasing sequence of convex polygonal domains $(D_n)_{n\geq 1}$ with $\cup_{n\geq 1} D_n = D$. By the results in [3], the reflecting Brownian motion in D_n converges weakly as $n \to \infty$ to the reflecting Brownian motion in D. This, together with the previous part of the proof shows that we have

$$p_D(t,x,y) = \lim_{n\to\infty} p_{D_n}(t,x,y) \geq p(t,x,y),$$

for all $t > 0$ and $x, y \in D$, concluding the proof. $\square$

The hypothesis on the convexity of the domain plays an important role for the validity of the inequality (8) in Theorem 1, as shown by the following example.

Example 2. *Let $D = \mathbb{R}^2 - \left\{\left(x^1, 0\right) : x^1 \leq 0\right\}$ denote the half-plane with the negative horizontal axis removed, and let $p_D\left(t, x, y\right)$ denote the corresponding Neumann heat kernel for D.*

Let $X_t = \left(X_t^1, X_t^2\right)$, $t \geq 0$, be a reflecting Brownian motion in D starting at $x = \left(x^1, x^2\right)$ with $x^2 > 0$. Since the normal to the boundary of D points in the vertical direction, it follows that the horizontal component $\left(X_t^1\right)_{t\geq 0}$ of $\left(X_t\right)_{t\geq 0}$ is a 1-dimensional (free) Brownian motion on $\mathbb{R}$, and it is easy to see that $\widetilde{X}_t = \left(X_t^1, \left|X_t^2\right|\right)$, $t \geq 0$, defines a reflecting Brownian motion in the upper half-plane $\mathbb{R}_+^2$ starting at $x \in \mathbb{R}_+^2$.

Let $y = \left(y^1, y^2\right) \in D$ with $y^2 > 0$, and let $\widetilde{y} = \left(y^1, -y^2\right)$ denote the symmetric of y with respect to the horizontal axis. For $r > 0$ sufficiently small so that the ball $B\left(y, r\right)$ does not intersect the horizontal axis, we have

$$P\left(X_t \in B\left(y, r\right)\right) + P\left(X_t \in B\left(\widetilde{y}, r\right)\right) = P\left(\widetilde{X}_t \in B\left(y, r\right)\right).$$

Dividing the previous equality by the volume $\left|B\left(y, \varepsilon\right)\right|$ of the ball $B\left(y, \varepsilon\right)$, and passing to the limit with $\varepsilon \searrow 0$, we obtain

$$p_D\left(t, x, y\right) + p_D\left(t, x, \widetilde{y}\right) = p_{\mathbb{R}_+^2}\left(t, x, y\right),$$

where $p_{\mathbb{R}_+^2}\left(t, x, y\right)$ denotes the Neumann heat kernel for the half-plane $\mathbb{R}_+^2$. It is known that the Neumann heat kernel for the half-space has the representation $p_{\mathbb{R}_+^2}\left(t, x, y\right) = p\left(t, x, y\right) + p\left(t, x, \widetilde{y}\right)$, so the previous equality becomes

$$p_D\left(t, x, y\right) + p_D\left(t, x, \widetilde{y}\right) = p\left(t, x, y\right) + p\left(t, x, \widetilde{y}\right). \tag{9}$$

The symmetry in y and $\widetilde{y}$ of the previous equality, together with the symmetry in the space variables of the heat kernels $p_D\left(t, x, y\right)$ and $p\left(t, x, y\right)$, show that in fact the above equality is true for all $t > 0$ and $x, y \in D$ ($\widetilde{y} = \left(y^1, -y^2\right)$ denotes the symmetric of $y = \left(y^1, y^2\right)$ with respect to the horizontal axis).

The equality (9) shows that we cannot have $p_D\left(t, x, y\right) \geq p\left(t, x, y\right)$ for all $x, y \in D$ and $t > 0$. This is so for otherwise from (9) we would obtain $p_D\left(t, x, y\right) \equiv p\left(t, x, y\right)$, which is impossible due to the Neumann boundary condition $\frac{\partial p_D}{\partial \nu} = 0$ on ∂D. Therefore, there exist $t > 0$ and $x, y \in D$ such that $p_D\left(t, x, y\right) < p\left(t, x, y\right)$, which shows that the inequality (8) in Theorem 1 may not hold without the assumption on the convexity of the domain.

We conclude with the remark that the inequality (8) in Theorem 1 may be viewed as the limiting case of Chavel's conjecture on the domain monotonicity of the Neumann heat kernel. Chavel's conjecture asserts that for smooth bounded convex domains $D \subset \Omega \subset \mathbb{R}^n$ and $x, y \in D$, the corresponding Neumann heat kernels satisfy the inequality

$$p_D\left(t, x, y\right) \geq p_\Omega\left(t, x, y\right), \qquad t > 0. \tag{10}$$

It is known that the conjecture is false as stated ([2]), but if in addition there exists a ball B centered at either x or y such that $D \subset B \subset \Omega$, then the inequality (10) is true (see [4] and [6], or [7] for a simple unifying proof).

The inequality (8) may be viewed as the limiting case of Chavel's conjecture: for $\Omega \nearrow \mathbb{R}^n$ we have $p_\Omega(t,x,y) \to p(t,x,y)$, and the inequality (10) becomes (8). However, the inequality (8) does not follow in general from Chavel's conjecture by using this argument. Chavel's conjecture is known to be true for smooth bounded convex domains for which the intermediate ball condition holds, so the argument above cannot be used for example in the case of unbounded convex domains.

Acknowledgements

The first and the last authors kindly acknowledge the support by a grant of the Romanian National Authority for Scientific Research, CNCS - UEFISCDI, project number PNII-ID-PCCE-2011-2-0015.

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