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ON A CLASS OF INTEGRAL OPERATORS

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Abstract

We study the property of uniform approximations for a class of integral operators and we illustrate considerations by an example.

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1 Introduction

In this paper, we introduce a class of integral linear positive operators. We give sufficient conditions for the property of uniform approximation of this type of operators and we also give an example of concrete operators.

Consider the sequence of operators $(L_n)_n$, $L_n : C[a,b] \to C[a,b]$, $n \in \mathbb{N}$ defined by:

$$L_n(f,x) = \int_a^b K_n(x,t) f(t)\, dt,\ f \in C[a,b],\ x \in [a,b],$$

where the *integral kernel* $K_n : [a,b] \times [a,b] \setminus \Delta \to [0,\infty)$ is given by

$$K_n(x,t) = \frac{e^{nR(x,t)}}{\int_a^b e^{nR(x,t)}\, dt},\ x,t \in [a,b],\ n \in \mathbb{N},$$

and

$$\Delta := \{(x,x) | x \in [a,b]\}.$$

We assume that $R : [a,b] \times [a,b] \setminus \Delta \to (-\infty, 0]$ satisfies the following conditions:

(A) *Let $x \in [a,b]$ be fixed. The function $t \mapsto R(x,t)$ is of class C^2 on $[a,b] \setminus \{x\}$, with*

$$\max_{t \in [a,b]} R(x,t) = 0,$$

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and the set $T(x)$ of its absolute maximum points is countably infinite, such that

$$T(x) = \{t_k(x) | k \geq 1\} \subset [a,b] \setminus \{x\}$$

and

$$\frac{\partial^2 R}{\partial t^2}(x, t_k(x)) < 0, \ \forall k \geq 1.$$

Consider, also, the open intervals $I_k(x) \subset \mathbb{R}$, $k \geq 1$ with the properties:
(a) $t_k(x) \in I_k(x)$, $\forall k \geq 1$;
(b) $I_k(x) \cap I_l(x) = \phi$, $\forall k \neq l$;
(c) $(a,b) \setminus \bigcup_{k\geq 1} I_k(x)$ *is a countably infinite set.*

(B) $\lim_{k\to\infty} t_k(x) = x$, *uniformly with respect to $x \in [a,b]$.*

(C) *For each $k \geq 1$,*

$$\lim_{n\to\infty} \int_{I_k(x)} e^{nR(x,t)}\, dt = A_k(x),$$

uniformly with respect to $x \in [a,b]$, and

$$A_k(x) \leq A < \infty, \ \forall x \in [a,b], \ \forall k \geq 1.$$

Remark 1. *According to Laplace's method ([2, §3.4]), for each $x \in [a,b]$ and $k \geq 1$,*

$$A_k(x) = \sqrt{\frac{-2\pi}{\frac{\partial^2 R}{\partial t^2}(x, t_k(x))}}.$$

(D) $\sum_{k\geq 1} A_k(x) = \infty$, *uniformly with respect to $x \in [a,b]$.*

2 Main results

The operators $(L_n)_n$ are linear and positive. We prove that these operators have the property of uniform approximation on $C[a,b]$.

Theorem 1. *In the conditions above, for each $f \in C[a,b]$ we have*

$$\lim_{n\to\infty} L_n(f,x) = f(x),$$

uniformly with respect to $x \in [a,b]$.

Proof. We use a known result from the theory of integral operators:

Theorem 2 ([1], Theorem 2.1)**.** *Assume conditions:*
(i) $\lim_{n\to\infty} \int_a^b K_n(x,t)\, dt = 1$, *uniformly with respect to $x \in [a,b]$;*
(ii) $\int_a^b |K_n(x,t)|\, dt \leq A < \infty$, $\forall x \in [a,b]$, $\forall n \in \mathbb{N}$;

(iii) for each $\delta > 0$, $\lim_{n\to\infty} \int_{|x-t|\geq\delta} |K_n(x,t)|\,dt = 0$, *uniformly with respect to* $x \in [a,b]$.

Then:

$$\lim_{n\to\infty} L_n(f,x) = f(x),\ \forall f \in C[a,b],$$

uniformly with respect to $x \in [a,b]$.

Thus, we shall prove that hypotheses (A), (B), (C) and (D) ensure conditions (i), (ii) and (iii). Conditions (i) and (ii) are obvious, because

$$\int_a^b K_n(x,t)\,dt = 1,\ \forall x \in [a,b],\ \forall n \in \mathbb{N}.$$

We prove (iii). Let $\delta \in (0, b-a)$. From (B), there is $k_0 = k_0(\delta)$ such that

$$|t_k(x) - x| < \delta,\ \forall k > k_0,\ \forall x \in [a,b].$$

Taking into account (A), there exist at most two indices $k > k_0$ for which $I_k(x)$ contains points outside the interval $(x-\delta, x+\delta)$. In this case, we denote k_0 an index k_0' greater than these. Thus, we can write

$$|u - x| < \delta,\ \forall u \in I_k(x),\ \forall k > k_0,\ \forall x \in [a,b].$$

Further, we can write:

$$\int_{|x-t|\geq\delta} |K_n(x,t)|\,dt \leq \frac{\sum_{k=1}^{k_0} \int_{I_k(x)} e^{nR(x,t)}\,dt}{\sum_{k\geq 1} \int_{I_k(x)} e^{nR(x,t)}\,dt},\ x \in [a,b]. \tag{1}$$

From (C), we have that

$$\lim_{n\to\infty} \sqrt{n} \sum_{k=1}^{k_0} \int_{I_k(x)} e^{nR(x,t)}\,dt = \sum_{k=1}^{k_0} A_k(x) \leq k_0 A, \tag{2}$$

uniformly with respect to $x \in [a,b]$.

It is important to notice that (C) and (D) ensure

$$\lim_{n\to\infty} \sqrt{n} \sum_{k\geq 1} \int_{I_k(x)} e^{nR(x,t)}\,dt = \infty, \tag{3}$$

uniformly with respect to $x \in [a,b]$. Indeed, let $M > 0$ be arbitrary. From (D), there exists $N_1 = N_1(M) \in \mathbb{N}$ such that for any $k_1 \geq N_1$ we have

$$\sum_{k=1}^{k_1} A_k(x) > M + 1,\ \forall x \in [a,b].$$

For a particular k_1 as above, from (C), there is $N_2 = N_2(k_1) \in \mathbb{N}$ such that for any $n \geq N_2$ we have

$$\left| \sqrt{n} \sum_{k=1}^{k_1} \int_{I_k(x)} e^{nR(x,t)}\,dt - \sum_{k=1}^{k_1} A_k(x) \right| < 1,\ \forall x \in [a,b].$$

Therefore, for $k_1 \geq N_1$ and $n \geq N_2$, we have

$$\sqrt{n}\sum_{k=1}^{k_1}\int_{I_k(x)} e^{nR(x,t)}\,dt > M,\ \forall x \in [a,b],$$

hence

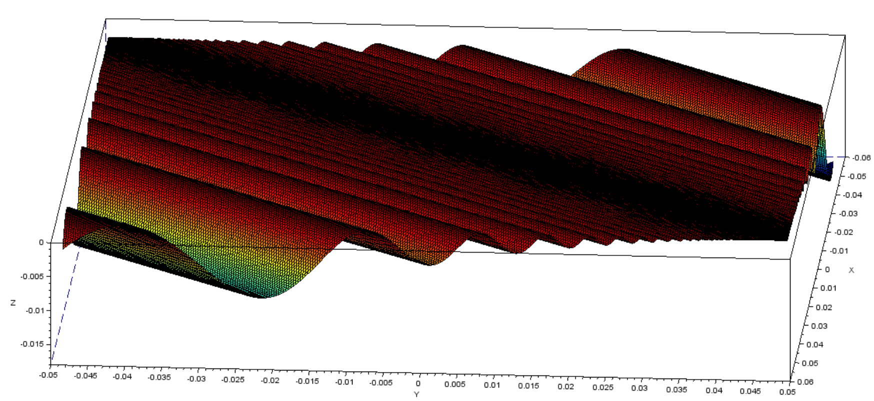
$$\sqrt{n}\sum_{k=1}^{\infty}\int_{I_k(x)} e^{nR(x,t)}\,dt > M,\ \forall x \in [a,b],$$

for any $n \geq N_2$. Because $M > 0$ is arbitrary, we obtain (3). Because δ is arbitrary, from (1), (2) and (3) we obtain (iii).

The proof is complete. $\square$

Example 1. *Let* $R : \left[-\frac{1}{2\pi}, \frac{1}{2\pi}\right] \times \left[-\frac{1}{2\pi}, \frac{1}{2\pi}\right] \setminus \Delta \to (-\infty, 0]$,

$$R(x,t) = (t-x)^2\left(\sin\frac{1}{t-x} - 1\right),\ x,t \in \left[-\frac{1}{2\pi}, \frac{1}{2\pi}\right],\ t \neq x.$$



Let $x \in [a,b]$ be fixed. Condition (A) is satisfied, with:

$$T(x) = \{t_k^+(x) | k \geq k^+(x)\} \cup \{t_k^-(x) | k \geq k^-(x)\} = T^+(x) \cup T^-(x),$$

where $T^+(x) \subset (x,b]$ and $T^-(x) \subset [a,x)$, and the indices $1 \leq k^{\pm}(x) \leq \infty$ are chosen such that $t_k^{\pm}(x) \in [a,b]$, $\forall k \geq k^{\pm}(x)$. We make the convention

$$k^+\left(\frac{1}{2\pi}\right) = \infty = k^-\left(-\frac{1}{2\pi}\right)$$

and, in this case

$$T^+\left(\frac{1}{2\pi}\right) = \phi = T^-\left(-\frac{1}{2\pi}\right).$$

We have

$$t_k^+(x) = \frac{2}{(4k+1)\pi} + x,\ k \geq k^+(x),\ t_k^-(x) = -\frac{2}{(4k+1)\pi} + x,\ k \geq k^-(x).$$

By simple calculi, we obtain

$$\frac{\partial^2 R}{\partial t^2}(x, t_k^{\pm}(x)) = -\frac{(4k+1)^2\pi^2}{4},\ k \geq k^{\pm}(x)$$

and consider

$$I_k^+(x) = \left(\frac{2}{(4k+3)\pi} + x, \frac{2}{(4k-1)\pi} + x\right),\ k \geq k^+(x),$$

$$I_k^-(x) = \left(-\frac{2}{(4k-1)\pi} + x, -\frac{2}{(4k+3)\pi} + x\right),\ k \geq k^-(x),$$

for which properties (a), (b) and (c) are satisfied.

The uniformity conditions in (B), (C) and (D) are obtained taking into account that

$$R(x,t) = r(t-x),\ x, t \in \left[-\frac{1}{2\pi}, \frac{1}{2\pi}\right],\ t \neq x,$$

where

$$r(u) := u^2\left(\sin\frac{1}{u} - 1\right),\ u \in \left[-\frac{1}{\pi}, \frac{1}{\pi}\right] \setminus \{0\}.$$

Also,

$$A_k^{\pm}(x) = \sqrt{\frac{8}{\pi}} \cdot \frac{1}{4k+1},\ x \in [a,b],\ k > k^{\pm}(x),$$

thus $A_k^{\pm}(x)$ are uniformly bounded and the series in (D) is divergent.

References

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