Bulletin of the *Transilvania* University of Braşov • Vol 6(55), No. 2 - 2013
Series III: Mathematics, Informatics, Physics, 1-10

A PRACTICAL APPROACH OF THE TRAFFIC FLOW USING FLUID DYNAMICS

Olivia FLOREA[1] **Carina IVAŞCU**[2]

Abstract

The mathematical modeling of the traffic flow from a crossroad can be approached using the fluid mechanics. In this paper, a problem with initial values for the flow traffic is solved analytically and numerically. The aim of this paper is to simulate the traffic inside a roundabout, seen as an intersection of 4 T-roads.

2000 *Mathematics Subject Classification*: 35L65, 90B20 .
Key words: Traffic flow, LWR model, hyperbolic systems.

1 Introduction

In 1950 using the link between the flux (q), density, concentration $\rho(x,t)$, and speed of vehicles (v), Lightwill, William and Richards (LWR) used the continuous approximation technique, in order to find the link between space and time in traffic (see[1]). In this approach, the flow of vehicles is seen as a one-dimensional compressible fluid, with the law for the traffic flow conservation. Taking into account the link between speed and density, the expression of the flow is given by: $q = q(\rho) = v \cdot \rho$. The scalar equation for the traffic flow conservation is:

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{1}$$

Experimental results show that modeling of traffic flow should be based on the following behavioral situations: for low densities, the flow is unimpeded and the LWR model can be used; for increasing densities, the traffic will be jammed and we

[1]Corresponding author - Transilvania University of Brasov, Faculty of Mathematics and Informatics, e-mail: olivia.florea@unitbv.ro,
[2]Transilvania University of Brasov, Faculty of Mathematics and Informatics, e-mail: carinaivascu@yahoo.com

can use a 2-dimensional model. A 2-dimensional model was introduced by Payne and Whitham (see[2]):

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial q}{\partial x} = 0 \Rightarrow \dfrac{\partial \rho}{\partial t} + \dfrac{\partial (\rho v)}{\partial x} = 0 \\ \dfrac{\partial q}{\partial t} + \dfrac{\partial (\rho v^2 + p(\rho))}{\partial x} = 0 \Rightarrow \dfrac{\partial (\rho v)}{\partial t} + \dfrac{\partial (\rho v^2 + p(\rho))}{\partial x} = 0 \end{cases} \tag{2}$$

where $p(\rho)$ is the pressure, used also in the equations of gas dynamics. The main flaw of this model is that it does not satisfy the two principles of a traffic flow model: the drivers react according to what happens in front of them, so the transmission of information is lower than the speed of cars, and the density and speed should remain positive and bounded.

These errors have been corrected by Aw and Rascle (see[6], [7]), using the substitution: $\dfrac{\partial p}{\partial x} \leftrightarrow \dfrac{\partial p}{\partial t} + v\dfrac{\partial p}{\partial x}$ we have the following model:

$$\begin{array}{l} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial (\rho v)}{\partial x} = 0 \\ \dfrac{\partial \left(\rho(v + p(\rho))\right)}{\partial t} + \dfrac{\partial (\rho v(v + p(\rho)))}{\partial x} = 0 \end{array} \tag{3}$$

Therefore, $p(\rho)$ can be considered an anticipation factor which takes into consideration the reactions of the drivers. Even so, the model is not well defined in the neighborhood of a null flow $q = 0$. Another model was introduced by Colombo (see[3]):

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial (\rho v)}{\partial x} = 0 \\ \dfrac{\partial q}{\partial t} + \dfrac{\partial ((q - Q)v)}{\partial x} = 0 \end{cases} \tag{4}$$

where q is a sort of "weighted momentum" in analogy to gas dynamics, Q is a parameter depending on the road under consideration and a closure law for $v = v(p, q)$ is given. This model displays a maximal car density, that is the only positive density at which velocity is zero. In other words, whenever a queue at zero speed forms, the maximal density is reached. Other else, cars cannot stop if maximal density is not reached.

2 Description of the model

Let:

$$\begin{cases} q_1 - \text{be the density of vehicles that neither exit, nor enter the road} \\ q_2 - \text{be the density of vehicles that exit the road} \\ q_3 - \text{be the density of vehicles that enter the road} \end{cases} \tag{5}$$

v- be the same speed law for all vehicles. Using relations (2) and considering the pressure negligible we have the second relation of the model described above on the following form:

$$\frac{\partial (\rho \cdot v)}{\partial t} + \frac{\partial (\rho \cdot v \cdot v(q))}{\partial x} = 0 \tag{6}$$

So, for the case when the cars exit the road $v(q) = v(q_1 + q_2)$ and for the case when the cars enter the road $v(q) = v(q_1 + q_3)$ Let us consider a T-road,as follows:

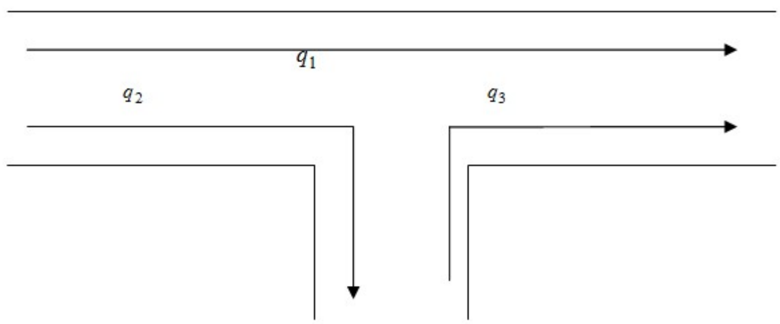

Figure 1: The general representation of a "T" - road

We have the following local conservation laws:

$$\begin{cases} \begin{cases} \dfrac{\partial q_1}{\partial t} + \dfrac{\partial(q_1 \cdot v(q_1+q_2))}{\partial x} = 0 \\ \dfrac{\partial q_2}{\partial t} + \dfrac{\partial(q_2 \cdot v(q_1+q_2))}{\partial x} = 0 \end{cases} , x < 0 \\ \begin{cases} \dfrac{\partial q_1}{\partial t} + \dfrac{\partial(q_1 \cdot v(q_1+q_3))}{\partial x} = 0 \\ \dfrac{\partial q_3}{\partial t} + \dfrac{\partial(q_3 \cdot v(q_1+q_3))}{\partial x} = 0 \end{cases} , x > 0 \end{cases} \tag{7}$$

We can add the initial data (see[7]):

$$\begin{array}{l} q_1(0,x) = q_{1,0}(x),\ x \in \mathbb{R} \\ q_2(0,x) = q_{2,0}(x), x < 0 \\ q_3(0,x) = q_{3,0}(x), x > 0 \end{array} \tag{8}$$

and the boundary conditions (see[7]):

$$\begin{array}{l} q_1 v(q_1+q_2)(t,0-) = q_1 v(q_1+q_3)(t,0+), \\ q_2 v(q_1+q_2)(t,0-) \leq o(t) \\ q_3 v(q_1+q_3)(t,0+) \leq i(t) \end{array} \tag{9}$$

where $o(t)$ is the output function, and $i(t)$ is the input function. As we mentioned before, our goal is to model the traffic in a roundabout, using the equations that characterize the 4 T-roads. We use the following notations: $q_{i,j.k}$ the flux of vehicles in the intersection, where: i represents the clasification of the road (5), and j is the rank, i.e. T-road is considered(the first,the second..), and

$$k = \begin{cases} 2, \text{ if the vehicle has to enter the roundabout} \\ 1, \text{if the vehicle does not have to enter the roundabout} \end{cases}$$

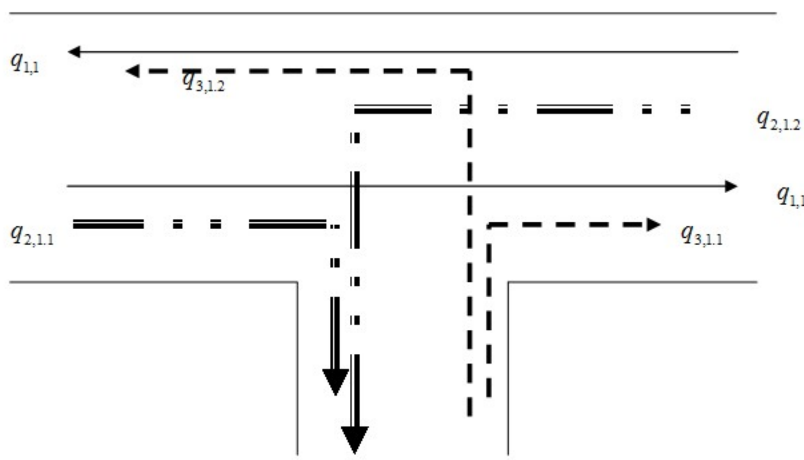

Figure 2: The first "T" - road considered

Using (6),we obtain:

$$x<0:I1\begin{cases}\dfrac{\partial q_{1,1}}{\partial t}+\dfrac{\partial(q_{1,1}\cdot v(q_{1,1}+q_{2,1.1}+q_{2,1.2}))}{\partial x}=0\\ \dfrac{\partial(q_{2,1.1}+q_{2,1.2})}{\partial t}+\dfrac{\partial((q_{2,1.1}+q_{2,1.2})\cdot v(q_{1,1}+q_{2,1.1}+q_{2,1.2}))}{\partial x}=0\end{cases} \tag{10}$$

$$x>0:I2\begin{cases}\dfrac{\partial q_{1,1}}{\partial t}+\dfrac{\partial(q_{1,1}\cdot v(q_{1,1}+q_{3,1.1}+q_{3,1.2}))}{\partial x}=0\\ \dfrac{\partial(q_{3,1.1}+q_{3,1.2})}{\partial t}+\dfrac{\partial((q_{3,1.1}+q_{3,1.2})\cdot v(q_{1,1}+q_{3,1.1}+q_{3,1.2}))}{\partial x}=0\end{cases} \tag{11}$$

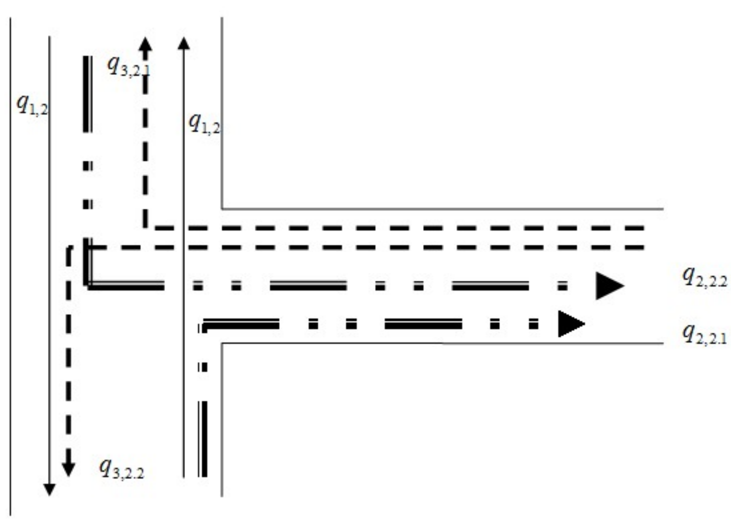

Figure 3: The second "T" - road considered

For the second case considered we obtain the following equations:

$$x<0:\begin{cases}\dfrac{\partial q_{1,2}}{\partial t}+\dfrac{\partial(q_{1,2}\cdot v(q_{1,2}+q_{2,2.1}+q_{2,2.2}))}{\partial x}=0\\ \dfrac{\partial(q_{2,2.1}+q_{2,2.2})}{\partial t}+\dfrac{\partial((q_{2,2.1}+q_{2,2.2})\cdot v(q_{1,2}+q_{2,2.1}+q_{2,2.2}))}{\partial x}=0\end{cases} \tag{12}$$

$$x>0:\begin{cases}\dfrac{\partial q_{1,2}}{\partial t}+\dfrac{\partial(q_{1,2}\cdot v(q_{1,2}+q_{3,2.1}+q_{3,2.2}))}{\partial x}=0\\ \dfrac{\partial(q_{3,2.1}+q_{3,2.2})}{\partial t}+\dfrac{\partial[(q_{3,2.1}+q_{3,2.2})\cdot v(q_{1,2}+q_{3,2.1}+q_{3,2.2})]}{\partial x}=0\end{cases} \tag{13}$$

By comparing this figure with the first one, we obtain the following relations:

$$\begin{cases}q_{2,2.1}=q_{3,1.1}\\ q_{3,2.2}=q_{2,1.2}\end{cases}$$

Therefore, (12) and (13) become:

$$x<0:\begin{cases}\dfrac{\partial q_{1,2}}{\partial t}+\dfrac{\partial(q_{1,2}\cdot v(q_{1,2}+q_{3,1.1}+q_{2,2.2}))}{\partial x}=0\\ \dfrac{\partial(q_{3,1.1}+q_{2,2.2})}{\partial t}+\dfrac{\partial((q_{3,1.1}+q_{2,2.2})\cdot v(q_{1,2}+q_{3,1.1}+q_{2,2.2}))}{\partial x}=0\end{cases} \tag{14}$$

$$x>0:\begin{cases}\dfrac{\partial q_{1,2}}{\partial t}+\dfrac{\partial(q_{1,2}\cdot v(q_{1,2}+q_{3,2.1}+q_{2,1.2}))}{\partial x}=0\\ \dfrac{\partial(q_{3,2.1}+q_{2,1.2})}{\partial t}+\dfrac{\partial((q_{3,2.1}+q_{2,1.2})\cdot v(q_{1,2}+q_{3,2.1}+q_{2,1.2}))}{\partial x}=0\end{cases} \tag{15}$$

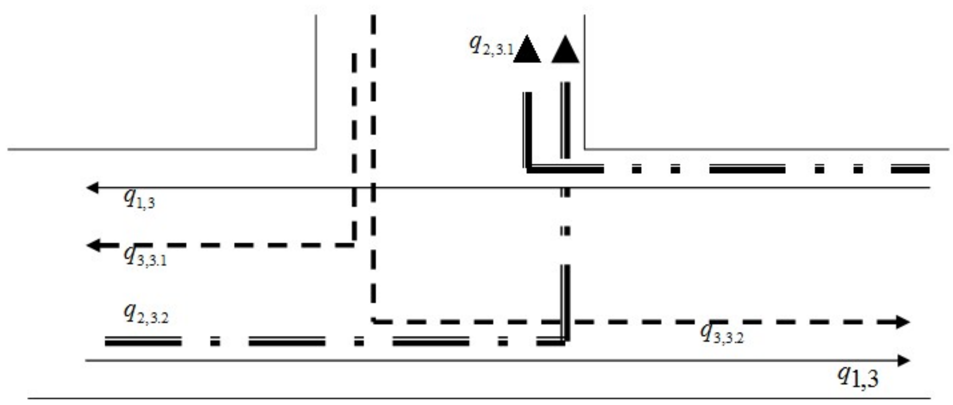

Figure 4: The third "T" - road considered

For the third case considered we obtain the following equations:

$$\begin{aligned}x<0&\begin{cases}\dfrac{\partial q_{1,3}}{\partial t}+\dfrac{\partial(q_{1,3}\cdot v(q_{1,3}+q_{2,3.1}+q_{2,3.2}))}{\partial x}=0\\ \dfrac{\partial(q_{2,3.1}+q_{2,3.2})}{\partial t}+\dfrac{\partial((q_{2,3.1}+q_{2,3.2})\cdot v(q_{1,3}+q_{2,3.1}+q_{2,3.2}))}{\partial x}=0\end{cases}\\ x>0&\begin{cases}\dfrac{\partial q_{1,3}}{\partial t}+\dfrac{\partial(q_{1,3}\cdot v(q_{1,3}+q_{3,3.1}+q_{3,3.2}))}{\partial x}=0\\ \dfrac{\partial(q_{3,3.1}+q_{3,3.2})}{\partial t}+\dfrac{\partial((q_{3,3.1}+q_{3,3.2})\cdot v(q_{1,3}+q_{3,3.1}+q_{3,3.2}))}{\partial x}=0\end{cases}\end{aligned} \tag{16}$$

By comparing all three figures, we obtain:

$$q_{1,3} = q_{1,1}$$
$$q_{2,3.1} = q_{3,2.1}$$
$$q_{2,2.2} = q_{3,3.2}$$

Therefore, (16) becomes:

$$x<0: \begin{cases} \dfrac{\partial q_{1,1}}{\partial t} + \dfrac{\partial(q_{1,1} \cdot v(q_{1,1} + q_{3,2.1} + q_{2,3.2}))}{\partial x} = 0 \\ \dfrac{\partial(q_{3,2.1} + q_{2,3.2})}{\partial t} + \dfrac{\partial((q_{3,2.1} + q_{2,3.2}) \cdot v(q_{1,1} + q_{3,2.1} + q_{2,3.2}))}{\partial x} = 0 \end{cases} \tag{17}$$

$$x>0: \begin{cases} \dfrac{\partial q_{1,1}}{\partial t} + \dfrac{\partial(q_{1,1} \cdot v(q_{1,1} + q_{3,3.1} + q_{2,2.2}))}{\partial x} = 0 \\ \dfrac{\partial(q_{3,3.1} + q_{2,2.2})}{\partial t} + \dfrac{\partial((q_{3,3.1} + q_{2,2.2}) \cdot v(q_{1,1} + q_{3,3.1} + q_{2,2.2}))}{\partial x} = 0 \end{cases} \tag{18}$$

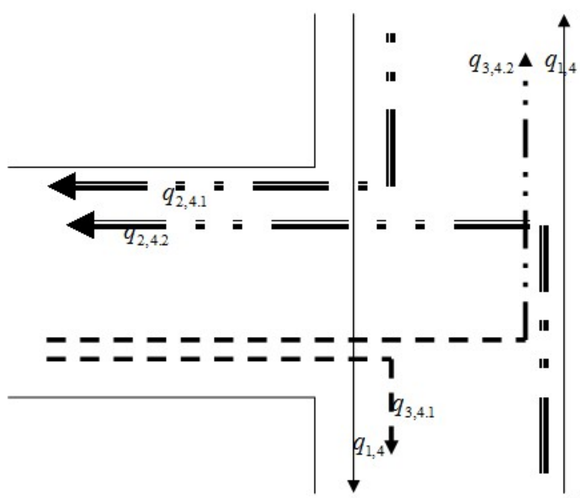

Figure 5: The fourth "T" - road considered

By analyzing the figure above, we obtain:

$$\begin{aligned} x<0 &\begin{cases} \dfrac{\partial q_{1,4}}{\partial t} + \dfrac{\partial(q_{1,4} \cdot v(q_{1,4} + q_{2,4.1} + q_{2,4.2}))}{\partial x} = 0 \\ \dfrac{\partial(q_{2,4.1} + q_{2,4.2})}{\partial t} + \dfrac{\partial(q_{2,4.1} + q_{2,4.2}) \cdot v(q_{1,4} + q_{2,4.1} + q_{2,4.2}))}{\partial x} = 0 \end{cases} \\ x>0 &\begin{cases} \dfrac{\partial q_{1,4}}{\partial t} + \dfrac{\partial(q_{1,4} \cdot v(q_{1,4} + q_{3,4.1} + q_{3,4.2}))}{\partial x} = 0 \\ \dfrac{\partial(q_{3,4.1} + q_{3,4.2})}{\partial t} + \dfrac{\partial(q_{3,4.1} + q_{3,4.2}) \cdot v(q_{1,4} + q_{3,4.1} + q_{3,4.2}))}{\partial x} = 0 \end{cases} \end{aligned} \tag{19}$$

By comparing all four figures, we obtain:

$$\begin{cases} q_{1,4} = q_{1,2} \\ q_{2,4.1} = q_{3,3.1} \\ q_{2,4.2} = q_{3,1.2} \\ q_{3,4.1} = q_{2,1.1} \\ q_{3,4.2} = q_{2,3.2} \end{cases}$$

Therefore, (19) becomes:

$$x < 0 \begin{cases} \frac{\partial q_{1,2}}{\partial t} + \frac{\partial (q_{1,2} \cdot v(q_{1,2} + q_{3,3.1} + q_{3,1.2}))}{\partial x} = 0 \\ \frac{\partial (q_{3,3.1} + q_{3,1.2})}{\partial t} + \frac{\partial (q_{3,3.1} + q_{3,1.2}) \cdot v(q_{1,2} + q_{3,3.1} + q_{3,1.2}))}{\partial x} = 0 \end{cases} \tag{20}$$

$$x > 0 \begin{cases} \frac{\partial q_{1,2}}{\partial t} + \frac{\partial (q_{1,2} \cdot v(q_{1,2} + q_{2,1.1} + q_{2,3.2}))}{\partial x} = 0 \\ \frac{\partial (q_{2,1.1} + q_{2,3.2})}{\partial t} + \frac{\partial ((q_{2,1.1} + q_{2,3.2}) \cdot v(q_{1,2} + q_{2,1.1} + q_{2,3.2}))}{\partial x} = 0 \end{cases} \tag{21}$$

The final equations for $x < 0$ are (see (10), (14), (17), (20)):

$$\begin{aligned} &\frac{\partial q_{1,1}}{\partial t} + \frac{\partial (q_{1,1} \cdot v(q_{1,1} + q_{2,1.1} + q_{2,1.2}))}{\partial x} = 0 \\ &\frac{\partial (q_{2,1.1} + q_{2,1.2} - q_{1,1})}{\partial t} + \frac{\partial ((q_{2,1.1} + q_{2,1.2} - q_{1,1}) \cdot v(q_{2,1.1} + q_{2,1.2} + q_{1,1})))}{\partial x} = 0 \\ &\frac{\partial q_{1,2}}{\partial t} + \frac{\partial (q_{1,2} \cdot v(q_{1,2} + q_{3,1.1} + q_{2,2.2}))}{\partial x} = 0 \\ &\frac{\partial (q_{3,1.1} + q_{2,2.2} - q_{1,2})}{\partial t} + \frac{\partial ((q_{3,1.1} + q_{2,2.2} - q_{1,2}) \cdot v(q_{1,2} + q_{3,1.1} + q_{2,2.2}))}{\partial x} = 0 \\ &\frac{\partial q_{1,1}}{\partial t} + \frac{\partial (q_{1,1} \cdot v(q_{1,1} + q_{3,2.1} + q_{2,3.2}))}{\partial x} = 0 \\ &\frac{\partial (q_{3,2.1} + q_{2,3.2} - q_{1,1})}{\partial t} + \frac{\partial ((q_{3,2.1} + q_{2,3.2} - q_{1,1}) \cdot v(q_{1,1} + q_{3,2.1} + q_{2,3.2}))}{\partial x} = 0 \\ &\frac{\partial q_{1,2}}{\partial t} + \frac{\partial (q_{1,2} \cdot v(q_{1,2} + q_{3,3.1} + q_{3,1.2}))}{\partial x} = 0 \\ &\frac{\partial (q_{3,3.1} + q_{3,1.2} - q_{1,2})}{\partial t} + \frac{\partial ((q_{3,3.1} + q_{3,1.2} - q_{1,2}) \cdot v(q_{1,2} + q_{3,3.1} + q_{3,1.2}))}{\partial x} = 0 \end{aligned} \tag{22}$$

Combining the equations from the (22) systems we obtain the final system of equations for the case when $x < 0$:

$$\begin{aligned}
&\frac{\partial(q_{1,1}\cdot v(q_{2,1.1}+q_{2,1.2}-q_{3,2.1}-q_{2,3.2}))}{\partial x}=0\\
&\frac{\partial(q_{2,1.1}+q_{2,1.2}-q_{3,2.1}-q_{2,3.2})}{\partial t}+\\
&+\frac{\partial((q_{2,1.1}+q_{2,1.2}-q_{3,2.1}-q_{2,3.2})\cdot v(q_{1,1}+q_{2,1.1}+q_{2,1.2}+q_{3,2.1}+q_{2,3.2})))}{\partial x}=0\\
&\frac{\partial(q_{1,2}\cdot v(q_{3,1.1}+q_{2,2.2}-q_{3,3.1}-q_{3,1.2}))}{\partial x}=0\\
&\frac{\partial((q_{3,1.1}+q_{2,2.2}-q_{3,3.1}-q_{3,1.2}))}{\partial t}+\\
&+\frac{\partial((q_{3,1.1}+q_{2,2.2}-q_{3,3.1}-q_{3,1.2})\cdot v(q_{3,1.1}+q_{2,2.2}+q_{3,3.1}+q_{3,1.2}+q_{1,2}))}{\partial x}=0
\end{aligned} \tag{23}$$

The final equations for $x > 0$ are (see (11), (15), (18), (21)):

$$\begin{aligned}
&\frac{\partial q_{1,1}}{\partial t}+\frac{\partial(q_{1,1}\cdot v(q_{1,1}+q_{3,1.1}+q_{3,1.2}))}{\partial x}=0\\
&\frac{\partial(q_{3,1.1}+q_{3,1.2}-q_{1,1})}{\partial t}+\frac{\partial((q_{3,1.1}+q_{3,1.2}-q_{1,1})\cdot v(q_{1,1}+q_{3,1.1}+q_{3,1.2}))}{\partial x}=0\\
&\frac{\partial q_{1,2}}{\partial t}+\frac{\partial(q_{1,2}\cdot v(q_{1,2}+q_{3,2.1}+q_{2,1.2}))}{\partial x}=0\\
&\frac{\partial(q_{3,2.1}+q_{2,1.2}-q_{1,2})}{\partial t}+\frac{\partial((q_{3,2.1}+q_{2,1.2}-q_{1,2})\cdot v(q_{1,2}+q_{3,2.1}+q_{2,1.2}))}{\partial x}=0\\
&\frac{\partial q_{1,1}}{\partial t}+\frac{\partial(q_{1,1}\cdot v(q_{1,1}+q_{3,3.1}+q_{2,2.2}))}{\partial x}=0\\
&\frac{\partial(q_{3,3.1}+q_{2,2.2}-q_{1,1})}{\partial t}+\frac{\partial((q_{3,3.1}+q_{2,2.2}-q_{1,1})\cdot v(q_{1,1}+q_{3,3.1}+q_{2,2.2}))}{\partial x}=0\\
&\frac{\partial q_{1,2}}{\partial t}+\frac{\partial(q_{1,2}\cdot v(q_{1,2}+q_{2,1.1}+q_{2,3.2}))}{\partial x}=0\\
&\frac{\partial(q_{2,1.1}+q_{2,3.2}-q_{1,2})}{\partial t}+\frac{\partial((q_{2,1.1}+q_{2,3.2}-q_{1,2})\cdot v(q_{1,2}+q_{2,1.1}+q_{2,3.2}))}{\partial x}=0
\end{aligned} \tag{24}$$

Combining the equations from the (24) systems we obtain the final system of equations for the case when $x > 0$:

$$\begin{aligned}
&\frac{\partial(q_{1,1}\cdot v(q_{3,1.1}+q_{3,1.2}-q_{3,3.1}-q_{2,2.2}))}{\partial x}=0\\
&\frac{\partial(q_{3,1.1}+q_{3,1.2}-q_{3,3.1}-q_{2,2.2})}{\partial t}+\\
&+\frac{\partial((q_{3,1.1}+q_{3,1.2}-q_{3,3.1}-q_{2,2.2})\cdot v(q_{1,1}+q_{3,3.1}+q_{2,2.2}+q_{3,1.1}+q_{3,1.2}))}{\partial x}=0\\
&\frac{\partial(q_{1,2}\cdot v(q_{3,2.1}+q_{2,1.2}-q_{2,1.1}-q_{2,3.2}))}{\partial x}=0\\
&\frac{\partial(q_{3,2.1}+q_{2,1.2}-q_{2,1.1}-q_{2,3.2})}{\partial t}+\\
&+\frac{\partial((q_{3,2.1}+q_{2,1.2}-q_{2,1.1}-q_{2,3.2})\cdot v(q_{1,2}+q_{2,1.1}+q_{2,3.2}+q_{3,2.1}+q_{2,1.2}))}{\partial x}=0
\end{aligned} \tag{25}$$

By collating the four previous T-roads, we obtain the following roundabout:

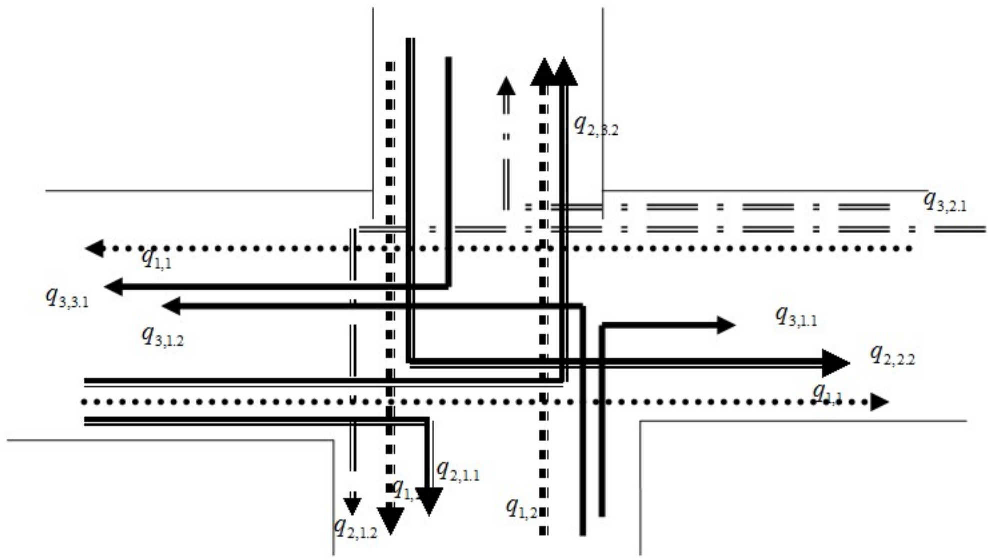

Figure 6: The entire intersection by overlapping the previous four parts

In the case $x > 0$ we have obtained the following graphic, representing the variation of the flux:

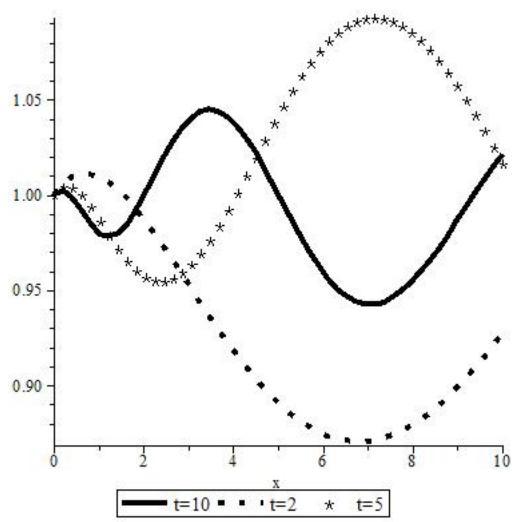

Figure 7: The flow variation for the positive direction

In the case $x < 0$,we have obtained the following graphic for the variation of the flux:

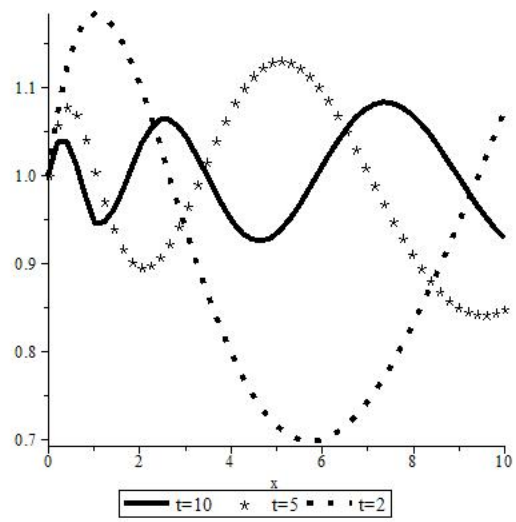

Figure 8: The flow variation for the negative direction

References

[1] Kim Y., *Online traffic flow model applying dynamic flow-density relation*,Fachgebiet Verkehrstechnik und Verkehrsplanung der Technischen Universit at M unchen, 2002.

[2] Bretti G., Natalini R., Piccoli B., *A fluid-dynamic traffic model on road networks*,Archives of Computational Methods in Engineering, **14** (2007), Issue 2, 139-172.

[3] Colombo R.M., *A 2 - 2 hyperbolic traffic flow model -Traffic flow modeling and simulation*, Math. Comput. Modeling, **35** (2002), 683-688.

[4] Bretti G., Natalini R., Piccoli B., *Numerical Approximations of a Traffic Flow Model on Networks*, Networks and Heterogeneous Media **1** (2006), 57 - 84.

[5] Mercier M., *Traffic flow modelling with junctions*, Journal of Mathematical Analysis and Applications, **350** (2009), 369-383.

[6] Payne H.J., *Models of freeway traffic and control*, In Simulation Council Proc. **28**, Math. Models Publ. Sys., (1971), 51-61.

[7] Colombo R.M., Goatin P., *A well posed conservation law with a variable unilateral constraint*, J. Differential Equations, **234(2)** (2007), 654-675.

[8] Vasseur A., *Strong traces for solutions of multidimensional scalar conservation laws*, Arch. Ration. Mech. Anal., **160** (2001), 181-193.