Bulletin of the *Transilvania* University of Braşov • Vol 6(55), No. 1 - 2013
Series III: Mathematics, Informatics, Physics, 89-100

CHERNOUSKO-LYUBUSHIN VERSION OF THE SUCCESIVE APPROXIMATION METHOD FOR OPTIMAL CONTROL PROBLEM REVISITED

Ernest SCHEIBER[1]

Abstract

The purpose of this paper is to emphesize the parallel property of a variant of the successive approximation method to solve the optimal control problem for ordinary differential equations. The obtained parallel algorithm combines synchronous and asynchronous computations.

2010 *Mathematics Subject Classification:* 49M05, 49M25, 68W10
Key words: optimal control problem, successive approximation method, parallel method.

1 Introduction

We recall a very well known variant of the successive approximation method to solve the optimal control problem for ordinary differential equations, [10, 11], elaborated more than 30 years ago. The initial version of the successive approximation method to solve the optimal control problem for ordinary differential equations are given in [8, 9].

This method belongs to the so called indirect approaches. It is based on the Pontryagin's maximum principle. In direct methods, the optimal control problem is discretized to obtain a finite dimensional optimization problem.

The drawback of the method presented is that it can not be used to solve optimal control problems with state constraints. Some partial solutions may be the usage of the penalization technique or to incorporate the control into the dynamic of the control problem, [5].

The purpose of this recall is to emphasize the parallel property of this method. This property simplifies the programming and brings advantages to a current computer. In order that this paper be self contained we shall give the proofs of the specific results.

[1]Faculty of Mathematics and Informatics, *Transilvania* University of Braşov, Romania, e-mail: scheiber@unitbv.ro

2 The optimal control problem

Consider the following optimal control problem for a fixed $T > 0$

$$\text{minimize } I(u) = G(x(T)) \tag{1}$$

subject to

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), t), \qquad t \in [0, T], \qquad x(0) = x_0, \quad &(2)\\ u(t) &\in \Omega, &(3) \end{aligned}$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^q$ are the state of the system and the control at time t, respectively. Ω is a given convex and compact subset of $\mathbb{R}^q$ with

$$\Omega \subseteq \{x \in \mathbb{R}^q : \|x\| \leq r\}.^2$$

Let U be the set of admissible controls, i.e. $U = \{u : [0, T] \to \Omega :$ continuous except at a countable number of points$\}$.

We assume the following hypotheses are satisfied:

(H$_1$) The functions $f : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R} \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{R}$ are continuous together with their partial derivatives $f_x, f_u, f_{xx}, f_{xu}, f_{uu}, G_x, G_{xx}$.

(H$_2$) There exists $M > 0$ such that

$$< x, f(x, u, t) > \leq M(1 + \|x\}^2), \quad \forall (x, u, t) \in \mathbb{R}^n \times \Omega \times [0, T].$$

Given $u \in U$, denote by $p^u : [0, T] \to \mathbb{R}^n$ the solution of the initial value problem

$$\dot{p}(t) = -H_x(x(t), u(t), p(t), t), \qquad p(T) = -G_x(x(T)), \tag{4}$$

where $H : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is the Hamiltonian defined by $H(x, u, p, t) =< p, f(x, u, t) >$.

From (H$_1$) and (H$_2$) it results that for any $u \in U$ the solutions x^u and p^u of the initial value problems (2) and (4), respectively, exists for any $t \in [0, T]$, and that there exists a positive constant M_1 such that

$$\|x^u(t)\|, \|p^u(t)\| \leq M_1, \qquad \forall\, t \in [0, T], \tag{5}$$

for any $u \in U$. Consequently the cost functional I is bounded below.

(H$_3$) For any $u \in U$, there exists $\bar{u} \in U$ such that

$$H(x^u(t), \bar{u}(t), p^u(t), t) = \max\{H(x^u(t), w, p^u(t), t) : w \in \Omega\}.$$

[2] Throughout this paper $\| \cdot \|$ and $< \cdot, \cdot >$ denote respectively the Euclidean norm and inner product.

We denote by $W(u,t)$ and $\mu(u)$ the functionals defined by

$$\begin{aligned} W(u,t) &= H(x^u(t),\bar{u}(t),p^u(t),t) - H(x^u(t),u(t),p^u(t),t), \\ \mu(u) &= \int_0^T W(u,t)\mathrm{d}t. \end{aligned}$$

We have $W(u,t) \geq 0,\ \forall\ t \in [0,T]$ and $\mu(u) \geq 0$, for any $u \in U$. If u_* is the optimal control of problem (1)-(3) then $W(u_*,t) = 0,\ \forall\ t \in [0,T]$ and $\mu(u_*) = 0$.

Let the metric $d : U \times U \to R$ be defined by

$$d(u_1,u_2) = \int_0^T \|u_1(t) - u_2(t)\|\mathrm{d}t.$$

The following properties are known

Theorem 2.1 *[6] The function $\mu(u)$ is uniformly continuous with respect to the metric d.*

Theorem 2.2 *[12] There exists a constant $C > 0$ such that for any $u, v \in U$*

$$I(v) - I(u) = -\int_0^T [H(x^u(t),v(t),p^u(t),t) - H(x^u(t),u(t),p^u(t),t)]\mathrm{d}t + \mathcal{R}$$

with $|\mathcal{R}| \leq Cd^2(u,v)$.

3 The successive approximation method

The idea of the successive approximation method is the control improvement: given a control function u find another control function v such that $I(v) < I(u)$. The control improvement will be obtained modifying the control function on a *small* interval. The parallel search of that interval optimizes the method.

Another approach of the control improvement is based on the perturbation of the Pontryagin's system and an iterative process to solve the corresponding boundary value problem, [2].

For any integer $m \geq 0$, let $\eta_m = \frac{T}{2^m}, t_{m,s} = s\eta_m,\ s \in \{0,1,\ldots,2^m\}$ and $J_j^m = [t_{m,2j-2}, t_{m,2j}],\ j \in \{1,2,\ldots,2^{m-1}\}$.

Theorem 3.1 *[10] For any $m > 0$ and any $u \in U$ there exists $j \in \{1,2,\ldots,2^{m-1}\}$ such that*

$$\frac{1}{2\eta_m}\int_{J_j^m} W(u,t)\mathrm{d}t \geq \frac{\mu(u)}{T}. \tag{6}$$

Proof. If we suppose that there exists an integer $m > 0$ and $u \in U$ such that

$$\frac{1}{2\eta_m}\int_{J_j^m} W(u,t)\mathrm{d}t < \frac{\mu(u)}{T}, \qquad \forall j \in \{1,2,\ldots,2^{m-1}\},$$

then we find

$$\mu(u) = \int_0^T W(u,t)\mathrm{d}t = \sum_{j=1}^{2^{m-1}} \int_{J_j^m} W(u,t)\mathrm{d}t < 2^{m-1} 2\eta_m \frac{\mu(u)}{T} = \mu(u). \quad \blacksquare$$

Let $m > 0,\ u \in U$ and $j \in \{1, 2, \ldots, 2^{m-1}\}$ satisfying (6). We define

$$\tilde{u}(t) = \begin{cases} \bar{u}(t) & t \in J_j^m \\ u(t) & t \in [0,T] \setminus J_j^m \end{cases}. \tag{7}$$

Theorem 3.2 *[10] If $\mu(u) \neq 0$ then there exist $m \in \mathbb{N}^*$ and $\alpha > 0$ such that*

$$I(\tilde{u}) \leq I(u) - \frac{\eta_m}{T}\mu(u) \leq I(u) - \alpha\mu^2(u).$$

Proof. Applying Theorem 2.2 we have

$$I(\tilde{u}) - I(u) = -\int_0^T [H(x^u(t), \tilde{u}(t), p^u(t), t) - H(x^u(t), u(t), p^u(t), t)]\mathrm{d}t + \mathcal{R} =$$

$$= -\int_{J_j^m} [H(x^u(t), \bar{u}(t), p^u(t), t) - H(x^u(t), u(t), p^u(t), t)]\mathrm{d}t + \mathcal{R} =$$

$$= -\int_{J_j^m} W(u,t)\mathrm{d}t + \mathcal{R} \leq -\frac{2\eta_m}{T}\mu(u) + \mathcal{R},$$

and

$$|\mathcal{R}| \leq C\left(\int_0^T \|\tilde{u}(t) - u(t)\|\mathrm{d}t\right)^2 = C\left(\int_{J_j^m} \|\bar{u}(t) - u(t)\|\mathrm{d}t\right)^2 \leq 16Cr^2\eta_m^2.$$

Thus

$$I(\tilde{u}) - I(u) \leq -\frac{2\eta_m}{T}\mu(u) + 16Cr^2\eta_m^2. \tag{8}$$

Let m be the smallest positive integer such that

$$16Cr^2\eta_m^2 < \frac{\eta_m}{T}\mu(u) \Leftrightarrow \eta_m = \frac{T}{2^m} < \frac{\mu(u)}{16Cr^2T} \Leftrightarrow 2^m > \frac{16Cr^2T^2}{\mu(u)}.$$

Then we have

$$2^m \leq 2\frac{16Cr^2T^2}{\mu(u)} \Leftrightarrow \eta_m \geq \frac{\mu(u)}{32Cr^2T}.$$

From (8), it follows that

$$I(\tilde{u}) - I(u) \leq -\frac{\eta_m}{T}\mu(u) \leq -\frac{1}{32Cr^2T^2}\mu^2(u). \quad \blacksquare$$

Now we can state an algorithm based on the above properties whose pseudocode is given in the table *Algorithm 1.*

The algorithm is adaptive in the sense that the mesh step decreases as the solution is approached.

The following convergence property is known:

Algorithm 1 Optimal Control by Successive Approximation Method

procedure OCOSAM($u^{(0)} \in U, \epsilon > 0, m_0 \in \mathbb{N}^*$)
  $k \leftarrow -1$
  $m \leftarrow m_0$
  repeat
    $k \leftarrow k + 1$
    Compute $x^{(k)}$ integrating (2) for $u = u^{(k)}$
    Compute $I_k = I(u^{(k)})$
    Compute $p^{(k)}$ integrating (4) for $u = u^{(k)}, x = x^{(k)}$
    Compute $\bar{u}^{(k)}$: $H(x^{(k)}(t), \bar{u}^{(k)}(t), p^{(k)}(t), t) = \max_{w \in \Omega} H(x^{(k)}(t), w, p^{(k)}(t), t)$
    if $\mu_k \geq \epsilon$ **then** ▷ Control improvement
      $end\_flag \leftarrow 0$
      repeat
        for $j = 1 : 2^{m-1}$ **do**
          if $\frac{1}{2\eta_m} \int_{J_j^m} W(u^{(k)}, t)\mathrm{d}t \geq \frac{\mu_k}{T}$ **then**
            Compute $\tilde{u}_j^{(k)}(t) = \begin{cases} \bar{u}^{(k)}(t) & t \in J_j^m \\ u^{(k)}(t) & t \in [0, T] \setminus J_j^m \end{cases}$
            Compute $\tilde{x}_j^{(k)}$ integrating (2) for $u = \tilde{u}_j^{(k)}$
            Compute $I_j^{(k)} = I(\tilde{u}_j^{(k)})$
            if $I_j^{(k)} \leq I_k - \frac{\eta_m}{T}\mu_k$ **then**
              $end\_flag \leftarrow 1$
              break
            end if
          end if
        end for
        if $end\_flag = 0$ **then**
          $m \leftarrow m + 1$ ▷ Sink $u^{(k)}$ into the new $\mathbb{R}^{2^m}$
          $end\_flag \leftarrow 1$
        else
          $u^{(k+1)} \leftarrow \tilde{u}_j^{(k)}$
        end if
      until $end\_flag = 1$
    end if
  until $\mu_k < \epsilon$
end procedure

Theorem 3.3 *[10] If the above algorithm generates an infinite sequence of controls* $(u^{(k)})_{k\in\mathbb{N}}$ *then* $\lim_{k\to\infty}\mu(u^{(k)})=0$.

Proof. From the inequalities $I(u^{(k+1)})\le I(u^{(k)})-\alpha\mu^2(u^{(k)}),\ k\in\mathbb{N}$ it follows

$$\sum_{k=0}^{\nu-1}\mu^2(u^{(k)})\le\frac{1}{\alpha}[I(u^{(0)})-I(u^{(\nu)})]\le\frac{1}{\alpha}[I(u^{(0)})-I^*],$$

where $I^*=\inf\{I(u):u\in U\}$. The convergence of the series implies the conclusion of the theorem. ∎

Theorem 3.4 *[10] If* $G(x)$ *is a convex differentiable function,* $f(x,u,t)=A(t)x+B(t)u$, *where the elements of the matrix* A *and* B *are continuous functions on* $[0,T]$, *then there exists a constant* $C>0$ *such that*

$$0\le I(u^{(k)})-I^*\le\frac{C}{k},\qquad k\in\mathbb{N},$$

where $I^*=\inf_{u\in U}I(u)$.

Proof. We denote u_* the optimal control, x_* its corresponding state, $I^*=I(u_*)$ and $\xi_k=I(u^{(k)})-I^*$. First we prove that $\xi_k\le\mu(u^{(k)})$.

From the convexity of G it results that

$$0\le\xi_k=G(x^{(k)}(T))-G(x_*(T))\le<G'(x^{(k)}(T)),x^{(k)}(T)-x_*(T)>. \qquad (9)$$

If $H(x,u,p,t)=<p,A(t)x+B(t)u>$ is the Hamiltonian of the optimal control problem, then $p^{(k)}$, the costate variable corresponding to the pair $(x^{(k)},u^{(k)})$, is the solution of the following initial value problem

$$\begin{aligned}p'(t)&=-A^T(t)p\\ p(T)&=-G'(x^{(k)}(T)).\end{aligned}$$

A standard computation leads to

$$<G'(x^{(k)}(T)),x^{(k)}(T)-x_*(T)>= \qquad (10)$$
$$=\int_0^T\Big[H(x^{(k)},u_*,p^{(k)},t)-H(x^{(k)},u^{(k)},p^{(k)},t)\Big]\,\mathrm{d}t.$$

But $H(x^{(k)},\bar{u},p^{(k)},t)\ge H(x^{(k)},w,p^{(k)},t)$ for any $w\in\Omega$. Using (9) and (10) we find

$$\xi_k\le\int_0^T\Big[H(x^{(k)},\bar{u},p^{(k)},t)-H(x^{(k)},u^{(k)},p^{(k)},t)\Big]\,\mathrm{d}t=$$
$$=\int_0^T W(u^{k)},t)\mathrm{d}t=\mu(u^{(k)}).$$

From Theorem 3.2 we obtain $I(u^{(k+1)})\le I(u^{(k)})-\alpha\mu^2(u^{(k)})$ and consequently

$$\alpha\xi_k^2\le\alpha\mu^2(u^{(k)})\le I(u^{(k)})-I(u^{(k+1)})=\xi_k-\xi_{k+1}.$$

Dividing by $\xi_k\xi_{k+1}$, it results $\alpha\le\alpha\frac{\xi_k}{\xi_{k+1}}\le\frac{1}{\xi_{k+1}}-\frac{1}{\xi_k}$. Summing this inequalities we obtain $\alpha k\le\frac{1}{\xi_k}-\frac{1}{\xi_0}\le\frac{1}{\xi_k}$ or $\xi_k\le\frac{C}{k}$, with $C=\frac{1}{\alpha}$. ∎

4 Parallel version of the method

Any step of a parallel algorithm of an iterative method contains 2 phases: (i) computation phase – the i process computes the attached component; (ii) communication phase – the i process sends to the other processors the computed component and receives the other components $j \neq i$. Depending on the available hardware / software, the communication phase may be programmed using shared memory or messages.

There are two versions to develop a parallel algorithm to generate an iterative sequence $(u^k)_{k\in\mathbb{N}}$:

- synchronous – After the computation phase, a barrier synchronization is required that is followed by the communication phase.

  The generated sequence $(u^k)_{k\in\mathbb{N}}$ coincides with that generated by the sequential algorithm
- asynchronous – The barrier synchronization is given up.

The cycle between the lines 14-24 may be programmed in parallel. The `break` instruction will be deleted and the control function may be modified on several intervals simultaneously and asynchronously.

Let $\mathcal{J}$ be the set of indexes $j \in \{1, 2, \ldots, 2^{m-1}\}$ satisfying (6). If $\tilde{u}$ is defined by

$$\tilde{u}(t) = \begin{cases} \bar{u}(t) & t \in J_j^m,\ j \in \mathcal{J} \\ u(t) & t \in [0, T] \setminus \bigcup_{j\in\mathcal{J}} J_j^m \end{cases} .$$

then

$$I(\tilde{u}) - I(u) = -\sum_{j\in\mathcal{J}} \int_{J_j^m} W(u, t)\mathrm{d}t + \mathcal{R} \leq -\mu(u) + 16Cr^2\eta_m^2 < 0,$$

for m sufficiently big.

If the $\mathcal{J}$ set is empty (the *end_flag* remains 0) then a barrier synchronization is required in order to sink the control function into a higher dimension space.

5 Numerical results

All the samples presented in [2], [3], [4] can be solved using this method. **Example 1.** The problem 6.3, from [4], requires to

$$\text{minimize} - x_1^2(T) - x_2^2(T) - x_3^2(T)$$

subject to

$$\begin{array}{rcll} \dot{x}_1 & = & -x_1 + x_2 & x_1(0) = 0 \\ \dot{x}_2 & = & -x_1 - x_2 - 10u & x_2(0) = 0 \\ \dot{x}_3 & = & x_2 & x_3(0) = -30 \end{array}$$

$$u(t) \in \Omega = \{u \in \mathbb{R} : -15 \leq u \leq 15\},$$

where $T = 2.8$.

The costate system is given by

$$\begin{array}{rclr} \dot{p}_1 & = & p_1 + p_2 & p_1(T) = 2x_1(T), \\ \dot{p}_2 & = & -p_1 + p_2 - p_3 & p_2(T) = 2x_2(T), \\ \dot{p}_3 & = & 0 & p_3(T) = 2x_3(T). \end{array}$$

and the expression for $\bar{u}$ and $W(u,t)$ are

$$\bar{u} = -15 \ \mathrm{sign}(p_2) \qquad W(u,t) = -10p_2(\bar{u} - u).$$

The value of the cost functional is -69449.99855. The plots of the control and the state functions are given in Fig.1.

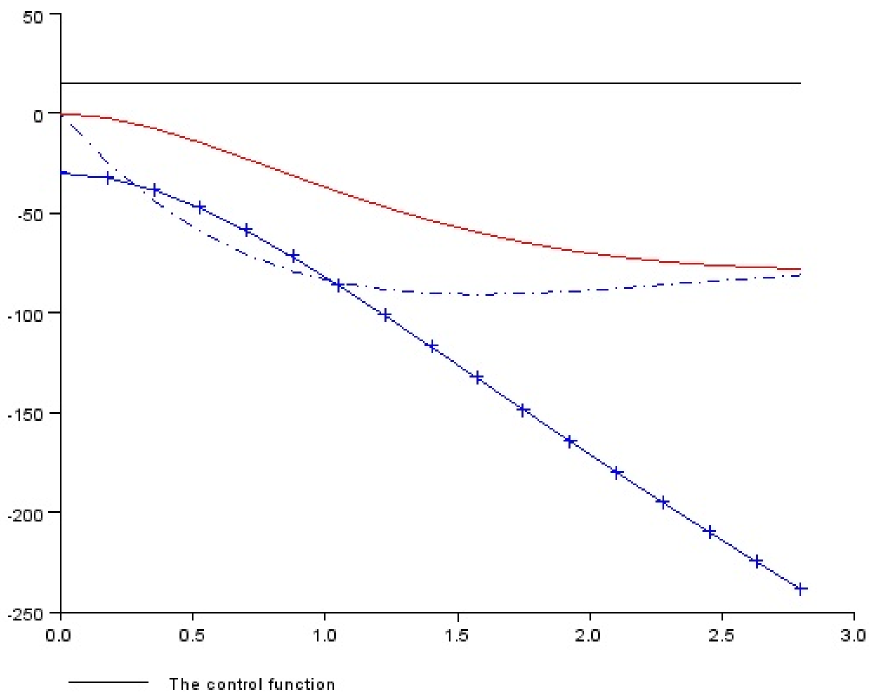

Fig. 1: Example 1- The plots of the control and state functions.

Example 2. A more interesting example is the finite-time optimal control of bilinear system [1], [7]

$$\text{minimize } I(u) = \frac{1}{2} < x(T), Fx(T) > + \frac{1}{2}\int_0^T (< x(t), Qx(t) > + < u(t), Ru(t) >)\, \mathrm{d}t \tag{11}$$

subject to

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + \left(\sum_{j=1}^{n} x_j(t) N_j\right) u(t), \qquad t \in [0, T], \\
x(0) &= x_0,
\end{aligned}
$$

with $T = 3, x_0 = (0.15, 0)^T$ and

$$
F = \begin{pmatrix} 1000 & 0 \\ 0 & 1000 \end{pmatrix}, \quad Q = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}, \quad R = 1,
$$

$$
A = \begin{pmatrix} \frac{13}{6} & \frac{5}{12} \\ -\frac{50}{3} & -\frac{8}{3} \end{pmatrix}, \quad B = \begin{pmatrix} -\frac{1}{8} \\ 0 \end{pmatrix}, \quad N_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
$$

Defining the additional state variable $x_3(t)$ by $\dot{x}_3(t) = \frac{1}{2}(< x(t), Qx(t) > + < u(t), Ru(t) >)$, $x_3(0) = 0$, the cost functional will be $I(u) = \frac{1}{2} < x(T), Fx(T) > + x_3(T)$ and the optimal control problem is reduced to the form (1)-(3).

We successfully solve the optimal control problem using a homotopy procedure for the cost functional $I(u, \lambda) = \frac{1}{2} x(T)^T F(\lambda) x(T) + x_3(T)$ with $F(\lambda) = (1000\lambda + 1 - \lambda) I_2$. For $\lambda = 0, \frac{1}{111}, \frac{100k-1}{999}, 1;\ k = 1, \ldots, 9$, the matrix F are, respectively, $I_2, 10I_2, 100kI_2, 1000I_2$. The control computed for a value of λ is used as the initial control for the next value of the homotopy parameter λ.

The value of the cost functional is 0.929753. The plots of the state functions and of the control are given in Fig. 2 and Fig 3.

6 Conclusion

A version of the successive approximation method for optimal control problem is revisited. The contribution of our work consists in an algorithm stated in pseudo code which highlights the parallel properties of this method. In the last years other methods have been suggested to solve optimal control problems of the same type [2],[3, 4], that already can be successfully solved by the successive approximation method.

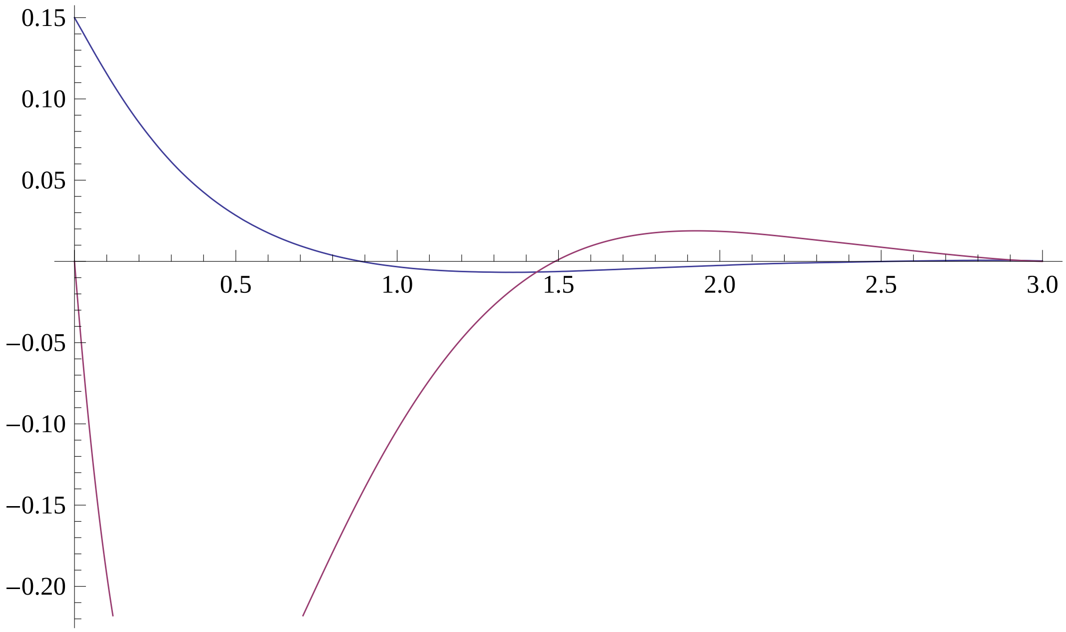

Fig. 2: Example 2- The plots of the state functions.

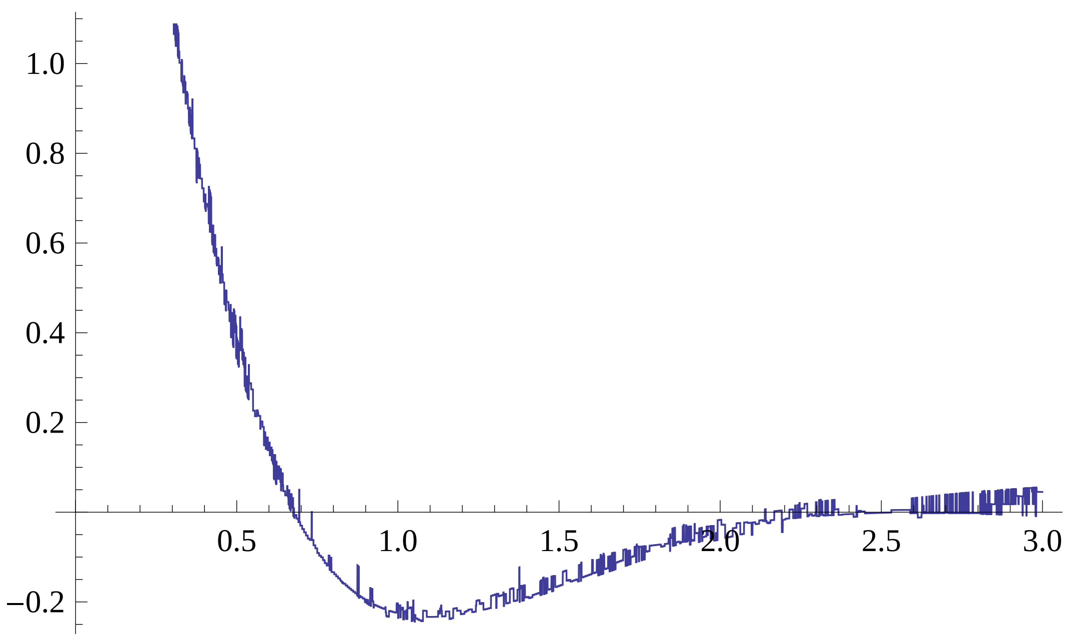

Fig. 3: Example 2- The plots of the control function.

References

[1] Aganovic, Z., Gajic, Z., *The successive approximation procedure for finite-time optimal control of bilinear systems.* IEEE Transactions on Automatic Control **39**, no. 9, (1994), 1932-1935.

[2] Buldaev, A.S., 2010, *Perturbation methos in optimal control problems.* Optimization and optimal control, Springer Serier Optimization and Its Applications **39** (2010) 325-373.

[3] Chinchuluun, A., Enkhbat, R., Pardalos, P., *A novel approach for nonconvex optimal control problems.* Optimization **58**, no. 7, (2009), 781-789.

[4] Enkhbat, R., *A Global method for some class of optimization and control problems.* Internat. J. Math. & Math. Sci. **23**, no. 9, (1998), 605-616.

[5] Graichen, K., Petit, N., *Incorporating a class of constraints into the dynamics of optimal control problems.* Optim. Control Appl. Meth. **30** (2009), 537-561.

[6] Mayne, D.Q., Polak, E., *First order strong variation algorithms for optimal control.* JOTA **16**, no. 3-4, (1975), 277-301.

[7] Ramezanpour, H., Setayeshi, S., Arabalibeik, H., Jajrami, A., 2012, *An iterative procedure for optimal control of bilinear systems.* Inter. J. of Instrumentation and Control Systems **2**, no. 1, (2012), 1-10.

[8] Крылов, И. А., Черноусько, Ф Л., *О методе последобательных приближения для решения задач оптимальього упрабления* Ж. вычисл. матем. и. матем. физ. **2**, no. 6, (1962), 1132-1139.

[9] Крылов, И. А., Черноусько, Ф Л., *Модификации и ицуледобание сходимости метода последобательных приближений для решения задач оптимальього упрабления* Ж. вычисл. матем. и. матем. физ. **12**, no. 1, (1972), 14-34.

[10] Любушин, А. А., *Модификации и исследование сходимости метода последовательных приближения для задач оптимального управления* Ж. вычисл. матем. и. матем. физ., **19**, no. 6, (1979), 1414-1421.

[11] Любушин, А. А., *О применений модификации метода последовательных приближений для решения задач оптимальього управления* Ж. вычисл. матем. и. матем. физ., **22**, no. 1, (1982), 30-35.

[12] Розоноэр, Л. И., *Принчип максимума Л. С. Понтрягина б теории оптимальных систем.* Автоматика и телемехан **20**, no. 10, (1959), 1320-1334; no. 11, (1959), 1441-1458.