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A GENERALIZATION OF THE UNIVALENCE CRITERION OF OZAKI AND NUNOKAWA

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Abstract

In this paper we obtain, by the method of subordination chains, a sufficient condition for the analyticity and the univalence of the functions defined by an integral operator. In a particular case we find the condition for univalence established by S. Ozaki and M. Nunokawa.

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1 Introduction

We denote by $U_r = \{ z \in \mathbb{C} : |z| < r\}$ the disk of z-plane, where $r \in (0,1]$, $U_1 = U$ and $I = [0,\infty)$. Let A be the class of functions f analytic in U such that $f(0) = 0,\ f'(0) = 1$.

Theorem 1. ([1]). *Let $f \in A$. If for all $z \in U$*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 , \tag{1}$$

then the function f is univalent in U.

2 Preliminaries

In order to prove our main result we need the theory of Löewner chains. A function $L : U \times I \longrightarrow \mathbb{C}$ is called a Löewner chain if it is analytic and univalent in U and $L(z,s)$ is subordinate to $L(z,t)$, for all $0 \le s \le t < \infty$. Recall that a function $f : U \longrightarrow \mathbb{C}$ is said to be subordinate to a function $g : U \longrightarrow \mathbb{C}$ (in symbols $f \prec g$) if there exists a function $w : U \longrightarrow U$ such that $f(z) = g(w(z))$ for all $z \in U$. We recall the basic result of this theory, from Pommerenke.

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Theorem 2. ([2]). *Let $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots, \quad a_1(t) \neq 0$ be analytic in U_r, for all $t \in I$, locally absolutely continuous in I and locally uniformly with respect to U_r.For almost all $t \in I$, suppose that*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}, \quad \forall z \in U_r,$$

where $p(z,t)$ is analytic in U and satisfies condition $Re\ p(z,t) > 0$, for all $z \in U$, $t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z,t)/a_1(t)\}$ forms a normal family in U_r, then for each $t \in I$, function $L(z,t)$ has an analytic and univalent extension to the whole disk U.

3 Main results

Theorem 3. *Let $f \in A$, α and β be complex numbers, $\Re\alpha > 0$, $\Re(\alpha+\beta) > 0$, $\Re\frac{\beta}{\alpha} > \frac{-1}{2}$, $2|\beta| \leq |\alpha+\beta|$. If the following inequalities*

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1 \tag{2}$$

and

$$\left|\left(\frac{z^2 f'(z)}{f^2(z)} - 1\right)|z|^{2(\alpha+\beta)} + \frac{1-|z|^{2(\alpha+\beta)}}{\alpha+\beta}\left[2\left(\frac{z^2 f'(z)}{f^2(z)} - 1\right) - \beta\right] + \right. \tag{3}$$

$$\left. \frac{(1-|z|^{2(\alpha+\beta)})^2}{(\alpha+\beta)^2|z|^{2(\alpha+\beta)}}\left[\left(\frac{z^2 f'(z)}{f^2(z)} - 1\right) + (1-\alpha)\left(\frac{f(z)}{z} - 1\right)\right]\right| \leq 1$$

are true for all $z \in U \setminus \{0\}$, then function F_α,

$$F_\alpha(z) = \left(\alpha\int_0^z u^{\alpha-1}f'(u)du\right)^{1/\alpha} \tag{4}$$

is analytic and univalent in U, where the principal branch is intended.

Proof. Let us prove that there exists a real number $r \in (0,1]$ such that function $L(z,t) : U_r \times I \longrightarrow \mathbb{C}$, defined formally by

$$L(z,t) = \left[(\alpha+\beta)\int_0^{e^{-t}z} u^{\alpha-1}f'(u)du + \frac{[e^{2(\alpha+\beta)t} - 1]e^{(2-\alpha)t}z^{\alpha-2}f^2(e^{-t}z)}{1 - \frac{e^{2(\alpha+\beta)t}-1}{\alpha+\beta}\left(\frac{f(e^{-t}z)}{e^{-t}z} - 1\right)}\right]^{1/\alpha} \tag{5}$$

is analytic in U_r, for all $t \in I$. Because $f \in A$, it is easy to see that the function

$$g_1(z,t) = (\alpha+\beta)\int_0^{e^{-t}z} u^{\alpha-1}f'(u)du\ ,$$

can be written as $g_1(z,t) = z^\alpha \cdot g_2(z,t)$, where $g_2(z,t)$ is analytic in U, for all $t \in I$ and $g_2(0,t) = \frac{\alpha+\beta}{\alpha} e^{-\alpha t}$. Let us consider function $g_3(z,t)$ given by

$$g_3(z,t) = 1 - \frac{e^{2(\alpha+\beta)t} - 1}{(\alpha+\beta)} \left(\frac{f(e^{-t}z)}{e^{-t}z} - 1 \right)$$

For all $t \in I$ and $z \in U$ we have $e^{-t}z \in U$ and because $f \in A$, function $g_3(z,t)$ is analytic in U and $g_3(0,t) = 1$. Then there is a disk $U_{r_1},\ 0 < r_1 < 1$ in which $g_3(z,t) \neq 0$, for all $t \in I$. It follows that the function

$$g_4(z,t) = g_2(z,t) + \frac{(e^{2(\alpha+\beta)t} - 1) \cdot e^{-\alpha t} \left(\frac{f(e^{-t}z)}{e^{-t}z} \right)^2}{g_3(z,t)}$$

is also analytic in U_{r_1} and

$$g_4(0,t) = e^{(\alpha+2\beta)t} \left[1 + \frac{\beta}{\alpha} e^{-2(\alpha+\beta)t} \right].$$

Let us prove that $g_4(0,t) \neq 0, \quad \forall t \in I$. We have $g_4(0,0) = 1 + \frac{\beta}{\alpha}$ and since $\Re \frac{\beta}{\alpha} > \frac{-1}{2}$ it follows that $g_4(0,0) \neq 0$. Assume now that there exists $t_0 > 0$ such that $g_4(0,t_0) = 0$. Then $e^{2(\alpha+\beta)t_0} = -\frac{\beta}{\alpha}$ and since $2|\beta| \leq |\alpha+\beta|$ implies $|\beta| \leq |\alpha|$, it follows that $e^{2(\alpha+\beta)t_0} \leq 1$. In view of $\Re(\alpha+\beta) > 0$, $t_0 > 0$, this inequality is imposible. Therefore, there is a disk $U_r,\ 0 < r \leq r_1$ in which $g_4(z,t) \neq 0$, for all $t \in I$ and we can choose an analytic branch of $[g_4(z,t)]^{1/\alpha}$, denoted by $g(z,t)$. We choose the uniform branch which is equal to $a_1(t) = e^{\frac{(\alpha+2\beta)t}{\alpha}} \left[1 + \frac{\beta}{\alpha} e^{-2(\alpha+\beta)t} \right]^{1/\alpha}$ at the origin, and for $a_1(t)$ we fix a determination.

From these considerations it follows that relation (5) may be written as

$$L(z,t) = z \cdot g(z,t) = a_1(t)z + a_2(t)z^2 + \ldots$$

and then function $L(z,t)$ is analytic in U_r. From $\Re(\alpha+\beta) > 0$, $\Re \frac{\beta}{\alpha} > \frac{-1}{2}$ we get $\lim_{t\to\infty} |a_1(t)| = \infty$. We saw also that $a_1(t) \neq 0$ for all $t \in I$.

From the analyticity of $L(z,t)$ in U_r, it follows that there is a number $r_2,\ 0 < r_2 < r$, and a constant $K = K(r_2)$ such that

$$|\ L(z,t)/a_1(t)\ | < K, \qquad \forall z \in U_{r_2}, \quad t \in I,$$

and then $\{L(z,t)/a_1(t)\}$ is a normal family in U_{r_2}. From the analyticity of $\partial L(z,t)/\partial t$, for all fixed numbers $T > 0$ and $r_3,\ 0 < r_3 < r_2$, there exists a constant $K_1 > 0$ (that depends on T and r_3) such that

$$\left| \frac{\partial L(z,t)}{\partial t} \right| < K_1, \qquad \forall z \in U_{r_3}, \quad t \in [0,T].$$

It follows that the function $L(z,t)$ is locally absolutely continuous in I, locally uniform with respect to U_{r_3}. We also have that the function

$$p(z,t) = z\frac{\partial L(z,t)}{\partial z} \Bigg/ \frac{\partial L(z,t)}{\partial t}$$

is analytic in U_{r_4}, $0 < r_4 < r_3$, for all $t \in I$.

To prove that function $p(z,t)$ has an analytic extension with positive real part in U, for all $t \in I$, it is sufficient to show that function $w(z,t)$ defined in U_{r_4} by

$$w(z,t) = \frac{p(z,t)-1}{p(z,t)+1}$$

can be continued analytically in U and that $|w(z,t)| < 1$ for all $z \in U$ and $t \in I$. After calculations, we obtain

$$w(z,t) = \left(\frac{e^{-2t}z^2 f'(e^{-t}z)}{f^2(e^{-t}z)} - 1\right) e^{-2(\alpha+\beta)t} + \frac{1-e^{-2(\alpha+\beta)t}}{\alpha+\beta}\left[2\left(\frac{e^{-2t}z^2 f'(e^{-t}z)}{f^2(e^{-t}z)}\right) - \beta\right].$$

$$+\frac{(1-e^{-2(\alpha+\beta)t})^2}{(\alpha+\beta)^2 e^{-2(\alpha+\beta)t}}\left[\left(\frac{e^{-2t}z^2 f'(e^{-t}z)}{f^2(e^{-t}z)} - 1\right) + (1-\alpha)\left(\frac{f(e^{-t}z)}{e^{-t}z} - 1\right)\right]. \quad (6)$$

From (2) and (3) we deduce that $f(z) \neq 0$ for all $z \in U$ and then function $w(z,t)$ is analytic in the unit disk. We have

$$|\, w(z,0)\,| = \left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1\,. \quad (7)$$

For $z = 0$, $t > 0$, from the hypothesis $\Re(\alpha+\beta) > 0$ and $2|\beta| \le |\alpha+\beta|$, we get

$$|\, w(0,t)\,| = \frac{|\beta|}{|\alpha+\beta|}\left|\,1 - e^{-2(\alpha+\beta)t}\,\right| < \frac{2|\beta|}{|\alpha+\beta|} \le 1. \quad (8)$$

Let now t be a fixed number, $t > 0$, $z \in U$, $z \neq 0$. In this case function $w(z,t)$ is analytic in $\overline{U}$ because $|e^{-t}z| \le e^{-t} < 1$ for all $z \in \overline{U} = \{z \in C : |z| \le 1\}$. Using the maximum modulus principle it follows that for each $t > 0$, arbitrary fixed, there exists $\theta = \theta(t) \in \mathbb{R}$ such that

$$|w(z,t)| < \max_{|\xi|=1} |w(\xi,t)| = |w(e^{i\theta},t)|, \quad (9)$$

We denote $u = e^{-t}\cdot e^{i\theta}$. Then $|u| = e^{-t} < 1$ and from (6) we get

$$w(e^{i\theta},t) = \left(\frac{u^2 f'(u)}{f^2(u)} - 1\right)|u|^{2(\alpha+\beta)} + \frac{1-|u|^{2(\alpha+\beta)}}{\alpha+\beta}\left[2\left(\frac{u^2 f'(u)}{f^2(u)} - 1\right) - \beta\right]$$

$$+\frac{(1-|u|^{2(\alpha+\beta)})^2}{(\alpha+\beta)^2|u|^{2(\alpha+\beta)}}\left[\left(\frac{u^2 f'(u)}{f^2(u)} - 1\right) + (1-\alpha)\left(\frac{f(u)}{u} - 1\right)\right].$$

Since $u \in U$, the inequality (3) implies $|w(e^{i\theta},t)| \le 1$ and from (7), (8) and (9) we conclude that $|w(z,t)| < 1$ for all $z \in U$ and $t \ge 0$.

From Theorem 2 it results that function $L(z,t)$ has an analytic and univalent extension to the whole disk U, for each $t \in I$. In particular, for $t = 0$, we conclude that function

$$L(z,0) = \left((\alpha+\beta) \int_0^z u^{\alpha-1} f'(u)du \right)^{1/\alpha}$$

is analytic and univalent in U and also function $F_\alpha(z)$ defined by (4) is analytic and univalent in U. □

Remark 1. Condition (2) of Theorem 3, which is just Ozaki-Nunokawa's univalence criterion, assures the univalence of function f, so Theorem 3 represents a generalization of this univalence criterion. For $\beta = 0$ we get a result from [3].

If in Theorem 3 we take $\alpha + \beta = 1$ we obtain the following

Corollary 1. *Let $f \in A$, $\alpha \in \mathbb{C}$, $|\alpha - 1| \le \frac{1}{2}$. If the following inequalities*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1 \tag{10}$$

and

$$\left| \left(\frac{z^2 f'(z)}{f^2(z)} - 1 \right) + (\alpha-1)(1-|z|^2) \left[|z|^2 - (1-|z|^2) \left(\frac{f(z)}{z} - 1 \right) \right] \right| \le |z|^2 \tag{11}$$

are true for all $z \in U \setminus \{0\}$, then the function $F_\alpha(z)$ defined by (4) is analytic and univalent in U.

Proof. In view of assumption $2|\beta| \le |\alpha+\beta|$ and since $\Re \frac{\beta}{\alpha} > \frac{-1}{2}$ is equivalent with $|\beta| < |\alpha+\beta|$, it follows $|\beta| \le \frac{1}{2}|\alpha+\beta| = \frac{1}{2}$ and then $|\alpha - 1| \le \frac{1}{2}$. From (3) we get immediately (11).

For $\alpha = 1$ and $\beta = 0$, the above Corollary reduces to the univalence criterion of Ozaki and Nunokawa [1].

Corollary 2. *Let $f \in A$. If for all $z \in U$, inequality (1) is true, then function f is univalent in U.*

Proof. For $\alpha = 1$ we have $F_1(z) = f(z)$ and inequality (11) becomes

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \le |z|^2 \,. \tag{12}$$

It is easy to check that if inequality (1) is true, then inequality (12) is also true. Indeed, function g,

$$g(z) = \frac{z^2 f'(z)}{f^2(z)} - 1$$

is analytic in U, $g(z) = b_2 z^2 + b_3 z^3 + \dots$, which shows that $g(0) = g'(0) = 0$. In view of (1) we have $|g(z)| < 1$ and using Schwarz's lemma we get $|g(z)| < |z|^2$.

Example 1. Let $\alpha \in \mathbb{C}$, $|\alpha - 1| \leq \frac{1}{2}$. We consider the function

$$f(z) = \frac{z}{1 - \frac{z^2}{a}}, \qquad \text{with } a > \frac{1}{1 - \sqrt{|\alpha - 1|}} \; . \tag{13}$$

Then f is univalent in U and F_α defined by (4) is analytic and univalent in U.

We have

$$\frac{z^2 f'(z)}{f^2(z)} - 1 = \frac{z^2}{a} \quad \text{and} \quad \frac{f(z)}{z} - 1 = \frac{z^2}{a - z^2} \; . \tag{14}$$

Since $a > 1$, it is clear that condition (10) of Corollary 1 is verified, and then f is univalent in U. Taking into account (14), from (11) we have that

$$\left| \frac{z^2}{a}\frac{1}{|z|^2} + (\alpha - 1)(1 - |z|^2) + \frac{(1 - |z|^2)^2}{|z|^2}(1 - \alpha)\frac{z^2}{a - z^2} \right| \leq$$

$$\frac{1}{a} + |\alpha - 1|(1 - |z|^2) + |\alpha - 1|\frac{(1 - |z|^2)^2}{a - 1}.$$

Because the greatest value of the function

$$g(x) = \frac{|\alpha - 1|}{a - 1}x^2 - |\alpha - 1|\frac{a + 1}{a - 1}x + \left(\frac{1}{a} + \frac{a}{a - 1}|\alpha - 1| \right) ,$$

for $x \in [0, 1]$ is taken for $x = 0$ and is

$$g(0) = \frac{1}{a} + \frac{a}{a - 1}|\alpha - 1|,$$

for $a > \frac{1}{1 - \sqrt{|\alpha - 1|}}$ we get $g(0) < 1$ and then all the conditions of Corollary 1 are satisfied. Therefore function F_α defined by (4) is analytic and univalent in U.

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