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THE EQUATIONS OF THE INDICATRIX OF A COMPLEX FINSLER SPACE

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Abstract

In this paper we extend the study of the indicatrix of a complex Finsler space initiated in [10, 11]. The equations that can be introduced on the indicatrix, which is studied as a hypersurface of a complex Finsler space, are investigated. In this manner, using the equations of Gauss-Weingarten, the link between the intrinsic and induced connection is deduced. The equations of Gauss, H-and A-Codazzi, and Ricci equations of the indicatrix are considered. Also, conditions for totally umbilical indicatrix are obtained.

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1 Introduction

The study of the indicatrix of a real Finsler space is one of interest, mainly because it is a compact and strictly convex set surrounding the origin. The geometry of (real) indicatrix as a hypersurface of a total space has been studied by Akbar-Zadeh in [2], where it is proved that it plays a special role in obtaining necessary and sufficient conditions for an isotropic Finsler manifold to be of constant sectional curvature. A comprehensive study of the indicatrix hypersurface could be found in [7]. In [5], a smooth compact and connected manifold with the properties of a indicatrix was called by Bryant with generalized Finsler structure.

The study of the indicatrix of a complex Finsler space was discussed in [10, 11], in which the general framework of the indicatrix bundle is established.

In the present paper, in Section 2, some preliminary properties of the n- dimensional complex Finsler space are recalled. The main relations of the intrinsic geometry of its indicatrix bundle are considered in Section 3. Since the approach of the indicatrix is as a complex hypersurface of $T'M$, it is natural to consider in Section 4 the equations of a subspace in this case and to analyse the link between the main induced and intrinsic connections considered. Some conditions for the indicatrix to be a totally umbilical submanifold of a complex Finsler space are obtained in Section 4.

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2 Complex Finsler spaces: settings

Firstly, we will make a short overview of the concepts and terminology used in complex Finsler geometry, for more see [1, 8]. Let M be a complex manifold and (z^k) the complex coordinates on a local chart.

The complexified of the real tangent bundle $T_{\mathbb{C}}M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$, i.e. $T_{\mathbb{C}}M = T'M \oplus T''M$. The bundle $T'M$ is in its turn a complex manifold and the local coordinates in a local chart are (z^k, η^k).

Definition 1. *A* complex Finsler space *is a pair* (M, F), *where* $\mathrm{F} : T'M \to \mathbb{R}^+$ *is a continuous function that satisfies the following conditions:*

i. $L := \mathrm{F}^2$ *is a smooth function on* $\widetilde{T'M} := T'M \setminus \{0\}$*;*

ii. $\mathrm{F}(z, \eta) \geq 0$, *, the equality holds iff* $\eta = 0$*;*

iii. $\mathrm{F}(z, \lambda\eta) = |\lambda|\mathrm{F}(z, \eta)$, $\forall \lambda \in \mathbb{C}$, *is the homogeneity condition of the Finsler function F;*

iv. *the Hermitian matrix* $\left(g_{i\bar{j}}(z, \eta)\right)$, *with* $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ *the fundamental metric tensor is positive definite. This means that the indicatrix* $\mathrm{I}_z = \{\eta \mid g_{i\bar{j}}(z, \eta)\eta^i \bar{\eta}^j = 1\}$ *is strongly pseudoconvex, for any* $z \in M$.

By applying Euler's formula for homogeneous functions, from iii. we get that:

$$\frac{\partial L}{\partial \eta^k}\eta^k = \frac{\partial L}{\partial \bar{\eta}^k}\bar{\eta}^k = L; \quad \frac{\partial g_{i\bar{j}}}{\partial \eta^k}\eta^k = \frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k}\bar{\eta}^k = 0 \quad \text{and} \quad L = g_{i\bar{j}}\eta^i \bar{\eta}^j. \tag{1}$$

Thus, the aim of the geometry of a complex Finsler space is to study the geometric objects of the complex manifold $T'M$ endowed with a Hermitian metric structure defined by $g_{i\bar{j}}$. Regarding this, the first step is the study of the sections of the complexified tangent bundle of $T'M$ which splits into the direct sum $T_{\mathbb{C}}(T'M) = T'(T'M) \oplus T''(T'M)$. Let $V(T'M) \subset T'(T'M)$ be the vertical bundle, locally spanned by $\left\{\frac{\partial}{\partial \eta^k}\right\}$ and let $V(T''M)$ be its conjugate.

The idea of complex nonlinear connection, briefly (c.n.c.), is fundamental in "linearization" of this geometry. A (c.n.c.) is a supplementary complex subbundle to $V(T'M)$ in $T'(T'M)$, i.e. $T'(T'M) = H(T'M) \oplus V(T'M)$. The horizontal distribution $H_u(T'M)$ is locally spanned by $\left\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\right\}$, where $N_k^j(z, \eta)$ are the coefficients of the (c.n.c.), which follow the local maps rule change, so that $\frac{\delta}{\delta z^k} = \frac{\partial z'^j}{\partial z^k}\frac{\delta}{\delta z'^j}$ is fulfilled. Obviously, we also have $\frac{\partial}{\partial \eta^k} = \frac{\partial z'^j}{\partial z^k}\frac{\partial}{\partial \eta'^j}$.

The pair $\left\{\delta_k := \frac{\delta}{\delta z^k}, \dot{\partial}_k := \frac{\partial}{\partial \eta^k}\right\}$ will be called the adapted frame of the (c.n.c.). By conjugation everywhere we get an adapted frame $\{\delta_{\bar{k}}, \dot{\partial}_{\bar{k}}\}$ on $T''_u(T'M)$. The dual adapted bases are $\left\{\mathrm{d}z^k, \delta\eta^k := \mathrm{d}\eta^k + N_j^k \mathrm{d}z^j\right\}$, respectively $\{\mathrm{d}\bar{z}^k, \delta\bar{\eta}^k\}$, where $\delta\bar{\eta}^k = \mathrm{d}\bar{\eta}^k + N_{\bar{j}}^{\bar{k}}\mathrm{d}\bar{z}^j$.

Let us consider the Sasaki type lift of the metric tensor $g_{i\bar{j}}$, as

$$G = g_{i\bar{j}} \mathrm{d}z^i \otimes \mathrm{d}\bar{z}^k + g_{i\bar{j}} \delta\eta^i \otimes \delta\bar{\eta}^j . \tag{2}$$

One main problem of this geometry is to determine a (c.n.c) related only by the fundamental function of a complex Finsler space (M, L); one almost classical now is the Chern-Finsler (c.n.c) ([1],[8]):

$$\overset{CF}{N^k_j} = g^{\bar{m}k} \frac{\partial g_{l\bar{m}}}{\partial z^j} \eta^l . \tag{3}$$

The next step is to specify the derivation law D on sections of $T_{\mathbb{C}}(T'M)$. A Hermitian connection D, of $(1,0)$-type, which satisfies $D_{JX}Y = JD_XY$, for all horizontal vectors X and the natural complex structure J on the manifold, will be the Chern-Finsler linear connection, in brief C-F, locally given by the next set of coefficients (notations from [8]):

$$L^i_{jk} = g^{\bar{l}i} \delta_k(g_{j\bar{l}}), \quad C^i_{jk} = g^{\bar{l}i} \dot{\partial}_k(g_{j\bar{l}}), \quad L^{\bar{i}}_{\bar{j}k} = 0, \quad C^{\bar{i}}_{\bar{j}k} = 0, \tag{4}$$

where $D_{\delta_k}\delta_j = L^i_{jk}\delta_i$, $D_{\delta_k}\dot{\partial}_j = L^i_{jk}\dot{\partial}_i$, $D_{\dot{\partial}_k}\dot{\partial}_j = C^i_{jk}\dot{\partial}_i$, $D_{\dot{\partial}_k}\delta_j = C^i_{jk}\delta_i$. Of course, there is also $\overline{D_XY} = D_{\bar{X}}\bar{Y}$. From the homogeneity conditions (1) it takes: $C^i_{jk}\eta^j = C^i_{jk}\eta^k = 0$. On the other hand, from $g^{\bar{m}i}g_{k\bar{m}} = \delta^i_k$ it follows that $\dfrac{\partial g^{\bar{m}i}}{\partial \eta^j} = -g^{\bar{m}p}g^{\bar{q}i}\dfrac{\partial g_{p\bar{q}}}{\partial \eta^j}$ and it is obtained that $\overset{CF}{L^i_{jk}}$ is of Berwald type, i.e.

$$\overset{CF}{L^i_{jk}} = \frac{\partial \overset{CF}{N^i_k}}{\partial \eta^j} . \tag{5}$$

Another (c.n.c.) with a special importance in the study of geodesics on a complex Finsler space, is the canonic (c.n.c.), given by:

$$\overset{c}{N^i_j} = \frac{1}{2}\dot{\partial}_j\left(\overset{CF}{N^i_k}\eta^k\right) = \frac{1}{2}\left[\dot{\partial}_j(\overset{CF}{N^i_k})\eta^k + \overset{CF}{N^i_k}\delta^k_j\right] = \frac{1}{2}\left(\overset{CF}{L^i_{jk}}\eta^k + \overset{CF}{N^i_j}\right) \tag{6}$$

It comes from a spray and using it the Berwald (c.l.c) can be introduced:

$$\overset{B}{D}\Gamma = \left(\overset{c}{N^i_k},\ \overset{B}{L^i_{jk}} = \frac{\partial \overset{c}{N^i_j}}{\partial \eta^k},\ \overset{B}{L^{\bar{i}}_{\bar{j}k}} = \frac{\partial \overset{c}{N^{\bar{i}}_{\bar{j}}}}{\partial \eta^k},\ \overset{B}{C^i_{jk}} = 0,\ \overset{B}{C^{\bar{i}}_{\bar{j}k}} = 0\right)$$

The Berwald (c.l.c) is defined on the vertical bundle $V(T'M)$ and it is easy to check that:

$$\overset{c}{N^i_k} = \frac{1}{2}\left(\overset{CF}{L^i_{k0}} + \overset{CF}{N^i_k}\right),\ \overset{B}{L^i_{jk}} = \frac{1}{2}\left(\overset{CF}{L^i_{jk}} + \overset{CF}{L^i_{kj}}\right) + \frac{1}{2}\dot{\partial}_j(\overset{CF}{L^i_{km}})\eta^m, \text{ and } \overset{B}{L^i_{jk}} = \overset{B}{L^i_{kj}} \tag{7}$$

where $\overset{CF}{L^i_{k0}} := \overset{CF}{L^i_{kj}}\eta^j$.

On $T_{\mathrm{C}}(T'M)$ the following 1-form: $\omega = \omega' + \omega" := \eta_k \mathrm{d}z^k + \bar{\eta}_k \mathrm{d}\bar{z}^k$ is well-defined, where $\eta_k := g_{k\bar{j}}\bar{\eta}^j = \frac{\partial L}{\partial \eta^k}$.

Further we will use the following notation $\bar{\eta}^j =: \eta^{\bar{j}}$ to denote a conjugate object.

3 The intrinsic geometry of complex indicatrix

Let M be a complex manifold, $\dim_{\mathbb{C}} M = n+1$, $\pi : T'M \to M$ be its holomorphic bundle and consider $(z^k, \eta^k)_{k=\overline{1,n+1}}$ the complex coordinates on the manifold $T'M$, $\dim_{\mathbb{C}} T'M = 2n+2$.

Consider $\mathrm{I}_z = \left\{ \eta \mid g_{i\bar{j}}(z,\eta)\eta^i\eta^j = 1 \right\}$ the indicatrix at z of a complex Finsler space (M, L) and $\pi_I : \mathrm{I} \to M$ the indicatrix bundle, where $\mathrm{I} = \bigcup\limits_{z \in M} \mathrm{I}_z$. $\mathrm{I} \subset T'M$ is a holomorphic subbundle and a complex and strictly connected in the 0 origin hypersurface of $T'M$, $\dim_{\mathbb{C}} \mathrm{I} = 2n+1$. Let $i : \mathrm{I} \to T'M$ be the inclusion map and $i_* : T_{\mathrm{C}}\mathrm{I} \to T_{\mathrm{C}}(T'M)$ be the extension of the tangent inclusion map to the complexified bundles.

Further on, we will study the geometry of the complex hypersurface I of the complex manifold $T'M$. If we consider $(\tilde{z}, \theta^\alpha)_{\alpha=\overline{1,n};\, k=\overline{1,n+1}}$ a parametric representation of the indicatrix hypersurface then we have the following local representation:

$$\tilde{z}^k = z^k \quad \text{and} \quad \eta^k = B^k_\alpha(z)\theta^\alpha, \text{ where } \mathrm{rang}(B^k_\alpha) = n,\ B^k_\alpha(z) = \frac{\partial \eta^k}{\partial \theta^\alpha}. \tag{8}$$

The involved immersion and holomorphy implies that $B^k_{\bar{\alpha}}(z) = \frac{\partial \eta^k}{\partial \bar{\theta}^\alpha} = 0$ and $B^{\bar{k}}_{\alpha}(z) = \frac{\partial \bar{\eta}^k}{\partial \theta^\alpha}$. Computing the Jacobi matrix in a point of the complexified tangent space $T_C I$, tangent vectors are obtained:

$$\frac{\partial}{\partial \tilde{z}^k} = \frac{\partial}{\partial z^k} + B^j_{\alpha k}\theta^\alpha \frac{\partial}{\partial \eta^j}, \quad \frac{\partial}{\partial \theta^\alpha} = B^k_\alpha \frac{\partial}{\partial \eta^k}, \quad \text{where } B^j_{\alpha k} = \frac{\partial B^j_\alpha}{\partial z^k}. \tag{9}$$

The vertical distribution $V\mathrm{I}$, spanned by $\left\{ \dot{\partial}_\alpha = \frac{\partial}{\partial \theta^\alpha} \right\}$, is a subdistribution of $V(T'M)$. We will note the tangent vectors $\frac{\partial}{\partial \bar{\tilde{z}}^k}, \frac{\partial}{\partial \bar{\theta}^\alpha}$ obtained by conjugation everywhere in the above relations. The dual bases are (by differentiation in (8)):

$$\mathrm{d}\tilde{z}^k = \mathrm{d}z^k \qquad \text{and} \qquad \mathrm{d}\eta^k = B^k_{\alpha j}\theta^\alpha \mathrm{d}\tilde{z}^j + B^k_\alpha \mathrm{d}\theta^\alpha. \tag{10}$$

The abbreviated formula $B^j_{0k} = B^j_{\alpha k}\theta^\alpha$ can be used above.

On the indicatrix I we have $L(z^k, \eta^k(\theta)) = g_{i\bar{j}}(z, \eta(\theta))\eta^i(\theta)\bar{\eta}^j(\bar{\theta}) = 1$ and by differentiation with respect to $\dot{\partial}_\alpha$ we obtain $\frac{\partial g_{i\bar{j}}}{\partial \eta^k} B^k_\alpha \eta^i \bar{\eta}^j + g_{i\bar{j}} B^i_\alpha \bar{\eta}^j = 0$. In terms of homogeneity conditions (1), it follows $g_{i\bar{j}} B^i_\alpha \bar{\eta}^j = 0$, i.e. the Liouville vector $\mathrm{N} := \eta^k \frac{\partial}{\partial \eta^k}$ is normal to the vertical distribution $V\mathrm{I}$ spanned by the tangent vectors

$\dot{\partial}_\alpha$ to the hypersurface I. Moreover, N is a unit vector, because $\eta_k\eta^k = 1$, where $\eta_k = g_{k\bar{j}}\bar{\eta}^j$.

Along $VT'M$ the frame $\mathcal{R} = \left\{\dot{\partial}_\alpha = B^k_\alpha \frac{\partial}{\partial\eta^k}, \mathrm{N} = \eta^k \frac{\partial}{\partial\eta^k}\right\}$ can be considered and be $\mathcal{R}^{-1} = \{B^\alpha_k\ \eta_k\}^t$ the inverse matrices of this base, i.e.:

$$B^\alpha_k B^k_\beta = \delta^\alpha_\beta, \quad B^\alpha_k \eta^k = 0, \quad B^k_\alpha \eta_k = 0, \quad B^k_\alpha B^\alpha_j + \eta^k\eta_j = \delta^k_j, \quad \eta_k\eta^k = 1. \tag{11}$$

The fundamental function $\tilde{L}(\tilde{z},\theta) = L(z,\eta(\theta))$ of the complex Finsler space defines a metric tensor $g_{\alpha\bar{\beta}}$ on the indicatrix I, $g_{\alpha\bar{\beta}} = B^j_\alpha B^{\bar{k}}_{\bar{\beta}} g_{i\bar{j}}$, where $B^{\bar{k}}_{\bar{\beta}} = \overline{B^k_\beta}$. It is easy to verify that $g^{\bar{\beta}\alpha} = g^{\bar{j}i} B^\alpha_i B^{\bar{\beta}}_{\bar{j}}$ is the inverse of $g_{\alpha\bar{\beta}}$ and $g^{\bar{j}i} = B^i_\alpha B^{\bar{j}}_{\bar{\beta}} g^{\bar{\beta}\alpha} + \eta^i\eta^{\bar{j}}$. Moreover, along (I,$\tilde{L}$) subspace $g_{k\bar{h}} = \tilde{g}_{k\bar{h}} + \eta_k\eta_{\bar{h}}$ takes place, where $\tilde{g}_{k\bar{h}} = B^\alpha_k B^{\bar{\beta}}_{\bar{h}} g_{\alpha\bar{\beta}}$. Also on the indicatrix I_z from $\eta^k\eta_k = 1$ it follows that $\theta_\alpha\theta^\alpha = 1$, where $\theta_\alpha = g_{\alpha\bar{\beta}}\theta^{\bar{\beta}}$.

On T'I the local frame $\left\{\tilde{\delta}_k = \frac{\partial}{\partial\tilde{z}^k} - \tilde{N}^\alpha_k \frac{\partial}{\partial\theta^\alpha}, \dot{\partial}_\alpha = \frac{\partial}{\partial\theta^\alpha}\right\}$ can be considered and its dual base is given by $\{\mathrm{d}\tilde{z}^k,\ \delta\theta^\alpha = \mathrm{d}\theta^\alpha + \tilde{N}^\alpha_j \mathrm{d}\tilde{z}^j\}$, where $\tilde{N}^\alpha_k$ will be called the coefficients of the induced (c.n.c) iff $\delta\theta^\alpha = B^\alpha_k \delta\eta^k$, i.e. $\mathrm{d}\theta^\alpha + \tilde{N}^\alpha_j \mathrm{d}\tilde{z}^j = B^\alpha_k(\mathrm{d}\eta^k + N^k_j \mathrm{d}z^j)$, and using (10), we have ([10])

$$\tilde{N}^\alpha_k = B^\alpha_k \left(B^k_{\beta j}\theta^\beta + N^k_j\right). \tag{12}$$

Let $\overset{CF}{N^k_j} = g^{m\bar{k}} \frac{\partial g_{l\bar{m}}}{\partial z^j}\eta^l$ be the Chern-Finsler (c.n.c) from (3), and

$$\overset{CF}{N^\alpha_j} = g^{\bar{\beta}\alpha} \frac{\partial g_{\gamma\bar{\beta}}}{\partial \tilde{z}^j}\theta^\gamma = g^{\bar{\beta}\alpha} \frac{\partial^2 \tilde{L}}{\partial\tilde{z}^j \partial\bar{\theta}^\beta}\theta^\gamma,$$

the intrinsic (c.n.c.) coefficients. Then it takes place (with complete demonstration in [11]):

Proposition 1. *The induced (c.n.c.)* $\overset{CF}{\tilde{N}^\alpha_k}$ *by the Chern-Finsler (c.n.c)* $\overset{CF}{N^k_j}$ *coincides with the intrinsic (c.n.c.)* $\overset{CF}{N^\alpha_j}$.

Further on, the problem of the induced canonical (c.n.c.) is being studied and using (12), from which $B^\alpha_k \overset{CF}{N^k_j} = \overset{CF}{\tilde{N}^\alpha_j} - B^\alpha_k B^k_{\beta j}\theta^\beta$, and (6) the link between this and the induced Chern-Finsler (c.n.c.) is obtained:

$$\overset{c}{\tilde{N}^\alpha_j} = \tfrac{1}{2} B^\alpha_k \left(B^k_{\beta j} + B^i_\beta \overset{CF}{L^k_{ji}} \right)\theta^\beta + \tfrac{1}{2}\overset{CF}{\tilde{N}^\alpha_j} \tag{13}$$

We note that in general $\left\{\tilde{\delta}_k := \dfrac{\delta}{\delta\tilde{z}^k} = \dfrac{\partial}{\partial\tilde{z}^k} - \tilde{N}^\alpha_k \dfrac{\partial}{\partial\theta^\alpha}\right\}$ are not d-tensor fields on $T'M$, i.e. they cannot change as vectors on the manifold. Also, by the inclusion tangent map, $i_*(\tilde{\delta}_k)$, which for convenience will be often identified with $\tilde{\delta}_k$ on $T'M$, using (9), (11) and (12), we have:

$$\tilde{\delta}_k = \delta_k + H_k^0 \mathrm{N} \quad \text{and} \quad \dot{\partial}_k = B_k^\alpha \dot{\partial}_\alpha + \eta_k \mathrm{N}; \quad \text{where } H_k^0 = (B_{\alpha k}^j \theta^\alpha + N_k^j)\eta_j. \quad (14)$$

The dual induced coframe $\tilde{\mathrm{d}}z^k = \mathrm{d}\tilde{z}^k$ and $\delta\theta^\alpha = \mathrm{d}\theta^\alpha + N_j^\alpha \mathrm{d}\tilde{z}^j$ may be also considered. The dual coframe from $T'M$ can be expressed by the elements of the induced dual coframe as:

$$\mathrm{d}z^k = \mathrm{d}\tilde{z}^k \quad \text{şi} \quad \delta\eta^k = B_\alpha^k \delta\theta^\alpha + \eta^k H_j^0 \mathrm{d}\tilde{z}^j,$$

The induced frame and co-frame on the whole $T_{\mathbb{C}}\mathrm{I}$ and the induced metric structure are obtained by conjugation everywhere:

$$\tilde{G} = g_{i\bar{j}} \tilde{\mathrm{d}}z^i \otimes \tilde{\mathrm{d}}\bar{z}^j + g_{\alpha\bar{\beta}} \delta\theta^\alpha \otimes \delta\bar{\theta}^\beta, \quad (15)$$

where $g_{i\bar{j}}(\tilde{z}, \eta(\theta))$ is the metric tensor of the space along the indicatrix points.

4 The equations of the indicatrix as hypersurface

In this section first will deduce the Gauss-Weingarten equations relative to the induced (c.n.c.) on the hypersurface space of the indicatrix, followed then by the equations of Gauss, H- and A-Codazzi, and Ricci equations.

To find the induced C-F or Berwald linear connections the Gauss-Weingarten equations of the hypersurface I will be considered , with respect to the Chern-Finsler complex linear connection, briefly C-F (c.l.c.), respectively Berwald (c.l.c.), of the space $T'M$.

Considering $\tilde{N}$ o fixed (c.n.c.) on I (let it be the one induced by (c.n.c.) N from $T'M$, by (12)), so that $T_C(\mathrm{I}) = H\mathrm{I} \oplus V\mathrm{I} \oplus \overline{H\mathrm{I}} \oplus \overline{V\mathrm{I}}$ takes place. Following the steps to define a d-(c.l.c.) on a complex space from [8], a linear connection on $T_C\mathrm{I}$ can be defined as a map

$$\tilde{D} : \Gamma(T_C\mathrm{I}) \to \Gamma(T_C\mathrm{I} \otimes T_C\mathrm{I}^*),$$

such that $\tilde{D}(fu) = u\tilde{\mathrm{d}}f + f\tilde{D}u$, $\forall f \in \mathcal{A}^0(\mathrm{I})$ and $u \in \Gamma(T_C\mathrm{I})$. Assuming $\tilde{D}$ conserves the above distributions, in the local frame $\{\tilde{\delta}_k, \dot{\partial}_\alpha, \tilde{\delta}_{\bar{k}}, \dot{\partial}_{\bar{k}}\}$ a d-(c.l.c.) is well defined by the next set of coefficients:

$$\begin{array}{llll}
\tilde{D}_{\tilde{\delta}_k} \tilde{\delta}_j = \tilde{L}^i_{jk} \tilde{\delta}_i, & \tilde{D}_{\dot{\partial}_\gamma} \tilde{\delta}_j = \tilde{C}^i_{j\gamma} \tilde{\delta}_i, & \tilde{D}_{\tilde{\delta}_{\bar{k}}} \tilde{\delta}_j = \tilde{L}^i_{j\bar{k}} \tilde{\delta}_i, & \tilde{D}_{\dot{\partial}_{\bar{\gamma}}} \tilde{\delta}_j = \tilde{C}^i_{j\bar{\gamma}} \tilde{\delta}_i, \\
\tilde{D}_{\tilde{\delta}_k} \dot{\partial}_\beta = \tilde{L}^\alpha_{\beta k} \dot{\partial}_\alpha, & \tilde{D}_{\dot{\partial}_\gamma} \dot{\partial}_\beta = \tilde{C}^\alpha_{\beta\gamma} \dot{\partial}_\alpha, & \tilde{D}_{\tilde{\delta}_{\bar{k}}} \dot{\partial}_\beta = \tilde{L}^\alpha_{\beta\bar{k}} \dot{\partial}_\alpha, & \tilde{D}_{\dot{\partial}_{\bar{\gamma}}} \dot{\partial}_\beta = \tilde{C}^\alpha_{\beta\bar{\gamma}} \dot{\partial}_\alpha, \\
\tilde{D}_{\tilde{\delta}_k} \tilde{\delta}_{\bar{j}} = \tilde{L}^{\bar{\imath}}_{\bar{j}k} \tilde{\delta}_{\bar{\imath}}, & \tilde{D}_{\dot{\partial}_\gamma} \tilde{\delta}_{\bar{j}} = \tilde{C}^{\bar{\imath}}_{\bar{j}\gamma} \tilde{\delta}_{\bar{\imath}}, & \tilde{D}_{\tilde{\delta}_{\bar{k}}} \tilde{\delta}_{\bar{j}} = \tilde{L}^{\bar{\imath}}_{\bar{j}\bar{k}} \tilde{\delta}_{\bar{\imath}}, & \tilde{D}_{\dot{\partial}_{\bar{\gamma}}} \tilde{\delta}_{\bar{j}} = \tilde{C}^{\bar{\imath}}_{\bar{j}\bar{\gamma}} \tilde{\delta}_{\bar{\imath}}, \\
\tilde{D}_{\tilde{\delta}_k} \dot{\partial}_{\bar{\beta}} = \tilde{L}^{\bar{\alpha}}_{\bar{\beta}k} \dot{\partial}_{\bar{\alpha}}, & \tilde{D}_{\dot{\partial}_\gamma} \dot{\partial}_{\bar{\beta}} = \tilde{C}^{\bar{\alpha}}_{\bar{\beta}\gamma} \dot{\partial}_{\bar{\alpha}}, & \tilde{D}_{\tilde{\delta}_{\bar{k}}} \dot{\partial}_{\bar{\beta}} = \tilde{L}^{\bar{\alpha}}_{\bar{\beta}\bar{k}} \dot{\partial}_{\bar{\alpha}}, & \tilde{D}_{\dot{\partial}_{\bar{\gamma}}} \dot{\partial}_{\bar{\beta}} = \tilde{C}^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}} \dot{\partial}_{\bar{\alpha}}.
\end{array}$$

It can be noticed that on the indicatrix space an $N-$ (*c.l.c.*) cannot be introduced, because the necessary condition is not fulfilled.

Let $\tilde{N}^\alpha_k$ be the induced (c.n.c.) on the indicatrix I. Then the tangent connection $D\tilde{\Gamma}$ induced by d-(c.l.c.) $D\Gamma = \left(N^i_j, \overset{1}{L}{}^i_{jk}, \overset{1}{C}{}^i_{jk}, \overset{3}{L}{}^i_{j\bar{k}}, \overset{3}{C}{}^i_{j\bar{k}}, \overset{2}{L}{}^i_{jk}, \overset{2}{C}{}^i_{jk}, \overset{4}{L}{}^i_{j\bar{k}}, \overset{4}{C}{}^i_{j\bar{k}} \right)$ will be a (c.l.c.) with respect to the induced connection and therefore the following decomposition occurs:

$$D_X Y = \tilde{D}_X Y + H(X,Y), \qquad \forall X, Y \in \Gamma(T_{\mathrm{C}}\mathrm{I}), \tag{16}$$

known as *Gauss's formula*, in which $\tilde{D}_X Y \in \Gamma(T_C\mathrm{I})$ is the induced tangent connection and $H(X,Y) \in \Gamma(T_C\mathrm{I}^{\perp})$ is the normal part of $D_X Y$. The map $H : \Gamma(T_C\mathrm{I}) \times \Gamma(T_C\mathrm{I}) \to \Gamma(T_C\mathrm{I}^{\perp})$ is $\mathcal{F}(I)$-bilinear and is called *the second fundamental form* of the indicatrix subspace.

On the adapted frame of (c.n.c.) on I and the normal frame formed only by N, the second fundamental form H is well-defined by the next set of coefficients:

$$\begin{array}{llll} H(\tilde{\delta}_j, \tilde{\delta}_i) = H_{ij}\mathrm{N}, & H(\tilde{\delta}_j, \tilde{\delta}_{\bar{\imath}}) = H_{\bar{\imath}j}\bar{\mathrm{N}}, & H(\dot{\partial}_\beta, \dot{\partial}_\alpha) = H_{\alpha\beta}\mathrm{N}, & H(\dot{\partial}_\beta, \dot{\partial}_{\bar{\alpha}}) = H_{\bar{\alpha}\beta}\bar{\mathrm{N}}, \\ H(\tilde{\delta}_j, \dot{\partial}_\alpha) = H_{\alpha j}\mathrm{N}, & H(\tilde{\delta}_j, \dot{\partial}_{\bar{\alpha}}) = H_{\bar{\alpha}j}\bar{\mathrm{N}}, & H(\dot{\partial}_\beta, \tilde{\delta}_i) = H_{i\beta}\mathrm{N}, & H(\dot{\partial}_\beta, \tilde{\delta}_{\bar{\imath}}) = H_{\bar{\imath}\beta}\bar{\mathrm{N}}. \end{array}$$

These coefficients are Hermitian ($\overline{H_{\alpha\beta}} = H_{\bar{\alpha}\bar{\beta}}$) and by a direct computation, taking into account that for the normal component occurs $G(D_X Y, \bar{\mathrm{N}}) = G(H(X,Y), \bar{\mathrm{N}})$, it takes:

$$\begin{aligned}
H_{ij} &= \delta_j(H_i^0) + H_j^0 \mathrm{N}(H_i^0) + H_i^0 H_j^0 - H_i^0 \eta_l (N_j^l - \eta^k \overset{2}{L}{}^l_{kj}) + H_i^0 H_j^0 \eta_k \eta^n \eta^l \overset{2}{C}{}^k_{nl}, \\
H_{\bar{\imath}j} &= \delta_j(H_{\bar{\imath}}^0) + H_{\bar{\imath}}^0 \overset{4}{L}{}^{\bar{k}}_{\bar{l}j} \eta^{\bar{l}} \eta_{\bar{k}} + H_j^0 \mathrm{N}(H_{\bar{\imath}}^0) + H_{\bar{\imath}}^0 H_j^0 \eta_{\bar{k}} \eta^l \eta^{\bar{n}} \overset{4}{C}{}^{\bar{k}}_{\bar{n}l}, \\
H_{\alpha\beta} &= B_\alpha^j B_\beta^k \overset{2}{C}{}^i_{jk} \eta_i, \qquad\qquad H_{\bar{\alpha}\beta} = B_{\bar{\alpha}}^{\bar{j}} B_\beta^k \overset{4}{C}{}^{\bar{\imath}}_{\bar{j}k} \eta_{\bar{\imath}}, \\
H_{\alpha j} &= \left(B_{\alpha j}^i + B_\alpha^k \overset{2}{L}{}^i_{kj} \right) \eta_i + H_j^0 B_\alpha^i \eta^l \eta_k \overset{2}{C}{}^k_{il}, \\
H_{\bar{\alpha}j} &= B_{\bar{\alpha}}^{\bar{\imath}} \overset{4}{L}{}^{\bar{k}}_{\bar{\imath}j} \eta_{\bar{k}} + H_j^0 B_{\bar{\alpha}}^{\bar{\imath}} \eta^l \eta_{\bar{k}} \overset{4}{C}{}^{\bar{k}}_{\bar{\imath}l}, \\
H_{i\beta} &= B_\beta^j \dot{\partial}_j(H_i^0) + B_\beta^j H_i^0 \eta^l \eta_k \overset{2}{C}{}^k_{lj}, \qquad H_{\bar{\imath}\beta} = B_\beta^j \dot{\partial}_j(H_{\bar{\imath}}^0) + B_\beta^j H_{\bar{\imath}}^0 \eta^{\bar{l}} \eta_{\bar{k}} \overset{4}{C}{}^{\bar{k}}_{\bar{l}j}.
\end{aligned} \tag{17}$$

Next, using the Gauss's formula (16), the coefficients of the induced d-(c.l.c.) are obtained:

$$\begin{aligned}
\tilde{L}^i_{jk} &= \overset{1}{L}{}^i_{jk} + H_k^0 \eta^l \overset{1}{C}{}^i_{jl}; \qquad \tilde{L}^{\bar{\imath}}_{\bar{j}k} = \overset{3}{L}{}^{\bar{\imath}}_{\bar{j}k} + H_k^0 \eta^l \overset{3}{C}{}^{\bar{\imath}}_{\bar{j}l}; \\
\tilde{C}^i_{j\gamma} &= B_\gamma^k \overset{1}{C}{}^i_{jk}; \qquad \tilde{C}^{\bar{\imath}}_{\bar{j}\gamma} = B_\gamma^k \overset{3}{C}{}^{\bar{\imath}}_{\bar{j}k}; \\
\tilde{L}^\alpha_{\beta k} &= \left(B_{\beta k}^i + B_\beta^l \overset{2}{L}{}^i_{lk} \right) B_i^\alpha + H_k^0 B_\beta^i B_p^\alpha \eta^l \overset{2}{C}{}^p_{il}; \\
\tilde{L}^{\bar{\alpha}}_{\bar{\beta}k} &= B_{\bar{p}}^{\bar{\alpha}} B_{\bar{\beta}}^{\bar{\imath}} \overset{4}{L}{}^{\bar{p}}_{\bar{\imath}k} + H_k^0 B_{\bar{p}}^{\bar{\alpha}} B_{\bar{\beta}}^{\bar{\imath}} \eta^l \overset{4}{C}{}^{\bar{p}}_{\bar{\imath}l}; \\
\tilde{C}^\alpha_{\beta\gamma} &= B_i^\alpha B_\beta^j B_\gamma^k \overset{2}{C}{}^i_{jk}; \qquad \tilde{C}^{\bar{\alpha}}_{\bar{\beta}\gamma} = B_{\bar{\imath}}^{\bar{\alpha}} B_{\bar{\beta}}^{\bar{j}} B_\gamma^k \overset{4}{C}{}^{\bar{\imath}}_{\bar{j}k}.
\end{aligned} \tag{18}$$

Similarly, following the settings of the general geometry of subspaces, a linear connection $D\Gamma(T'M)$ induces a normal connection $D^{\perp}\Gamma(\mathrm{I})$. For $X \in \Gamma(T_C\mathrm{I})$ and $W \in \Gamma(T_C\mathrm{I}^{\perp})$, we have

$$D_X W = -A_W X + D^{\perp}_X W, \tag{19}$$

where $A_W X \in \Gamma(T_C\mathrm{I})$ and $D^{\perp}_X W \in \Gamma(T_C\mathrm{I}^{\perp})$. This formula is called *Weingarten's formula.*

The map $A : \Gamma(T_C\mathrm{I}^{\perp}) \times \Gamma(T_C\mathrm{I}) \to \Gamma(T_C\mathrm{I})$ is $\mathcal{F}(\mathrm{I})$-bilinear, $A_W X = A(W, X)$, and A_W is called the *shape operator* (or Weingarten operator). Also $T_C\mathrm{I}^{\perp}$ is spanned by N, $\bar{\mathrm{N}}$, namely it has only the vertical component and thus it can be concluded that $D^{\perp}_X W \in \Gamma(V_C\mathrm{I}^{\perp})$ and $A : \Gamma(V_C\mathrm{I}^{\perp}) \times \Gamma(T_C\mathrm{I}) \to \Gamma(V_C\mathrm{I})$. Thus, as before, the action of the shape operator may be expressed $A_{\mathrm{N}}(X) := A(X) \in V\mathrm{I}$ on $\tilde{\delta}_k$ and $\dot{\partial}_\alpha$ as:

$$\begin{aligned} A_{\mathrm{N}}(\tilde{\delta}_k\,) &= A^{\alpha}_{k}\dot{\partial}_\alpha; \qquad A_{\mathrm{N}}(\dot{\partial}_\beta) = A^{\alpha}_{\beta}\dot{\partial}_\alpha;\\ A_{\mathrm{N}}(\tilde{\delta}_{\bar{k}}\,) &= A^{\alpha}_{\bar{k}}\dot{\partial}_\alpha; \qquad A_{\mathrm{N}}(\dot{\partial}_{\bar{\beta}}) = A^{\alpha}_{\bar{\beta}}\dot{\partial}_\alpha, \end{aligned}$$

these coefficients being Hermitian, i.e. $\overline{A^{\alpha}_{k}} = A^{\bar{\alpha}}_{\bar{k}}$. Thus, considering $G(D_X\mathrm{N},\dot{\partial}_{\bar{\beta}}) = -G(A(X),\dot{\partial}_{\bar{\beta}})$, $\dot{\partial}_{\bar{\beta}} = B^{\bar{k}}_{\bar{\beta}}\dot{\partial}_{\bar{k}}$ and bilinear G, we obtain the following relation $G(D_X\mathrm{N},\dot{\partial}_{\bar{k}}) = -G(A(X),\dot{\partial}_{\bar{k}})$. Thereby

$$\begin{aligned} A^{\alpha}_{k} &= B^{\alpha}_{i}\left(N^{i}_{k} - \eta^{j}\overset{2}{L}{}^{i}_{jk} - H^{0}_{k}\eta^{l}\eta^{j}\overset{2}{C}{}^{i}_{jl}\right); \qquad A^{\alpha}_{\beta} = -B^{\alpha}_{i}\left(B^{i}_{\beta} + B^{k}_{\beta}\eta^{j}\overset{2}{C}{}^{i}_{jk}\right);\\ A^{\alpha}_{\bar{k}} &= -B^{\alpha}_{i}\left(\eta^{j}\overset{4}{L}{}^{i}_{j\bar{k}} + H^{0}_{\bar{k}}\eta^{\bar{l}}\eta^{j}\overset{4}{C}{}^{i}_{j\bar{l}}\right); \qquad A^{\alpha}_{\bar{\beta}} = -B^{\alpha}_{i}B^{\bar{k}}_{\bar{\beta}}\eta^{j}\overset{4}{C}{}^{i}_{j\bar{k}}. \end{aligned} \tag{20}$$

Next, using these, a relation between the induced and intrinsic particular connections introduced on the indicatrix bundle will be obtained.

On $T'M$ a Hermitian N-(c.l.c.) D of $(1,0)-$type can be introduced, known as Chern-Finsler (c.l.c), locally given by the following set of coefficients:

$$\overset{CF}{D}\Gamma = \left(\overset{CF}{N}{}^{i}_{j} = g^{\bar{m}i}\frac{\partial g_{l\bar{m}}}{\partial z^{j}}\eta^{l},\ \overset{CF}{L}{}^{i}_{jk} = g^{\bar{m}i}\frac{\delta g_{j\bar{m}}}{\delta z^{k}},\ \overset{CF}{C}{}^{i}_{jk} = g^{\bar{m}i}\frac{\partial g_{j\bar{m}}}{\partial \eta^{k}},\ \overset{CF}{L}{}^{\bar{i}}_{\bar{j}k} = 0,\ \overset{CF}{C}{}^{\bar{i}}_{\bar{j}k} = 0\right).$$

Considering (12), Proposition 1, the tangent connection $\overset{CF}{D}\tilde{\Gamma}$ induced by $\overset{CF}{D}\Gamma$ will be a complex linear connection, so the Gauss formula (16) can be applied and from (18) and homogeneity conditions $\overset{CF}{C}{}^{i}_{jk}\eta^{j} = \overset{CF}{C}{}^{i}_{jk}\eta^{k} = 0$, the induced d-(c.l.c.) coefficients may be calculated:

$$\overset{CF}{D}\tilde{\Gamma} = \left(\begin{array}{l} \overset{CF}{\tilde{N}^{\alpha}_{j}} = g^{\bar{\beta}\alpha}\frac{\partial g_{\gamma\bar{\beta}}}{\partial \bar{z}^{j}}\theta^{\gamma};\ \overset{CF}{\tilde{L}^{i}_{jk}} = \overset{CF}{L^{i}_{jk}};\ \overset{CF}{\tilde{C}^{i}_{j\gamma}} = B^{k}_{\gamma}\overset{CF}{C^{i}_{jk}};\ \overset{CF}{\tilde{L}^{\bar{\imath}}_{\bar{j}k}} = 0;\ \overset{CF}{\tilde{C}^{\bar{\imath}}_{\bar{j}\gamma}} = 0; \\ \overset{CF}{\tilde{L}^{\alpha}_{\beta k}} = B^{\alpha}_{i}\left(B^{i}_{\beta k} + B^{j}_{\beta}\overset{CF}{L^{i}_{jk}}\right);\ \overset{CF}{\tilde{C}^{\alpha}_{\beta\gamma}} = B^{\alpha}_{i}B^{j}_{\beta}B^{k}_{\gamma}\overset{CF}{C^{i}_{jk}};\ \overset{CF}{\tilde{L}^{\bar{\alpha}}_{\bar{\beta}k}} = 0;\ \overset{CF}{\tilde{C}^{\bar{\alpha}}_{\bar{\beta}\gamma}} = 0 \end{array} \right). \tag{21}$$

On a complex Finsler space relation (5) takes place, that is $\overset{CF}{L^{i}_{jk}} = \dot{\partial}_{j}\left(\overset{CF}{N^{i}_{k}}\right)$. For the induced connection, using (12), this is preserved:

$$\dot{\partial}_{\beta}\overset{CF}{\tilde{N}^{\alpha}_{k}} = \dot{\partial}_{\beta}\left\{B^{\alpha}_{j}(z)\left(B^{j}_{\gamma k}\theta^{\gamma} + \overset{CF}{N^{j}_{k}}\right)\right\} = B^{\alpha}_{j}\left(B^{j}_{\beta k} + B^{i}_{\beta}\overset{CF}{L^{j}_{ik}}\right) = \overset{CF}{\tilde{L}^{\alpha}_{\beta k}}$$

Using (17), the homogeneity condition $\overset{CF}{C^{i}_{jk}}\eta^{j} = \overset{CF}{C^{i}_{jk}}\eta^{k} = 0$ and $\eta^{j}\overset{CF}{L^{i}_{jk}} = N^{i}_{k}$, obtained from $\overset{CF}{L^{i}_{jk}} = \frac{\partial \overset{CF}{N^{i}_{k}}}{\partial \eta^{j}}$ and 1-homogeneity of $\overset{CF}{N^{i}_{k}}$, the coefficients of the second fundamental form H for the induced C-F d-(c.l.c.):

$$\begin{array}{ll} H_{ij} = \delta_{j}(H^{0}_{i}) + H^{0}_{j}\mathrm{N}(H^{0}_{i}) + H^{0}_{i}H^{0}_{j}, & H_{\bar{\imath}j} = \delta_{j}(H^{0}_{\bar{\imath}}) + H^{0}_{j}\mathrm{N}(H^{0}_{\bar{\imath}}), \\ H_{\alpha\beta} = B^{j}_{\alpha}B^{k}_{\beta}\overset{CF}{C^{i}_{jk}}\eta_{i}, & H_{\bar{\alpha}\beta} = 0, \\ H_{\alpha j} = B^{i}_{\alpha j}\eta_{i} + B^{k}_{\alpha}\overset{CF}{L^{i}_{kj}}\eta_{i}, & H_{\bar{\alpha}j} = 0, \\ H_{i\beta} = B^{j}_{\beta}\dot{\partial}_{j}(H^{0}_{i}), & H_{\bar{\imath}\beta} = B^{j}_{\beta}\dot{\partial}_{j}(H^{0}_{\bar{\imath}}). \end{array}$$

Similar, using these conditions and Weingarten formula (19), the coefficients of the shape operator can be expressed:

$$\begin{array}{lll} A^{\alpha}_{k} = 0; & & A^{\alpha}_{\beta} = -\delta^{\alpha}_{\beta}; \\ A^{\alpha}_{\bar{k}} = 0; & & A^{\alpha}_{\bar{\beta}} = 0. \end{array} \tag{22}$$

Using the good vertical connection technique, an intrinsic (c.n.c.) N^{α}_{k} can be determined on I and some of the d-(c.l.c.) coefficients, defined on the vertical bundle $D : T_{C}\mathrm{I} \times V_{C}\mathrm{I} \to V_{C}\mathrm{I}$, $D\Gamma(N) = (L^{\alpha}_{\beta k}, L^{\bar{\alpha}}_{\bar{\beta}k}, C^{\alpha}_{\beta\gamma}, C^{\bar{\alpha}}_{\bar{\beta}\gamma})$. For example, in the C-F (c.n.c.) $\overset{CF}{N^{\alpha}_{j}} = g^{\bar{\beta}\alpha}\frac{\partial g_{\gamma\bar{\beta}}}{\partial \bar{z}^{j}}\theta^{\gamma} = g^{\bar{\beta}\alpha}\frac{\partial^{2}L}{\partial \bar{z}^{j}\partial\theta^{\beta}}$, we can introduce on the vertical fibers

$$\overset{CF}{D}\Gamma\left(\overset{CF}{N^{\alpha}_{j}}\right) = \left(\overset{CF}{L^{\alpha}_{\beta k}} = g^{\bar{\sigma}\alpha}\frac{\delta g_{\beta\bar{\sigma}}}{\delta \bar{z}^{k}}, \overset{CF}{C^{\alpha}_{\beta\gamma}} = g^{\bar{\sigma}\alpha}\frac{\partial g_{\beta\bar{\sigma}}}{\partial\theta^{\gamma}}, \overset{CF}{L^{\bar{\alpha}}_{\bar{\beta}k}} = \overset{CF}{C^{\bar{\alpha}}_{\bar{\beta}\gamma}} = 0\right)$$

Considering that the C-F intrinsic and induced (c.n.c.) coincide according to Proposition 1, using (14), the homogeneity condition $\frac{\partial g_{j\bar{m}}}{\partial \eta^{l}}\eta^{l} = 0$ and $g^{\bar{n}i}\eta_{\bar{n}}B^{\alpha}_{i} =$

$g^{\bar{n}i}g_{j\bar{n}}\eta^j B_i^\alpha = \delta^i_j \eta^j B_i^\alpha = \eta^i B_i^\alpha = 0$, we obtained that the corresponding coefficients coincide too: $\overset{CF}{L}{}^\alpha_{\beta k} = \overset{CF}{\tilde{L}}{}^\alpha_{\beta k}$ and $\overset{CF}{C}{}^\alpha_{\beta\gamma} = \overset{CF}{\tilde{C}}{}^\alpha_{\beta\gamma}$. Similarly, the coefficients on the horizontal fibers can be defined:

$$\overset{CF}{L}{}^i_{jk}\Big|_{\mathrm{I}} = g^{\bar{m}i}\frac{\delta g_{j\bar{m}}}{\delta \tilde{z}^k},\ \overset{CF}{C}{}^i_{j\gamma} = g^{\bar{m}i}\frac{\partial g_{j\bar{m}}}{\partial \theta^\gamma},\ \overset{CF}{L}{}^{\bar{i}}_{\bar{j}k}\Big|_{\mathrm{I}} = \overset{CF}{C}{}^{\bar{i}}_{\bar{j}\gamma} = 0$$

and they coincide with the corresponding ones: $\overset{CF}{L}{}^i_{jk}\Big|_{\mathrm{I}} = \overset{CF}{\tilde{L}}{}^i_{jk}$ and $\overset{CF}{C}{}^i_{j\gamma} = \overset{CF}{\tilde{C}}{}^i_{j\gamma}$, and so we have proved that

Proposition 2. *On the indicatrix bundle, the induced and the intrinsic C-F d-(c.l.c.) coincide, i.e. the (21) relation concur with*

$$\overset{CF}{D}\Gamma\Big|_{\mathrm{I}} = \left(\overset{CF}{N}{}^\alpha_j = g^{\bar{\beta}\alpha}\frac{\partial g_{\gamma\bar{\beta}}}{\partial \tilde{z}^j}\theta^\gamma;\ \overset{CF}{L}{}^i_{jk}\Big|_{\mathrm{I}} = g^{\bar{m}i}\frac{\delta g_{j\bar{m}}}{\delta \tilde{z}^k},\ \overset{CF}{C}{}^i_{j\gamma} = g^{\bar{m}i}\frac{\partial g_{j\bar{m}}}{\partial\theta^\gamma},\ \overset{CF}{L}{}^{\bar{i}}_{\bar{j}k}\Big|_{\mathrm{I}} = \overset{CF}{C}{}^{\bar{i}}_{\bar{j}\gamma} = 0,\right.$$
$$\left.\overset{CF}{L}{}^\alpha_{\beta k} = g^{\bar{\sigma}\alpha}\frac{\delta g_{\beta\bar{\sigma}}}{\delta\tilde{z}^k},\ \overset{CF}{C}{}^\alpha_{\beta\gamma} = g^{\bar{\sigma}\alpha}\frac{\partial g_{\beta\bar{\sigma}}}{\partial\theta^\gamma},\ \overset{CF}{L}{}^{\bar{\alpha}}_{\bar{\beta}k} = \overset{CF}{C}{}^{\bar{\alpha}}_{\bar{\beta}\gamma} = 0\right).$$

From the general theory of sprays on a manifold M (see [8]) from the coefficients of a spray $\overset{c}{G} := \frac{1}{2}\overset{CF}{N}{}^i_k\eta^k = \frac{1}{2}\overset{CF}{N}{}^i_0$ a (c.n.c.) can determined $\overset{c}{N}{}^i_k = \frac{\partial \overset{c}{G}{}^i}{\partial\eta^k}$, called the canonical (c.n.c.). Correspondingly, a complex linear connection can be associated known as Berwald type complex connection, locally given by the set of coefficients:

$$B\Gamma = \left(\overset{c}{N}{}^i_j = \frac{1}{2}\dot{\partial}_j\left(\overset{CF}{N}{}^i_k\eta^k\right),\ \overset{B}{L}{}^i_{jk} = \dot{\partial}_k\overset{c}{N}{}^i_j = \overset{B}{L}{}^i_{kj},\ \overset{B}{L}{}^i_{j\bar{k}} = \dot{\partial}_{\bar{k}}\overset{c}{N}{}^i_j,\ \overset{B}{C}{}^i_{jk} = 0,\ \overset{B}{C}{}^i_{j\bar{k}} = 0\right).$$

with (6), (7) properties and, moreover, $\overset{B}{L}{}^i_{j\bar{k}}\eta^{\bar{k}} = 0$ takes place (see [3], Lemma 2.2.a.).

Considering (12), $\overset{c}{\tilde{N}}{}^\alpha_j = B^\alpha_k\left(B^k_{\beta j}\theta^\beta + \overset{c}{N}{}^k_j\right)$, the tangent connection $B\tilde{\Gamma}$ induced by $B\Gamma$ is a (c.l.c), therefore the Gauss formula (16) can be applied and from (18) the induced d-(c.l.c.) coefficients can be estimated:

$$B\tilde{\Gamma} = \left(\overset{c}{\tilde{N}}{}^\alpha_j = B^\alpha_k\left(B^k_{\beta j}\theta^\beta + \overset{c}{N}{}^k_j\right);\ \overset{B}{\tilde{L}}{}^i_{jk} = \overset{B}{L}{}^i_{jk};\ \overset{B}{\tilde{L}}{}^i_{j\bar{k}} = \overset{B}{L}{}^i_{j\bar{k}};\ \overset{B}{\tilde{C}}{}^i_{j\gamma} = 0;\ \overset{B}{\tilde{C}}{}^i_{j\bar{\gamma}} = 0;\right.$$
$$\left.\overset{B}{\tilde{L}}{}^\alpha_{\beta k} = B^\alpha_i\left(B^i_{\beta k} + B^j_\beta\overset{B}{L}{}^i_{jk}\right);\ \overset{B}{\tilde{L}}{}^\alpha_{\beta\bar{k}} = B^\alpha_i B^j_\beta \overset{B}{L}{}^i_{j\bar{k}};\ \overset{B}{\tilde{C}}{}^\alpha_{\beta\gamma} = 0;\ \overset{B}{\tilde{C}}{}^\alpha_{\beta\bar{\gamma}} = 0\right). \tag{23}$$

From (17) and (20), using $\eta^j \overset{B}{L}{}^i_{jk} = \overset{B}{N}{}^i_k$, the coefficients of the second fundamental form H and the coefficients of the shape operator can be obtained for the induced Berwald d-(c.l.c.):

$$H_{ij} = \delta_j(H^0_i) + H^0_j \mathrm{N}(H^0_i) + H^0_i H^0_j, \qquad H_{\bar{\imath}j} = \delta_j(H^0_{\bar{\imath}}) + H^0_j \mathrm{N}(H^0_{\bar{\imath}}) + H^0_{\bar{\imath}} \overset{B}{L}{}^{\bar{k}}_{\bar{l}j}\eta^{\bar{l}}\eta_{\bar{k}},$$
$$H_{\alpha\beta} = 0, \qquad H_{\bar{\alpha}\beta} = 0,$$
$$H_{\alpha j} = B^i_{\alpha j}\eta_i + B^k_\alpha \overset{B}{L}{}^i_{kj}\eta_i, \qquad H_{\bar{\alpha}j} = B^{\bar{\imath}}_{\bar{\alpha}} \overset{B}{L}{}^{\bar{k}}_{\bar{\imath}j}\eta_{\bar{k}},$$
$$H_{i\beta} = B^j_\beta \dot{\partial}_j(H^0_i), \qquad H_{\bar{\imath}\beta} = B^j_\beta \dot{\partial}_j(H^0_{\bar{\imath}}).$$

$$A^\alpha_k = 0; \qquad A^\alpha_\beta = -\delta^\alpha_\beta;$$
$$A^\alpha_{\bar{k}} = -B^\alpha_i \eta^j \overset{B}{L}{}^i_{j\bar{k}}; \qquad A^\alpha_{\bar{\beta}} = 0.$$

Similarly as in the intrinsic C-F d-(c.l.c.) case, first we introduce the coefficients of the intrinsic Berwald d-(c.l.c.) on the vertical fibers

$$B\Gamma\left(\overset{c}{N}{}^\alpha_j\right) = \left(\overset{B}{L}{}^\alpha_{\beta k} = \frac{\partial \overset{c}{N}{}^\alpha_k}{\partial \theta^\beta},\ \overset{B}{L}{}^\alpha_{\beta\bar{k}} = \frac{\partial\left(B^j_\beta \overset{c}{N}{}^\alpha_j\right)}{\partial \eta^{\bar{k}}},\ \overset{B}{C}{}^\alpha_{\beta\gamma} = 0,\ \overset{B}{C}{}^\alpha_{\beta\bar{\gamma}} = 0\right) \tag{24}$$

where $\overset{c}{N}{}^\alpha_j = \frac{1}{2}\frac{\partial\left(\overset{CF}{N}{}^\alpha_i \eta^i\right)}{\partial \eta^j}$ is the intrinsic canonical (c.n.c.) on the indicatrix bundle. Then it can be proved:

Proposition 3. *The canonical (c.n.c.)* $\overset{c}{\tilde{N}}{}^\alpha_j$ *induced on* I *from the canonical (c.n.c.)* $\overset{c}{N}{}^i_j$ *of the base manifold* $T'M$ *coincides with the intrinsic canonical (c.n.c.)* $\overset{c}{N}{}^\alpha_j$ *of the indicatrix bundle.*

So, it can be easily verified that the coefficients of the vertical fields of the intrinsic and induced Berwald d-(c.l.c.) coincide. Using (24) and (13) it can be checked that the induced Berwald connection is of Berwald type, i.e.

Proposition 4. *The induced Berwald connection coincides with the intrinsic Berwald connection of the indicatrix bundle, namely:*

$$\overset{B}{\tilde{L}}{}^\alpha_{\beta j} = \frac{\partial \overset{c}{\tilde{N}}{}^\alpha_j}{\partial \theta^\beta}.$$

On the horizontal fibers can be introduced:

$$\left.\overset{B}{L}{}^i_{jk}\right|_{\mathrm{I}} = \frac{\partial \overset{c}{N}{}^i_k}{\partial \eta^j} = \dot{\partial}_k \overset{c}{N}{}^i_j,\quad \left.\overset{CF}{L}{}^i_{j\bar{k}}\right|_{\mathrm{I}} = \dot{\partial}_{\bar{k}} \overset{c}{N}{}^i_j,\quad \overset{CF}{C}{}^i_{j\gamma} = 0,\ \overset{CF}{C}{}^{\bar{\imath}}_{\bar{j}\gamma} = 0$$

and we can state:

Proposition 5. *On the indicatrix bundle, the induced and the intrinsic Berwald type d-(c.l.c.) coincide, i.e. the (23) relation concur with*

$$B\Gamma|_{\mathrm{I}} = \left(\overset{c}{N}{}^{\alpha}_{j} = \frac{1}{2}\frac{\partial\left(\overset{CF}{N}{}^{\alpha}_{i}\eta^{i}\right)}{\partial\eta^{j}};\ \ \overset{B}{L}{}^{i}_{jk}\Big|_{\mathrm{I}} = \dot{\partial}_{j}\overset{c}{N}{}^{i}_{k},\ \ \overset{B}{L}{}^{i}_{j\bar{k}}\Big|_{\mathrm{I}} = \dot{\partial}_{\bar{k}}\overset{c}{N}{}^{i}_{j},\ \ \overset{B}{C}{}^{i}_{j\gamma} = 0,\ \overset{B}{C}{}^{i}_{j\bar{\gamma}} = 0, \right.$$
$$\left. \overset{B}{L}{}^{\alpha}_{\beta k} = \dot{\partial}_{\beta}\overset{c}{N}{}^{\alpha}_{k},\ \ \overset{B}{L}{}^{\alpha}_{\beta\bar{k}} = \dot{\partial}_{\bar{k}}\left(B^{j}_{\beta}\overset{c}{N}{}^{\alpha}_{j}\right),\ \ \overset{B}{C}{}^{\alpha}_{\beta\gamma} = 0,\ \overset{B}{C}{}^{\bar{\alpha}}_{\bar{\beta}\gamma} = 0 \right).$$

In order to introduce Gauss, Codazzi and Ricci equations on the indicatrix hypersurface let us consider D a N-(c.l.c.) on $T'M$ and $\tilde{D}$, $D^{\perp}$ $\tilde{N}$- the induced tangent and normal connection on I, as above. Let $\tilde{v}$ and $\tilde{h}$ denote the projectors on VI and HI distributions, respectively, and through $\overline{\tilde{v}}$ and $\overline{\tilde{h}}$ the projectors on conjugate distributions will be denoted. Without the tilde the same projectors on $T_{C}T'M$ will be noted.

To get a link between curvatures $R(X,Y)Z$ of D connection and $\tilde{R}(X,Y)Z$ of $\tilde{D}$ connection, for $X,Y,Z \in \Gamma(T_{C}\mathrm{I})$ we act similar steps as in [4] for real Finsler manifolds and in [9] for complex Finsler space. First, the covariant derivative of the second fundamental form is defined:

$$\left(D_{X}H\right)(Y,Z) = D^{\perp}_{X}\left(H(Y,Z)\right) - H\left(\tilde{D}_{X}Y,Z\right) - H\left(Y,\tilde{D}_{X}Z\right).$$

Now, using the curvature definition $R\left(X,Y\right)Z = D_{X}D_{Y}Z - D_{Y}D_{X}Z - D_{[X,Y]}$, the torsion definition $T(X,Y) = D_{X}Y - D_{Y}X - [X,Y]$, for $X,Y,Z \in \Gamma(T_{C}\mathrm{I})$ and applying the Gauss-Weingarten formulae (16) and (19), we get:

$$\begin{aligned} R(X,Y)Z \ &= \tilde{R}(X,Y)Z + A\left(H(X,Z),Y\right) - A\left(H(Y,Z),X\right) + \left(D_{X}H\right)(Y,Z) - \\ &- \left(D_{Y}H\right)(X,Z) + H\left(\tilde{T}(X,Y),Z\right) \end{aligned}$$

Equating the components from $T_{C}\mathrm{I}$ and $T^{\perp}_{C}\mathrm{I}$ with the help of the metric structures $\mathbf{G}$ and $\tilde{\mathbf{G}}$ introduced in previous sections, we obtain

$$\mathbf{G}\left(R(X,Y)Z,U\right) = \tilde{\mathbf{G}}\left(\tilde{R}(X,Y)Z,U\right) + \tilde{\mathbf{G}}\left(A_{H(X,Z)}Y - A_{H(Y,Z)}X,U\right)$$

where $U \in \Gamma\left(\overline{T'\mathrm{I}}\right)$, and respectively, using that $T_{C}\mathrm{I}^{\perp}$ is spanned only by N, $\bar{\mathrm{N}}$,

$$\mathbf{G}\left(R(X,Y)Z,\bar{\mathrm{N}}\right) \ = \mathbf{G}\left(\left(D_{X}H\right)(Y,Z) - \left(D_{Y}H\right)(Y,Z),\bar{\mathrm{N}}\right) + \mathbf{G}\left(H\left(\tilde{T}(X,Y),Z\right),\bar{\mathrm{N}}\right)$$

called *the Gauss equations*, respectively *H-Codazzi* equations of (I,$\tilde{L}$) subspace.

Analogously, for normal curvatures $R\left(X,Y\right)\mathrm{N}$ and $\tilde{R}\left(X,Y\right)\mathrm{N}$, defining the covariant derivative of the shape operator

$$(D_X A)(\mathrm{N}, Y) = \tilde{D}_X (A_{\mathrm{N}} Y) - A\left(D_X^{\perp}\mathrm{N}, Y\right) - A\left(\mathrm{N}, \tilde{D}_X Y\right),$$

and the curvature form $R^{\perp}$ of the normal Finsler connection, $R^{\perp}(X,Y)\mathrm{N} = D_X^{\perp}\left(D_Y^{\perp}\mathrm{N}\right) - D_Y^{\perp}\left(D_X^{\perp}\mathrm{N}\right) - D_{[X,Y]}^{\perp}\mathrm{N}$, using the Gauss-Weingarten equations (16) and (19) it is obtained that:

$$\begin{aligned} R(X,Y)\mathrm{N} &= R^{\perp}(X,Y)\mathrm{N} + H(Y, A_{\mathrm{N}}X) - H(X, A_{\mathrm{N}}Y) + (D_Y A)(\mathrm{N}, X) - \\ &- (D_X A)(\mathrm{N}, Y) - A_{\mathrm{N}}\left(\tilde{T}(X,Y)\right). \end{aligned}$$

Equating their components from $T_C\mathrm{I}$ and $T_C^{\perp}\mathrm{I}$, we have

$$\mathbf{G}(R(X,Y)\mathrm{N}, Z) = \tilde{\mathbf{G}}\left((D_Y A)(\mathrm{N}, X) - (D_X A)(\mathrm{N}, Y), Z\right) - \tilde{\mathbf{G}}\left(A_{\mathrm{N}}\left(\tilde{T}(X,Y)\right), Z\right)$$

where $X, Y \in \Gamma(T_C T'\mathrm{I})$, $Z \in \Gamma\left(\overline{T'\mathrm{I}}\right)$, and

$$\mathbf{G}\left(R(X,Y)\mathrm{N}, \bar{\mathrm{N}}\right) = \mathbf{G}\left(R^{\perp}(X,Y)\mathrm{N}, \bar{\mathrm{N}}\right) + \mathbf{G}\left(H(Y, A_{\mathrm{N}}X) - H(X, A_{\mathrm{N}}Y), \bar{\mathrm{N}}\right)$$

called the *A-Codazzi equations*, respectively *Ricci equations* of $(\mathrm{I}, \tilde{L})$ subspace.

Further on, we will try to give some conditions when the indicatrix hypersurface is an umbilical submanifold.

Roughly speaking, a submanifold of a Riemannian manifold is *totally umbilical*, or simply umbilical, if it is equally curved in all tangent directions. A point $x \in M$ is called an umbilical point of the indicatrix if the shape operator A is proportional to the identity transformation for all vector fields from $T_C\mathrm{I}^{\perp}$, i.e. for $W \in T^{\perp}\mathrm{I}$, the Weingarten operator satisfies:

$$A_W X = \lambda X, \quad \text{where } \lambda \in \mathbb{R},\ \forall W \in T_C\mathrm{I}^{\perp}.$$

The submanifold is said to be totally umbilical if every point of the submanifold is an umbilical point.

Considering that $T_C\mathrm{I}^{\perp}$ is spanned only by $\mathrm{N}, \bar{\mathrm{N}}$, and given the fact that

$$\begin{aligned} A_{\mathrm{N}}(\tilde{\delta}_k) &= A_k^{\alpha}\dot{\partial}_{\alpha}; \qquad A_{\mathrm{N}}(\dot{\partial}_{\beta}) = A_{\beta}^{\alpha}\dot{\partial}_{\alpha}; \\ A_{\mathrm{N}}(\tilde{\delta}_{\bar{k}}) &= A_{\bar{k}}^{\alpha}\dot{\partial}_{\alpha}; \qquad A_{\mathrm{N}}(\dot{\partial}_{\bar{\beta}}) = A_{\bar{\beta}}^{\alpha}\dot{\partial}_{\alpha}, \end{aligned}$$

where the shape operator coefficients are given by (20), for the indicatrix to be an umbilical manifold we must have $A_k^{\alpha} = A_{\bar{k}}^{\alpha} = A_{\bar{\beta}}^{\alpha} = 0$ and $A_{\beta}^{\alpha} = \lambda\delta_{\beta}^{\alpha}$, where $\lambda \in \mathbb{R}$. It can be noticed that if the induced C-F d-(c.l.c.) is considered, relation (22) confirms that the indicatrix is umbilical with $\lambda = -1$, and we may conclude that in this case the indicatrix is a totally umbilical hypersurface of constant mean curvature.

In [6], the definition of an extrinsic sphere is given as a submanifold of a Riemannian manifold that is a totally umbilical submanifold with a nonzero parallel mean curvature vector. So, in the case of the C-F N-(c.l.c.) connection considered on the complex Finsler space (M, L), the indicatrix I_x is an extrinsic sphere of $T'M$.

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