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SOME UNIVALENCE CONDITIONS FOR AN INTEGRAL OPERATOR

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Abstract

In this work we define a general integral operator for analytic functions in the open unit disk and we derive some conditions for univalence of this integral operator.

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1 Introduction

Let $\mathcal{P}$ be the class of functions p of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} b_k z^k,$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$, with $Re\, p(z) > 0$, for all $z \in \mathcal{U}$.

We denote by $\mathcal{A}$ the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in $\mathcal{U}$.

We consider $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent in $\mathcal{U}$.

In this work we consider a general integral operator, which is defined by

$$V_n(z) = \left\{ \beta \int_0^z u^{\beta-1} (p_1(u))^{\gamma_1} \dots (p_n(u))^{\gamma_n} du \right\}^{\frac{1}{\beta}}, \tag{1.1}$$

for functions $p_j \in \mathcal{P}$ and $\beta,\ \gamma_j$ be complex numbers, $\beta \neq 0$ and $j = \overline{1, n}$.

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2 Preliminary results

Lemma 2.1. *[3]. Let α be a complex number, $Re\ \alpha > 0$ and $f \in \mathcal{A}$. If*

$$\frac{1-|z|^{2Re\ \alpha}}{Re\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \tag{2.1}$$

for all $z \in \mathcal{U}$, then for any complex number β, $Re\ \beta \geq Re\ \alpha$, the function

$$F_\beta(z) = \left\{\beta \int_0^z u^{\beta-1} f'(u)\right\}^{\frac{1}{\beta}} \tag{2.2}$$

is regular and univalent in $\mathcal{U}$.

Lemma 2.2. *(Schwarz [1]). Let f be the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$, M fixed. If $f(z)$ has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m,\ (z \in \mathcal{U}_R), \tag{2.3}$$

the equality (in the inequality (2.3) for $z \neq 0$) can hold only if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,$$

where θ is constant.

Lemma 2.3. *[2]. If function f is regular in $\mathcal{U}$ and $|f(z)| < 1$ in $\mathcal{U}$, then for all $\xi \in \mathcal{U}$ and $z \in \mathcal{U}$ the following inequalities hold*

$$\left|\frac{f(\xi)-f(z)}{1-\overline{f(z)}f(\xi)}\right| \leq \frac{|\xi - z|}{|1-\overline{z}\xi|}, \tag{2.4}$$

$$|f'(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2}, \tag{2.5}$$

the equalities hold only in the case $f(z) = \frac{\epsilon(z+u)}{1+\overline{u}z}$, where $|\epsilon| = 1$ and $|u| < 1$.

Remark 2.4. *[2]. For $z = 0$, from inequality (2.4)*

$$\left|\frac{f(\xi)-f(0)}{1-\overline{f(0)}f(\xi)}\right| \leq |\xi| \tag{2.6}$$

and, hence

$$|f(\xi)| \leq \frac{|\xi| + |f(0)|}{1+|f(0)||\xi|}. \tag{2.7}$$

Considering $f(0) = a$ and $\xi = z$, we have

$$|f(z)| \leq \frac{|z|+|a|}{1+|a||z|}, \tag{2.8}$$

for all $z \in \mathcal{U}$.

3 Main results

Theorem 3.1. *Let β, γ_j be complex numbers, $j=\overline{1,n}$, M_j positive real numbers and $p_j \in \mathcal{P}$, $p_j(z)=1+b_{1j}z+b_{2j}z^2+\ldots$, $j=\overline{1,n}$.*

If

$$\left|\frac{p_j'(z)}{p_j(z)}\right| \le M_j, \quad (z\in\mathcal{U},\ j=\overline{1,n}), \tag{3.1}$$

$$M_1|\gamma_1|+M_2|\gamma_2|+\ldots+M_n|\gamma_n| \le \frac{1}{\max_{|z|\le 1}\left[(1-|z|^2)|z|\frac{|z|+|c|}{1+|c||z|}\right]}, \tag{3.2}$$

where

$$c=\frac{b_{11}\gamma_1+b_{12}\gamma_2+\ldots+b_{1n}\gamma_n}{M_1|\gamma_1|+M_2|\gamma_2|+\ldots+M_n|\gamma_n|}, \tag{3.3}$$

then for every complex number β, Re $\beta \ge 1$, the integral operator V_n given by (1.1) is in class $\mathcal{S}$.

Proof. Let's us consider the function

$$f_n(z)=\int_0^z (p_1(u))^{\gamma_1}\ldots(p_n(u))^{\gamma_n}\,du, \tag{3.4}$$

for $p_j\in\mathcal{P}$, $j=\overline{1,n}$. Function f_n is regular in $\mathcal{U}$ and $f_n(0)=f_n'(0)-1=0$.

We have

$$\frac{f_n''(z)}{f_n'(z)}=\sum_{j=1}^{n}\gamma_j\frac{p_j'(z)}{p_j(z)}, \quad (z\in\mathcal{U}). \tag{3.5}$$

We consider the function

$$K(z)=\frac{1}{M_1|\gamma_1|+M_2|\gamma_2|+\ldots+M_n|\gamma_n|}\frac{f_n''(z)}{f_n'(z)}, \quad (z\in\mathcal{U}). \tag{3.6}$$

From (3.5) and (3.6) we obtain

$$\begin{aligned} K(z) &= \frac{\gamma_1}{M_1|\gamma_1|+\ldots+M_n|\gamma_2|}\cdot\frac{p_1'(z)}{p_1(z)}+\ldots+ \\ &+ \frac{\gamma_n}{M_1|\gamma_1|+\ldots+M_n|\gamma_2|}\cdot\frac{p_n'(z)}{p_n(z)}, \end{aligned} \tag{3.7}$$

for all $z\in\mathcal{U}$.

From (3.1) and (3.7) we obtain $|K(z)|<1$, $z\in\mathcal{U}$.

We have

$$K(0)=\frac{b_{11}\gamma_1+b_{12}\gamma_2+\ldots+b_{1n}\gamma_n}{M_1|\gamma_1|+M_2|\gamma_2|+\ldots+M_n|\gamma_n|}=c$$

By Remark 2.4 we obtain

$$|K(z)| \leq \frac{|z| + |c|}{1 + |c||z|}, \quad (z \in \mathcal{U}), \tag{3.8}$$

where

$$|c| = \frac{|b_{11}\gamma_1 + b_{12}\gamma_2 + \ldots + b_{1n}\gamma_n|}{M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n|}.$$

From (3.6), (3.8) we get

$$\frac{1}{M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n|} \left| \frac{f_n''(z)}{f_n'(z)} \right| \leq \frac{|z| + |c|}{1 + |c||z|}, \quad (z \in \mathcal{U}). \tag{3.9}$$

Using (3.9) we obtain

$$(1 - |z|^2) \left| \frac{z f_n''(z)}{f_n'(z)} \right| \leq$$

$$\leq (M_1|\gamma_1| + \ldots + M_n|\gamma_n|) \max_{|z| \leq 1} \left[(1 - |z|^2)|z| \frac{|z| + |c|}{1 + |c||z|} \right], \tag{3.10}$$

for all $z \in \mathcal{U}$.

From (3.2) and (3.10) we have

$$(1 - |z|^2) \left| \frac{z f_n''(z)}{f_n'(z)} \right| \leq 1, \quad (z \in \mathcal{U}). \tag{3.11}$$

Since $f_n'(z) = (p_1(z))^{\gamma_1} \ldots (p_n(z))^{\gamma_n}$, from (3.11) and Lemma 2.1, for $Re\ \alpha = 1$, it results that the integral operator V_n defined by (1.1) is in class $\mathcal{S}$. $\square$

Theorem 3.2. *Let α, β, γ_j be complex numbers, $j = \overline{1, n}$, $Re\ \alpha > 0$, M_j positive real numbers and $p_j \in \mathcal{P}$, $p_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots$, $j = \overline{1, n}$.*

If

$$\left| \frac{z p_j'(z)}{p_j(z)} \right| \leq M_j, \quad (z \in \mathcal{U},\ j = \overline{1, n}), \tag{3.12}$$

$$M_1|\gamma_1| + \ldots + M_n|\gamma_n| \leq \frac{3\sqrt{3}}{2} Re\ \alpha,\ for\ 0 \leq Re\ \alpha \leq Re\ \beta \leq 1 \tag{3.13}$$

or

$$M_1|\gamma_1| + \ldots + M_n|\gamma_n| \leq \frac{3\sqrt{3}}{2},\ for\ Re\ \beta > Re\ \alpha > 1, \tag{3.14}$$

then the integral operator V_n belongs to class $\mathcal{S}$.

Proof. We consider the function

$$f_n(z)=\int_0^z (p_1(u))^{\gamma_1}\dots(p_n(u))^{\gamma_n}\,du,\quad (p_j\in\mathcal{P};\ j=\overline{1,n}). \tag{3.15}$$

Function f_n is regular in $\mathcal{U}$ and $f_n(0)=f_n'(0)-1=0$.

We have

$$\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}\left|\frac{zf_n''(z)}{f_n'(z)}\right|\le\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}\sum_{j=1}^n|\gamma_j|\left|\frac{zp_j'(z)}{p_j(z)}\right|, \tag{3.16}$$

for all $z\in\mathcal{U}$.

By (3.12), applying Lemma 2.2 we obtain

$$\left|\frac{zp_j'(z)}{p_j(z)}\right|\le M_j|z|,\quad (z\in\mathcal{U};\ j=\overline{1,n}). \tag{3.17}$$

From (3.17) and (3.16) we have

$$\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}\left|\frac{zf_n''(z)}{f_n'(z)}\right|\le\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}|z|\sum_{j=1}^n M_j|\gamma_j|,\quad (z\in\mathcal{U}). \tag{3.18}$$

For $0<Re\,\alpha\le 1$ we have $1-|z|^{2Re\,\alpha}\le 1-|z|^2$, for all $z\in\mathcal{U}$ and from (3.18) we obtain

$$\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}\left|\frac{zf_n''(z)}{f_n'(z)}\right|\le\frac{1-|z|^2}{Re\,\alpha}|z|\sum_{j=1}^n M_j|\gamma_j|,\quad (z\in\mathcal{U}). \tag{3.19}$$

Since

$$\max_{|z|\le 1}(1-|z|^2)|z|=\frac{2}{3\sqrt{3}}, \tag{3.20}$$

using (3.13) and (3.20) we get

$$\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}\left|\frac{zf_n''(z)}{f_n'(z)}\right|\le 1\quad (z\in\mathcal{U};0<Re\,\alpha\le 1). \tag{3.21}$$

For $Re\,\alpha>1$ we have $\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}\le 1-|z|^2$, for all $z\in\mathcal{U}$ and from (3.18) we obtain

$$\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}\left|\frac{zf_n''(z)}{f_n'(z)}\right|\le(1-|z|^2)|z|\sum_{j=1}^n M_j|\gamma_j|,\quad (z\in\mathcal{U}), \tag{3.22}$$

and by (3.14), (3.20) and using (3.22) it results that

$$\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}\left|\frac{zf_n''(z)}{f_n'(z)}\right|\le 1\quad (z\in\mathcal{U};Re\,\alpha>1). \tag{3.23}$$

From (3.21), (3.23) and Lemma 2.1 we obtain that $V_n\in\mathcal{S}$. □

4 Some particular cases

Corollary 4.1. *Let β, γ_j be complex numbers, $j = \overline{1,n}$, $Re\ \beta \geq 1$, M_j positive real numbers and $p_j \in \mathcal{P}$, $p_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots$, $j = \overline{1,n}$.*

If

$$\left|\frac{p_j'(z)}{p_j(z)}\right| \leq M_j, \quad (z \in \mathcal{U},\ j = \overline{1,n}), \tag{4.1}$$

$$|b_{11}\gamma_1 + b_{12}\gamma_2 + \ldots + b_{1n}\gamma_n| = M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n|, \tag{4.2}$$

and

$$|b_{11}\gamma_1 + b_{12}\gamma_2 + \ldots + b_{1n}\gamma_n| \leq \frac{3\sqrt{3}}{2}, \tag{4.3}$$

then the integral operator $V_n \in \mathcal{S}$.

Proof. From (4.2) and Theorem 3.1, by (3.3) we obtain $|c| = 1$. Using (4.2) and (3.2) it results that

$$|b_{11}\gamma_1 + b_{12}\gamma_2 + \ldots + b_{1n}\gamma_n| \leq \frac{1}{\max_{|z|\leq 1}[(1-|z|^2)|z|]}. \tag{4.4}$$

Since

$$\max_{|z|\leq 1}[(1-|z|^2)|z|] = \frac{2}{3\sqrt{3}},$$

from (4.4) we have condition (4.3) and hence, Corollary 4.1. □

Corollary 4.2. *Let β, γ_j be complex numbers, $j = \overline{1,n}$, $Re\ \beta \geq Re\ \alpha > 0$, M_j positive real numbers and $p_j \in \mathcal{P}$, $p_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots$, $j = \overline{1,n}$, $b_{11}\gamma_1 + b_{12}\gamma_2 + \ldots + b_{1n}\gamma_n = 0$.*

If

$$\left|\frac{p_j'(z)}{p_j(z)}\right| \leq M_j, \quad (z \in \mathcal{U},\ j = \overline{1,n}), \tag{4.5}$$

$$M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n| \leq 4, \tag{4.6}$$

then the integral operator $V_n \in \mathcal{S}$.

Proof. For $b_{11}\gamma_1 + b_{12}\gamma_2 + \ldots + b_{1n}\gamma_n = 0$, from Theorem 3.1 we obtain $c = 0$ and by (3.2) we have

$$M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n| \leq \frac{1}{\max_{|z|\leq 1}[(1-|z|^2)|z|^2}. \tag{4.7}$$

Because

$$\max_{|z|\leq 1}\left[(1-|z|^2)|z|^2\right] = \frac{1}{4},$$

from (4.7) we get condition (4.6) and Corollary 4.2. □

Corollary 4.3. *Let γ_j be complex numbers, $j=\overline{1,n}$, M_j positive real numbers and $p_j \in \mathcal{P}$, $p_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots$, $j=\overline{1,n}$.*

If

$$\left|\frac{p_j'(z)}{p_j(z)}\right| \le M_j, \quad (z \in \mathcal{U},\ j=\overline{1,n}), \tag{4.8}$$

$$M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n| \le \frac{1}{\max_{|z|\le 1}\left[(1-|z|^2)|z|\frac{|z|+|c|}{1+|c||z|}\right]}, \tag{4.9}$$

where

$$c = \frac{b_{11}\gamma_1 + b_{12}\gamma_2 + \ldots + b_{1n}\gamma_n}{M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n|}, \tag{4.10}$$

then the integral operator defined by

$$H_n(z) = \int_0^z (p_1(u))^{\gamma_1} \ldots (p_n(u))^{\gamma_n}\, du \tag{4.11}$$

is in class $\mathcal{S}$.

Proof. We take $\beta = 1$ in Theorem 3.1. □

Corollary 4.4. *Let α, γ_j be complex numbers, $j=\overline{1,n}$, $0 < Re\ \alpha \le 1$, M_j positive real numbers and $p_j \in \mathcal{P}$, $p_j(z) = 1 + b_{1j}z + b_{2j}z^2 + \ldots$, $j=\overline{1,n}$.*

If

$$\left|\frac{p_j'(z)}{p_j(z)}\right| \le M_j, \quad (z \in \mathcal{U},\ j=\overline{1,n}), \tag{4.12}$$

$$M_1|\gamma_1| + M_2|\gamma_2| + \ldots + M_n|\gamma_n| \le \frac{3\sqrt{3}}{2} Re\ \alpha,$$

then the integral operator H_n given by (4.11) belongs to class $\mathcal{S}$.

Proof. For $\beta = 1$, from Theorem 3.2 we obtain Corollary 4.4. □

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