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ABOUT INTRINSIC FINSLER CONNECTIONS FOR THE HOMOGENEOUS LIFT TO THE OSCULATOR BUNDLE OF A FINSLER METRIC

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Abstract

In this article we present a study of the subspaces of the manifold OscM, the total space of the osculator bundle of a real manifold M. We obtain the induced connections of the canonical metrical N-linear connection determined by the homogeneous prolongation of a Finsler metric to the manifold OscM. We present the relation between the induced and the intrinsic geometric objects of the associated osculator submanifold.

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1 Introduction

The Sasaki N-prolongation $\mathbb{G}$ to the osculator bundle without the null section $\widetilde{OscM} = OscM \setminus \{0\}$ of a Finslerian metric g_{ab} on the manifold M given by

$$\mathbb{G} = g_{ab}\left(x,y\right)dx^a \otimes dx^b + g_{ab}\left(x,y\right)\delta y^a \otimes \delta y^b$$

is a Riemannian structure on $\widetilde{OscM}$, which depends only on the metric g_{ab}.

The tensor $\mathbb{G}$ is not invariant with respect to the homothetis on the fibres of $\widetilde{OscM}$, because $\mathbb{G}$ is not homogeneous with respect to the variable y^a.

In this paper, we use a new kind of prolongation $\mathring{\mathbb{G}}$ to $\widetilde{OscM}$, [3], which depends only on the metric g_{ab}. Thus, $\mathring{\mathbb{G}}$ determines on the manifold $\widetilde{OscM}$ a Riemannian structure which is 0-homogeneous on the fibres of $OscM$.

Some geometrical properties of $\mathring{\mathbb{G}}$ are studied: the canonical metrical N-linear connection, the induced linear connections etc.

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2 Preliminaries

Let us consider $F^n = (M, F)$ a Finsler space ([12]), and $F : TM = OscM \to \mathbb{R}$ the fundamental function. F is a C^∞ function on the manifold $OscM$ and it is continuous on the null section of the projection $\pi : OscM \to M$. The fundamental tensor on F^n is

$$g_{ab}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^a \partial y^b}, \ \ \forall (x, y) \in OscM.$$

The lagrangian $F^2(x, y)$ determines the canonical spray $S = y^a \frac{\partial}{\partial x^a} - 2G^a \frac{\partial}{\partial y^a}$ with the coefficients $G^a = \frac{1}{2}\gamma^a_{bc}(x, y)\, y^b y^c$, where $\gamma^a_{bc}(x, y)$ are the Christoffels symbols of the metric tensor $g_{ab}(x, y)$. The Cartan nonlinear connection N of the space F^n has the coefficients

$$N^a{}_b = \frac{\partial G^a}{\partial y^b}. \tag{2.1}$$

N determines a distribution on the manifold $\widetilde{OscM}$, ([12],[11]), which is supplementary to the vertical distribution V. We have the next decomposition

$$T_w\widetilde{OscM} = N_w \oplus V_w, \forall w = (x, y) \in \widetilde{OscM}. \tag{2.2}$$

The adapted basis of this decomposition is $\left\{\frac{\delta}{\delta x^a}, \frac{\partial}{\partial y^a}\right\}$, $(a = 1, .., n)$ and its dual basis is $(dx^a, \delta y^a)$, where

$$\begin{cases} \dfrac{\delta}{\delta x^a} = \dfrac{\partial}{\partial x^a} - N^b{}_a \dfrac{\delta}{\delta y^b}, \\ \\ \dfrac{\partial}{\partial y^a} = \dfrac{\partial}{\partial y^a} \end{cases} \tag{2.3}$$

and

$$\begin{cases} dx^a = dx^a, \\ \delta y^a = dy^a + N^a{}_b dx^b. \end{cases} \tag{1.1.7}$$

We use the next notations:

$$\delta_a = \frac{\delta}{\delta x^a}, \ \ \dot{\partial}_{1a} = \frac{\partial}{\partial y^a}.$$

The fundamental tensor g_{ab} determines on the manifold $\widetilde{OscM}$ the homogeneous N-lift $\overset{0}{\mathbb{G}}$,[10],

$$\overset{0}{\mathbb{G}} = g_{ab}(x, y)\, dx^a \otimes dx^b + h_{ab}(x, y)\, \delta y^a \otimes \delta y^b, \tag{2.4}$$

where

$$h_{ab}(x,y) = \frac{p^2}{\|y\|^2} g_{ab}(x,y), \tag{1.1.9}$$

$$\|y\|^2 = g_{ab}(x,y)\, y^a y^b.$$

This is homogeneous with respect to y, and p is a positive constant required by applications in order that the physical dimensions of the terms of $\mathring{\mathbb{G}}$ be the same.

Let $\check{M}$ be a real, m-dimensional manifold, immersed in M through the immersion $i : \check{M} \to M$. Locally, i can be given in the form

$$x^a = x^a\left(u^1, ..., u^m\right), \qquad rank \left\| \frac{\partial x^a}{\partial u^\alpha} \right\| = m.$$

The indices a, b, c,....run over the set $\{1, ..., n\}$ and $\alpha, \beta, \gamma, ...$ run on the set $\{1, ..., m\}$. We assume $1 < m < n$. We take the immersed submanifold $Osc\check{M}$ of the manifold $OscM$, by the immersion Osc$i$: $Osc\check{M} \to OscM$. The parametric equations of the submanifold $Osc\check{M}$ are

$$\begin{cases} x^a = x^a\left(u^1, ..., u^m\right), rang \left\| \dfrac{\partial x^a}{\partial u^\alpha} \right\| = m \\ \\ y^a = \dfrac{\partial x^a}{\partial u^\alpha} v^\alpha \end{cases} \tag{2.5}$$

The restriction of the fundamental function F to the submanifold $\widetilde{Osc\check{M}}$ is

$$\check{F}(u,v) = F(x(u), y(u,v))$$

and we call $\check{F}^m = \left(\check{M}, \check{F}\right)$ the **induced Finsler subspaces** of F^n and $\check{F}$ the **induced fundamental function**.

Let $B^a_\alpha(u) = \dfrac{\partial x^a}{\partial u^\alpha}$ and $g_{\alpha\beta}$ the induced fundamental tensor,

$$g_{\alpha\beta}(u,v) = g_{ab}(x(u), y(u,v))\, B^a_\alpha B^b_\beta. \tag{2.6}$$

We obtain a system of d-vectors $\{B^a_\alpha, B^a_{\bar{\alpha}}\}$ wich determines a moving frame $\mathcal{R} = \{(u,v); B^a_\alpha(u), B^a_{\bar{\alpha}}(u,v)\}$ in $OscM$ along to the submanifold $Osc\check{M}$.

Its dual frame will be denoted by $\mathcal{R}^* = \{B^\alpha_a(u,v), B^{\bar{\alpha}}_a(u,v)\}$. This is also defined on an open set $\check{\pi}^{-1}\left(\check{U}\right) \subset Osc\check{M}$, $\check{U}$ being a domain of a local chart on the submanifold $\check{M}$.

The conditions of duality are given by:

$$B^a_\beta B^\alpha_a = \delta^\alpha_\beta, \quad B^a_\beta B^{\bar{\alpha}}_a = 0, \quad B^\alpha_a B^a_{\bar{\beta}} = 0, \quad B^{\bar{\alpha}}_a B^a_{\bar{\beta}} = \delta^{\bar{\alpha}}_{\bar{\beta}}$$

$$B^a_\alpha B^\alpha_b + B^a_{\bar{\alpha}} B^{\bar{\alpha}}_b = \delta^a_b.$$

The restriction of the nonlinear connection N to $\widetilde{Osc\check{M}}$ uniquely determines an induced nonlinear connection $\check{N}$ on $\widetilde{Osc\check{M}}$

$$\check{N}^{\alpha}{}_{\beta} = B^{\alpha}_{a}\left(B^{a}_{0\beta} + N^{a}{}_{b}B^{b}_{\beta}\right). \tag{2.7}$$

The cobasis $\left(dx^{i}, \delta y^{a}\right)$ restricted to $Osc\check{M}$ is uniquely represented in the moving frame $\mathcal{R}$ in the following form:

$$\begin{cases} dx^{a} = B^{a}_{\beta}du^{\beta} \\ \\ \delta y^{a} = B^{a}_{\alpha}\delta v^{\alpha} + B^{a}_{\bar{\alpha}}K^{\bar{\alpha}}_{\beta}du^{\beta} \end{cases} \tag{2.8}$$

where

$$K^{\bar{\alpha}}_{\beta} = B^{\bar{\alpha}}_{a}\left(B^{a}_{0\beta} + M^{a}_{b}B^{b}_{\beta}\right),\ B^{a}_{0\beta} = B^{a}_{\alpha\beta}v^{a}.$$

A linear connection D on the manifold $OscM$ is called **metrical N-linear connection** with respect to $\overset{\circ}{\mathbb{G}}$, if $D\overset{\circ}{\mathbb{G}}=0$ and D preserves by parallelism the distributions N and V. The coefficients of the N-linear connections $D\Gamma(N)$ will be denoted with $\left(\underset{(00)}{\overset{H}{L}}{}^{a}_{bc}, \underset{(10)}{\overset{V}{L}}{}^{a}_{bc}, \underset{(01)}{\overset{H}{C}}{}^{a}_{bc}, \underset{(11)}{\overset{V}{C}}{}^{a}_{bc}\right)$.

Theorem 1.1([10]) *There exist metrical N-linear connections $D\Gamma(N)$ on $\widetilde{OscM}$, with respect to the homogeneous prolongation $\overset{\circ}{\mathbb{G}}$, wich depend only on the metric $g_{ab}(x,y)$. One of these connections has*
the "horizontal" coefficients

$$\begin{aligned} \underset{(00)}{\overset{H}{L}}{}^{a}_{bc} &= \frac{1}{2}g^{ad}\left(\delta_{b}g_{dc} + \delta_{c}g_{bd} - \delta_{d}g_{bc}\right) \\ \underset{(10)}{\overset{V}{L}}{}^{a}_{bc} &= \frac{1}{2}h^{ad}\left(\delta_{b}h_{dc} + \delta_{c}h_{bd} - \delta_{d}h_{bc}\right) \end{aligned} \tag{2.9}$$

and the "vertical" coefficients:

$$\begin{aligned} \underset{(01)}{\overset{H}{C}}{}^{a}_{bc} &= \frac{1}{2}g^{ad}\left(\dot{\partial}_{b}g_{dc} + \dot{\partial}_{c}g_{bd} - \dot{\partial}_{d}g_{bc}\right) \\ \underset{(11)}{\overset{V}{C}}{}^{a}_{bc} &= \frac{1}{2}h^{ad}\left(\dot{\partial}_{b}h_{dc} + \dot{\partial}_{c}h_{bd} - \dot{\partial}_{d}h_{bc}\right). \end{aligned} \tag{2.10}$$

It is called the **Cartan metrical N-linear connection**. This linear connection will be used throughout this paper.

For this N-linear connection, we have the operators $\overset{H}{D}$ and $\overset{V}{D}$ which are given by the following relations

$$\left.\begin{aligned} \overset{H}{D}X^{a} &= dX^{a} + \overset{H}{\omega}{}^{a}_{b}X^{b} \\ \overset{V}{D}X^{a} &= dX^{a} + \overset{V}{\omega}{}^{a}_{b}X^{b} \end{aligned}\right. \quad \forall X \in \mathcal{F}\left(\widetilde{OscM}\right). \tag{2.11}$$

We call these operators the **horizontal** and **vertical covariant differentials**. The 1-forms which define these operators will be called the **horizontal** and **vertical 1-form**, where

$$\begin{aligned}\overset{H}{\omega}{}^a_b &= \underset{(00)}{\overset{H}{L}}{}^a_{bc}dx^c + \underset{(01)}{\overset{H}{C}}{}^a_{bc}\delta y^c\\ \overset{V}{\omega}{}^a_b &= \underset{(10)}{\overset{V}{L}}{}^a_{bc}dx^c + \underset{(11)}{\overset{V}{C}}{}^a_{bc}\delta y^c.\end{aligned} \tag{2.12}$$

We have

Theorem 1.2 *The d-tensors of torsion of the Cartan metrical N-linear connection D have the next expresions:*

$$\begin{aligned}&\underset{(00)}{\overset{H}{T}}{}^a_{bc} = \underset{(00)}{\overset{H}{L}}{}^a_{bc} - \underset{(00)}{\overset{H}{L}}{}^a_{cb}, && \underset{(01)}{\overset{V}{T}}{}^a_{bc} = R^a_{bc},\\ &\underset{(10)}{\overset{H}{P}}{}^a_{bc} = \underset{(01)}{\overset{H}{C}}{}^a_{bc}, && \underset{(11)}{\overset{V}{P}}{}^a_{bc} = \underset{(11)}{B}{}^a_{bc} - \underset{(10)}{\overset{V}{L}}{}^a_{cb}\\ &\underset{(11)}{\overset{V}{S}}{}^a_{bc} = \underset{(11)}{\overset{V}{C}}{}^a_{bc} - \underset{(11)}{\overset{V}{C}}{}^a_{cb}.\end{aligned} \tag{2.13}$$

Theorem 1.3 *The Cartan metrical N-linear connection D has, in the adapted bases*

$\left\{\delta_a, \dot{\partial}_{1a}\right\}$, *the following d-tensors of curvature "horizontals"*

$$\begin{aligned}\underset{(00)}{\overset{H}{R}}{}_b{}^a{}_{cd} &= \delta_d \underset{(00)}{\overset{H}{L}}{}^a_{bc} - \delta_c \underset{(00)}{\overset{H}{L}}{}^a_{bd} + \underset{(00)}{\overset{H}{L}}{}^f_{bc}\underset{(00)}{\overset{H}{L}}{}^a_{fd} - \underset{(00)}{\overset{H}{L}}{}^f_{bd}\underset{(00)}{\overset{H}{L}}{}^a_{fc} +\\ &\quad + \underset{(01)}{\overset{H}{C}}{}^a_{bf}R^f_{cd},\\ \underset{(10)}{\overset{H}{P}}{}_b{}^a{}_{cd} &= \dot{\partial}_{1d} \underset{(00)}{\overset{H}{L}}{}^a_{bc} - \underset{(01)}{\overset{H}{C}}{}^a_{bd|0c} + \underset{(01)}{\overset{H}{C}}{}^a_{bf}\underset{(11)}{\overset{H}{P}}{}^f_{cd},\\ \underset{(10)}{\overset{H}{S}}{}_b{}^a{}_{cd} &= \dot{\partial}_{1d} \underset{(01)}{\overset{H}{C}}{}^a_{bc} - \dot{\partial}_{1c} \underset{(01)}{\overset{H}{C}}{}^a_{bd} + \underset{(01)}{\overset{H}{C}}{}^f_{bc}\underset{(01)}{\overset{H}{C}}{}^a_{fd} - \underset{(01)}{\overset{H}{C}}{}^f_{bd}\underset{(01)}{\overset{H}{C}}{}^a_{fc},\end{aligned} \tag{2.14}$$

and the "verticals"

$$\begin{aligned}
\underset{(01)}{\overset{V}{R}}{}_{b}{}^{a}{}_{cd} &= \delta_d \underset{(10)}{\overset{V}{L}}{}^{a}_{bc} - \delta_c \underset{(10)}{\overset{V}{L}}{}^{a}_{bd} + \underset{(10)}{\overset{V}{L}}{}^{f}_{bc} \underset{(10)}{\overset{V}{L}}{}^{a}_{fd} - \underset{(10)}{\overset{V}{L}}{}^{f}_{bd} \underset{(10)}{\overset{V}{L}}{}^{a}_{fc} + \\
&\quad + \underset{(11)}{\overset{V}{C}}{}^{a}_{bf} R^{f}_{cd}, \\
\underset{(11)}{\overset{V}{P}}{}_{b}{}^{a}{}_{cd} &= \dot{\partial}_{1d} \underset{(10)}{\overset{V}{L}}{}^{a}_{bc} - \underset{(11)}{\overset{V}{C}}{}^{a}_{bd|1c} + \underset{(11)}{\overset{V}{C}}{}^{a}_{bf} \underset{(11)}{\overset{V}{P}}{}^{f}_{cd}, \\
\underset{(11)}{\overset{V}{S}}{}_{b}{}^{a}{}_{cd} &= \dot{\partial}_{1d} \underset{(11)}{\overset{V}{C}}{}^{a}_{bc} - \dot{\partial}_{1c} \underset{(11)}{\overset{V}{C}}{}^{a}_{bd} + \underset{(11)}{\overset{V}{C}}{}^{f}_{bc} \underset{(11)}{\overset{V}{C}}{}^{a}_{fd} - \underset{(11)}{\overset{V}{C}}{}^{f}_{bd} \underset{(11)}{\overset{V}{C}}{}^{a}_{fc}.
\end{aligned} \tag{2.15}$$

3 The relative covariant derivatives

Let $D\Gamma(N)$, the Cartan metrical N-linear connection of the manifold $OscM$. A classical method to determine the laws of derivation on a Finsler submanifold is the type of the coupling.

Theorem 2.1 *The coupling of the N-linear connection D to the induced nonlinear connection $\check{N}$ along $\widetilde{Osc\check{M}}$ is locally given by the set of coefficients* $\check{D}\Gamma\left(\check{N}\right) = \left(\underset{(00)}{\overset{H}{\check{L}}}{}^{a}_{b\delta}, \underset{(10)}{\overset{V}{\check{L}}}{}^{a}_{b\delta}, \underset{(01)}{\overset{H}{\check{C}}}{}^{a}_{b\delta}, \underset{(11)}{\overset{V}{\check{C}}}{}^{a}_{b\delta} \right)$, *where*

$$\begin{cases}
\underset{(00)}{\overset{H}{\check{L}}}{}^{a}_{b\delta} = \underset{(00)}{\overset{H}{L}}{}^{a}_{bd} B^{d}_{\delta} + \underset{(01)}{\overset{H}{C}}{}^{a}_{bd} B^{d}_{\bar{\delta}} K^{\bar{\delta}}_{\delta} \\
\underset{(10)}{\overset{V}{\check{L}}}{}^{a}_{b\delta} = \underset{(10)}{\overset{V}{L}}{}^{a}_{bd} B^{d}_{\delta} + \underset{(11)}{\overset{V}{C}}{}^{a}_{bd} B^{d}_{\bar{\delta}} K^{\bar{\delta}}_{\delta} \\
\underset{(01)}{\overset{H}{\check{C}}}{}^{a}_{b\delta} = \underset{(01)}{\overset{H}{C}}{}^{a}_{bd} B^{d}_{\delta} \\
\underset{(11)}{\overset{V}{\check{C}}}{}^{a}_{b\delta} = \underset{(11)}{\overset{V}{C}}{}^{a}_{bd} B^{d}_{\delta}.
\end{cases} \tag{3.1}$$

Definition 2.2 *We call the **induced tangent connection** on $\widetilde{Osc\check{M}}$ by the metrical N-linear connection D, the couple of the operators $\overset{H}{D}{}^{\top}$, $\overset{V}{D}{}^{\top}$ which are defined by*

$$\begin{aligned}
\overset{H}{D}{}^{\top} X^{\alpha} &= B^{\alpha}_{b} \overset{H}{\check{D}} X^{b}, \\
& \qquad\qquad \textit{for } X^{a} = B^{a}_{\gamma} X^{\gamma} \\
\overset{V}{D}{}^{\top} X^{\alpha} &= B^{\alpha}_{b} \overset{V}{\check{D}} X^{b},
\end{aligned}$$

where

$$\overset{H}{D}{}^{\top} X^{\alpha} = dX^{\alpha} + X^{\beta} \overset{H}{\omega}{}^{\alpha}_{\beta}$$

$$\overset{V}{D}{}^{\top} X^{\alpha} = dX^{\alpha} + X^{\beta} \overset{V}{\omega}{}^{\alpha}_{\beta}$$

and $\overset{H}{\omega}{}^{\alpha}_{\beta}$, $\overset{V}{\omega}{}^{\alpha}_{\beta}$ *are called the* ***tangent connection 1-forms***.

We have

Theorem 2.3 *The tangent connections 1-forms are as follows:*

$$\begin{aligned} \overset{H}{\omega}{}^{\alpha}_{\beta} &= \underset{(00)}{\overset{H}{L}}{}^{\alpha}_{\beta\delta} du^{\delta} + \underset{(01)}{\overset{H}{C}}{}^{\alpha}_{\beta\delta} \delta v^{\delta} \\ \overset{V}{\omega}{}^{\alpha}_{\beta} &= \underset{(10)}{\overset{V}{L}}{}^{\alpha}_{\beta\delta} du^{\delta} + \underset{(11)}{\overset{V}{C}}{}^{\alpha}_{\beta\delta} \delta v^{\delta}, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} \underset{(00)}{\overset{H}{L}}{}^{\alpha}_{\beta\delta} &= B^{\alpha}_{d} \left(B^{d}_{\beta\delta} + B^{f}_{\beta} \underset{(00)}{\overset{H}{\check{L}}}{}^{d}_{f\delta} \right), \\ \underset{(10)}{\overset{V}{L}}{}^{\alpha}_{\beta\delta} &= B^{\alpha}_{d} \left(B^{d}_{\beta\delta} + B^{f}_{\beta} \underset{(10)}{\overset{V}{\check{L}}}{}^{d}_{f\delta} \right), \\ \underset{(01)}{\overset{H}{C}}{}^{\alpha}_{\beta\delta} &= B^{\alpha}_{d} B^{f}_{\beta} \underset{(01)}{\overset{H}{\check{C}}}{}^{d}_{f\delta}, \\ \underset{(11)}{\overset{V}{C}}{}^{\alpha}_{\beta\delta} &= B^{\alpha}_{d} B^{f}_{\beta} \underset{(11)}{\overset{V}{\check{C}}}{}^{d}_{f\delta}. \end{aligned} \tag{3.3}$$

Definition 2.4 *We call the* ***induced normal connection*** *on* $\widetilde{Osc\check{M}}$ *by the metrical N-linear connection D, the couple of the operators* $\overset{H}{D}{}^{\perp}$, $\overset{V}{D}{}^{\perp}$ *which are defined by*

$$\left.\begin{aligned} \overset{H}{D}{}^{\perp} X^{\overline{\alpha}} &= B^{\alpha}_{b} \overset{H}{\check{D}} X^{b} \\ \overset{V}{D}{}^{\perp} X^{\overline{\alpha}} &= B^{\alpha}_{b} \overset{V}{\check{D}} X^{b}, \end{aligned}\right. \quad \textit{for } X^{a} = B^{a}_{\bar{\gamma}} X^{\bar{\gamma}}$$

where

$$\overset{H}{D}{}^{\perp} X^{\overline{\alpha}} = dX^{\overline{\alpha}} + X^{\overline{\beta}} \overset{H}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}}$$

$$\overset{V}{D}{}^{\perp} X^{\overline{\alpha}} = dX^{\overline{\alpha}} + X^{\overline{\beta}} \overset{V}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}}$$

and $\overset{H}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}}$, $\overset{V}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}}$ *are called the* ***normal connection 1-forms***.

We have

Theorem 2.5 *The normal connections 1-forms are as follows:*

$$\begin{aligned} \overset{H}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}} &= \underset{(00)}{\overset{H}{L}}{}^{\overline{\alpha}}_{\overline{\beta}\delta} du^\delta + \underset{(01)}{\overset{H}{C}}{}^{\overline{\alpha}}_{\overline{\beta}\delta} \delta v^\delta \\ \overset{V}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}} &= \underset{(10)}{\overset{V}{L}}{}^{\overline{\alpha}}_{\overline{\beta}\delta} du^\delta + \underset{(11)}{\overset{V}{C}}{}^{\overline{\alpha}}_{\overline{\beta}\delta} \delta v^\delta, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} \underset{(00)}{\overset{H}{L}}{}^{\overline{\alpha}}_{\overline{\beta}\delta} &= B^{\overline{\alpha}}_d \left(\frac{\delta B^d_{\overline{\beta}}}{\delta u^\delta} + B^f_{\overline{\beta}} \underset{(00)}{\overset{H}{\check{L}}}{}^d_{f\delta} \right), \\ \underset{(10)}{\overset{V}{L}}{}^{\overline{\alpha}}_{\overline{\beta}\delta} &= B^{\overline{\alpha}}_d \left(\frac{\delta B^d_{\overline{\beta}}}{\delta u^\delta} + B^f_{\overline{\beta}} \underset{(10)}{\overset{V}{\check{L}}}{}^d_{f\delta} \right), \\ \underset{(01)}{\overset{H}{C}}{}^{\overline{\alpha}}_{\overline{\beta}\delta} &= B^{\overline{\alpha}}_d \left(\frac{\partial B^d_{\overline{\beta}}}{\partial v^\delta} + B^f_{\overline{\beta}} \underset{(01)}{\overset{H}{\check{C}}}{}^d_{f\delta} \right), \\ \underset{(11)}{\overset{V}{C}}{}^{\overline{\alpha}}_{\overline{\beta}\delta} &= B^{\overline{\alpha}}_d \left(\frac{\partial B^d_{\overline{\beta}}}{\partial v^\delta} + B^f_{\overline{\beta}} \underset{(11)}{\overset{V}{\check{C}}}{}^d_{f\delta} \right). \end{aligned} \tag{3.5}$$

Now, we can define the relative (or mixed) covariant derivatives $\overset{H}{\nabla}$ and $\overset{V}{\nabla}$.

Theorem 2.6 *The relative covariant (mixed) derivatives in the algebra of mixed d-tensor fields are the operators* $\overset{H}{\nabla}, \overset{V}{\nabla}$ *for which the following properties hold*:

$$\left.\begin{aligned} \overset{H}{\nabla} f &= df, \\ \overset{V}{\nabla} f &= df, \end{aligned}\right. \qquad \forall f \in \mathcal{F}\left(\widetilde{Osc\check{M}}\right)$$

$$\overset{H}{\nabla} X^a = \overset{H}{\check{D}} X^a, \quad \overset{H}{\nabla} X^\alpha = \overset{H}{D^\top} X^\alpha, \quad \overset{H}{\nabla} X^{\overline{\alpha}} = \overset{H}{D^\perp} X^{\overline{\alpha}},$$

$$\overset{V}{\nabla} X^a = \overset{V}{\check{D}} X^a, \quad \overset{V}{\nabla} X^\alpha = \overset{V}{D^\top} X^\alpha, \quad \overset{V}{\nabla} X^{\overline{\alpha}} = \overset{H}{D^\perp} X^{\overline{\alpha}}.$$

$\overset{H}{\check{\omega}}{}^a_b, \overset{V}{\check{\omega}}{}^a_b, \overset{H}{\omega}{}^\alpha_\beta, \overset{V}{\omega}{}^\alpha_\beta, \overset{H}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}}, \overset{V}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}}$ *are called the* ***connection 1-forms*** *of* $\overset{H}{\nabla}, \overset{V}{\nabla}$.

4 A comparison between the induced and intrinsic geometrical objects

It is known that, in the case of Finsler or pseudo-Finsler spaces ([6], [5], [14], [15], [13], [16]), the intrinsic nonlinear connection of the submanifold is different from the induced nonlinear connection by the nonlinear connection on the manifold. Moreover, the induced Finsler connection is different from the induced Finsler connection.

In this section, we present a comparison between the the induced and intrinsic geometric objects on the submanifold $\widetilde{Osc\check{M}}$ with respect to the Cartan metrical N-linear connection determined by the homogeneous lift $\overset{0}{\mathbb{G}}$ (2.4).

Let $\widetilde{\overset{0}{\mathbb{G}}}$, the homogeneous lift to the submanifold $\widetilde{Osc\check{M}}$ of the induced fundamental tensor (2.6),

$$\widetilde{\overset{0}{\mathbb{G}}} = g_{\alpha\beta}(u,v)\, du^{\alpha} \otimes du^{\beta} + h_{\alpha\beta}(u,v)\, \delta v^{\alpha} \otimes \delta v^{\beta}, \tag{4.1}$$

where

$$h_{\alpha\beta}(u,v) = \frac{p^2}{\|v\|^2} g_{\alpha\beta}(u,v)$$

$$\|v\|^2 = g_{\alpha\beta} v^{\alpha} v^{\beta}$$

and $\mathring{N}$, the intrinsic Cartan nonlinear connection

$$\mathring{N}^{\alpha}{}_{\beta} = \frac{\partial G^{\alpha}}{\partial v^{\beta}},$$

where $G^{\alpha} = \frac{1}{2}\gamma^{\alpha}_{\beta\gamma}(u,v)\, v^{\beta} v^{\gamma}$ and $\gamma^{\alpha}_{\beta\gamma}(u,v)$ are the Christoffel symbols of $g_{\alpha\beta}$.

Let $\mathring{D}\Gamma\left(\mathring{N}\right) = \left(\overset{H}{\underset{(00)}{\mathring{L}}}{}^{\alpha}_{\beta\gamma}, \overset{V_1}{\underset{(10)}{\mathring{L}}}{}^{\alpha}_{\beta\gamma}, \overset{H}{\underset{(01)}{\mathring{C}}}{}^{\alpha}_{\beta\gamma}, \overset{V_1}{\underset{(11)}{\mathring{C}}}{}^{\alpha}_{\beta\gamma}\right)$ the intrinsic Cartan metrical N-linear connection of the submanifold $\widetilde{Osc\check{M}}$ with the "*horizontal*" coeficients

$$\begin{aligned} \overset{H}{\underset{(00)}{\mathring{L}}}{}^{\alpha}_{\beta\gamma} &= \frac{1}{2} g^{\alpha\delta}\left(\delta_{\beta} g_{\delta\gamma} + \delta_{\gamma} g_{\beta\delta} - \delta_{\delta} g_{\beta\gamma}\right) \\ \overset{V}{\underset{(10)}{\mathring{L}}}{}^{\alpha}_{\beta\gamma} &= \frac{1}{2} h^{\alpha\delta}\left(\delta_{\beta} h_{\delta\gamma} + \delta_{\gamma} h_{\beta\delta} - \delta_{\delta} h_{\beta\gamma}\right) \end{aligned} \tag{4.2}$$

and the "*vertical*" coeficients

$$\begin{aligned}\overset{H}{\mathring{C}}{}_{(01)}{}^{\alpha}_{\beta\gamma} &= \frac{1}{2} g^{\alpha\delta}\left(\dot{\partial}_{1\beta} g_{\delta\gamma} + \dot{\partial}_{1\gamma} g_{\beta\delta} - \dot{\partial}_{1\delta} g_{\beta\gamma}\right) \\ \overset{V}{\mathring{C}}{}_{(11)}{}^{\alpha}_{\beta\gamma} &= \frac{1}{2} h^{\alpha\delta}\left(\dot{\partial}_{1\beta} h_{\delta\gamma} + \dot{\partial}_{1\gamma} h_{\beta\delta} - \dot{\partial}_{1\delta} h_{\beta\gamma}\right).\end{aligned} \tag{4.3}$$

Proposition 3.1 *The Lie brackets of the vector fields of the adapted basis* $\left\{\delta_\alpha, \dot{\partial}_{1\alpha}\right\}$ *are given by:*

$$\begin{aligned} [\delta_\alpha, \delta_\beta] &= -R^{\sigma}_{\alpha\beta}\dot{\partial}_{1\sigma}, \\ \left[\delta_\alpha, \dot{\partial}_{1\beta}\right] &= \underset{(11)}{B}{}^{\sigma}_{\alpha\beta}\dot{\partial}_{1\sigma}, \\ \left[\dot{\partial}_{1\alpha}, \dot{\partial}_{1\beta}\right] &= 0, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} R^{\sigma}_{\alpha\beta} &= \delta_\alpha N^{\sigma}{}_{\beta} - \delta_\beta N^{\sigma}{}_{\alpha} \\ \underset{(11)}{B}{}^{\sigma}_{\alpha\beta} &= \dot{\partial}_{1\beta} N^{\sigma}{}_{\alpha}. \end{aligned}$$

For any d-vector field $X \in X(\widetilde{OscM})$ expressed in the adapted basis $\left\{\delta_a, \dot{\partial}_{1a}\right\}$ we have

$$X = \overset{0}{X}{}^a \frac{\delta}{\delta x^a} + \overset{1}{X}{}^a \frac{\partial}{\partial y^a}, X \in \mathcal{X}\left(\widetilde{OscM}\right).$$

We consider h and v, the horizontal and the vertical projectors associated to the nonlinear connection N. Denote by

$$\begin{aligned} X^H &= hX = \overset{0}{X}{}^a \frac{\delta}{\delta x^a} \\ X^V &= vX = \overset{1}{X}{}^a \frac{\partial}{\partial y^a}. \end{aligned}$$

For any d-vector field $\check{X} \in X(\widetilde{Osc\check{M}})$ expressed in the adapted basis $\left\{\delta_\alpha, \dot{\partial}_{1\alpha}\right\}$ we have

$$\check{X} = \overset{0}{\check{X}}{}^\alpha \frac{\mathring{\delta}}{\mathring{\delta} u^\alpha} + \overset{1}{\check{X}}{}^\alpha \frac{\partial}{\partial v^\alpha}, \forall \check{X} \in \mathcal{X}\left(\widetilde{Osc\check{M}}\right)$$

and consider $\mathring{h}$ and $\mathring{v}$, the horizontal and the vertical projectors associated to the intrinsic nonlinear connection $\overset{\circ}{N}$. Denote by

$$\left.\begin{aligned} \check{X}^{\overset{\circ}{H}} &= \overset{\circ}{h}\check{X} = \overset{0}{\check{X}}{}^{\alpha}\frac{\overset{\circ}{\delta}}{\overset{\circ}{\delta} u^{\alpha}} \\ \check{X}^{\overset{\circ}{V}} &= v\check{X} = \overset{1}{\check{X}}{}^{a}\frac{\partial}{\partial v^{a}} \end{aligned}\right. \quad \forall \check{X} \in \mathcal{X}\left(\widetilde{Osc\check{M}}\right).$$

Proposition 3.2 *Let $\overset{\circ}{N}$, the intrinsic Cartan nonlinear connection and $\check{N}$, the induced nonlinear connection on the submanifold $\widetilde{Osc\check{M}}$ by the Cartan nonlinear connection N. Then the following relations hold*:

1° The coefficients of the nonlinear connections $\overset{\circ}{N}$ and $\check{N}$ are related by ([5])

$$\overset{\circ}{N}{}^{\alpha}{}_{\beta} = \check{N}^{\alpha}{}_{\beta} + D^{\alpha}{}_{\beta}.$$

2° There exist the following relations between the components of the adapted bases of $\overset{\circ}{N}$ and $\check{N}$

$$\overset{\circ}{\delta}_{\alpha} = \delta_{\alpha} - D^{\beta}{}_{\alpha}\dot{\partial}_{1\beta},$$

$$\overset{\circ}{\partial}_{1\alpha} = \dot{\partial}_{1\alpha}.$$

3° There exist the following relations between the coefficients of the Lie brackets of the adapted bases of $\mathbb{R}N$ and $\check{N}$

$$\overset{\circ}{R}{}^{\alpha}_{\beta\gamma} = R^{\alpha}_{\beta\gamma} + \underset{00}{\overset{1}{D}}{}^{\alpha}_{\beta\gamma}$$

$$\underset{(11)}{\overset{\circ}{B}}{}^{\alpha}_{\beta\gamma} = \underset{(11)}{B}{}^{\alpha}_{\beta\gamma} + \underset{01}{\overset{1}{D}}{}^{\alpha}_{\beta\gamma},$$

where

$$D^{\alpha}{}_{\beta} \quad = \quad g_{\bar{\alpha}}{}^{\alpha}{}_{\beta}K^{\bar{\alpha}}_{\beta}v^{\beta} \tag{4.5}$$

$$\underset{00}{\overset{1}{D}}{}^{\alpha}_{\beta\gamma} = \delta_{\gamma}D^{\alpha}{}_{\beta} - \delta_{\beta}D^{\alpha}{}_{\gamma} + D^{\delta}{}_{\beta}\dot{\partial}_{1\delta}\left(N^{\alpha}{}_{\gamma} + D^{\alpha}{}_{\gamma}\right) - D^{\delta}{}_{\gamma}\dot{\partial}_{1\delta}\left(N^{\alpha}{}_{\beta} + D^{\alpha}{}_{\beta}\right) \tag{4.6}$$

$$\underset{01}{\overset{1}{D}}{}^{\alpha}_{\beta\gamma} = \dot{\partial}_{1\gamma}D^{\alpha}{}_{\beta}. \tag{4.7}$$

Proposition 3.3 *The local coefficients of the intrinsic Cartan metrical N-linear* *connection* $\overset{\circ}{D}\Gamma\left(\overset{\circ}{N}\right)$ *and of the induced tangent connection of the Cartan metrical* N*-linear connection* $D\Gamma\,(N)$ *are related by*:

$$\underset{(00)}{\overset{H}{\overset{\circ}{L}}}{}^{\alpha}_{\beta\delta} = \underset{(00)}{\overset{H}{L}}{}^{\alpha}_{\beta\delta} + \underset{(00)}{\overset{H}{\Delta}}{}^{\alpha}_{\beta\delta}$$

$$\underset{(10)}{\overset{V}{\overset{\circ}{L}}}{}^{\alpha}_{\beta\delta} = \underset{(10)}{\overset{V}{L}}{}^{\alpha}_{\beta\delta} + \underset{(10)}{\overset{V}{\Delta}}{}^{\alpha}_{\beta\delta}$$

$$\underset{(01)}{\overset{H}{\overset{\circ}{C}}}{}^{\alpha}_{\beta\delta} = \underset{(01)}{\overset{H}{C}}{}^{\alpha}_{\beta\delta},$$

$$\underset{(11)}{\overset{V}{\overset{\circ}{C}}}{}^{\alpha}_{\beta\delta} = \underset{(11)}{\overset{V}{C}}{}^{\alpha}_{\beta\delta},$$

where

$$\begin{aligned}
\underset{(00)}{\overset{H}{\Delta}}{}^{\alpha}_{\beta\delta} &= \frac{1}{2}\left[g_{\bar{\delta}}{}^{\alpha}{}_{\delta}K^{\bar{\delta}}_{\beta} - g_{\bar{\delta}\beta\delta}K^{\bar{\delta}\alpha}\right] \\
&\quad -\frac{1}{2}g^{\alpha\delta}\left[D^{\varepsilon}{}_{\beta}\delta_{1\varepsilon}g_{\delta\gamma} + D^{\varepsilon}{}_{\gamma}\delta_{1\varepsilon}g_{\delta\beta} - D^{\varepsilon}{}_{\delta}\delta_{1\varepsilon}g_{\beta\gamma}\right], \\
\underset{(10)}{\overset{V}{\Delta}}{}^{\alpha}_{\beta\delta} &= -\frac{1}{2h}g^{\alpha\sigma}\left[B^{b}_{\bar{\beta}}K^{\bar{\beta}}_{\sigma}\left(\dot{\partial}_{1b}h\right)g_{\beta\delta} - B^{b}_{\bar{\beta}}K^{\bar{\beta}}_{\beta}\left(\dot{\partial}_{1b}h\right)g_{\sigma\delta}\right] \\
&\quad +\frac{1}{2h}g^{\alpha\sigma}\left[D^{\varepsilon}{}_{\sigma}\dot{\partial}_{1\varepsilon}g_{\beta\delta} - D^{\varepsilon}{}_{\beta}\dot{\partial}_{1\varepsilon}g_{\sigma\delta} - D^{\varepsilon}{}_{\delta}\dot{\partial}_{1\varepsilon}g_{\beta\sigma}\right] \\
&\quad +\underset{(10)}{\Delta}{}^{\alpha}_{\beta\delta},
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
\underset{(10)}{\Delta}{}^{\alpha}_{\beta\gamma} &= \frac{1}{2}\left(\dot{\partial}_{1\beta}B^{\alpha}_{a}\right)\left(B^{a}_{0\gamma} + B^{c}_{\gamma}\underset{1}{N}{}^{a}{}_{c}\right) - \frac{1}{2}B^{\alpha}_{a}B^{a}_{\beta\gamma} \\
&\quad -\frac{1}{2}g^{af}B^{\alpha}_{a}B^{b}_{\beta}B^{d}_{\bar{\delta}}\left(\dot{\partial}_{1d}g_{bf}\right)K^{\bar{\delta}}{}_{\gamma} - \frac{1}{2}g^{\alpha\delta}B^{\sigma}_{a}B^{a}_{\gamma\delta}g_{\sigma\beta} \\
&\quad -\frac{1}{2}g^{\alpha\delta}g_{\sigma\beta}\left(\dot{\partial}_{1\gamma}B^{\sigma}_{a}\right)\left(B^{a}_{0\delta} + B^{d}_{\delta}N^{a}{}_{d}\right) \\
&\quad +g^{\alpha\delta}g_{bd}B^{b}_{\beta}B^{a}_{\delta\gamma} + \frac{1}{2}\underset{01}{\overset{1}{D}}{}^{\alpha}_{\gamma\beta} - \frac{1}{2}g^{\alpha\delta}\left[D^{\varepsilon}{}_{\gamma}\dot{\partial}_{1\varepsilon}g_{\beta\delta} + \underset{01}{\overset{1}{D}}{}^{\sigma}_{\gamma\delta}g_{\sigma\beta}\right].
\end{aligned} \tag{4.9}$$

From the proposition 3.3 we get that $\overset{\circ}{D}$, the intrinsic Cartan metrical N-linear connection is not identical with $D^{\top}$, the induced tangent connection of the Cartan metrical N-linear connection $D\Gamma(N)$. From this fact, [5], there exists $\overset{\top}{D}$, the **deformation tensor** of the pair $(\overset{\circ}{D}, D^{\top})$. For $\check{X}, \check{Y} \in \check{\mathcal{X}}\left(\widetilde{Osc\check{M}}\right)$ we get

$$\begin{aligned}
&\overset{\circ}{D}_{\check{X}^{\mathring{V}}} \check{Y}^{\mathring{H}} = D^{\top}_{\check{X}^{\mathring{V}}} \check{Y}^{\mathring{H}} \\
&\overset{\circ}{D}_{\check{X}^{\mathring{V}}} \check{Y}^{\mathring{V}} = D^{\top}_{\check{X}^{\mathring{V}}} \check{Y}^{\mathring{V}} \\
&\overset{\circ}{D}_{\check{X}^{\mathring{H}}} \check{Y}^{\mathring{H}} = D^{\top}_{\check{X}^{\mathring{H}}} \check{Y}^{\mathring{H}} + \overset{\top}{D}\left(\check{X}^{\mathring{H}}, \check{Y}^{\mathring{H}}\right) \qquad \forall \check{X}, \check{Y} \in \check{\mathcal{X}}\left(\widetilde{Osc\check{M}}\right). \\
&\overset{\circ}{D}_{\check{X}^{\mathring{H}}} \check{Y}^{\mathring{V}} = D^{\top}_{\check{X}^{\mathring{H}}} \check{Y}^{\mathring{V}} + \overset{\top}{D}\left(\check{X}^{\mathring{H}}, \check{Y}^{\mathring{V}}\right),
\end{aligned}$$

If we express $\overset{\top}{D}$ in the adapted bases of $\overset{\circ}{N}$, we get:

$$\overset{\top}{D}\left(\frac{\overset{\circ}{\delta}}{\overset{\circ}{\delta} u^{\gamma}}, \frac{\overset{\circ}{\delta}}{\overset{\circ}{\delta} u^{\beta}}\right) = \underset{00}{\overset{\top}{D}}{}^{\mathring{H}\alpha}_{\ \beta\gamma} \frac{\overset{\circ}{\delta}}{\overset{\circ}{\delta} u^{\alpha}} + \underset{00}{\overset{\top}{D}}{}^{\mathring{V}\alpha}_{\ \beta\gamma} \frac{\partial}{\partial v^{a}}$$

$$\overset{\top}{D}\left(\frac{\overset{\circ}{\delta}}{\overset{\circ}{\delta} u^{\gamma}}, \frac{\partial}{\partial v^{\beta}}\right) = \underset{10}{\overset{\top}{D}}{}^{\mathring{H}\alpha}_{\ \beta\gamma} \frac{\overset{\circ}{\delta}}{\overset{\circ}{\delta} u^{\alpha}} + \underset{10}{\overset{\top}{D}}{}^{\mathring{V}\alpha}_{\ \beta\gamma} \frac{\partial}{\partial v^{a}}.$$

We have the next

Proposition 3.4 *The components of the deformation tensor $\overset{\top}{D}$ are given by the formula*:

$$\begin{aligned}
\underset{00}{\overset{\top}{D}}{}^{\mathring{H}\alpha}_{\ \beta\gamma} &= \underset{(00)}{\overset{H}{\Delta}}{}^{\alpha}_{\beta\gamma} + D^{\varphi}_{\gamma} \underset{(01)}{\overset{H}{C}}{}^{\alpha}_{\beta\varphi} \\
\underset{00}{\overset{\top}{D}}{}^{\mathring{V}\varepsilon}_{\ \beta\gamma} &= \underset{00}{\overset{\top}{D}}{}^{\mathring{H}\varepsilon}_{\ \beta\gamma} D^{\alpha}_{\varepsilon} + \delta_{\gamma} D^{\alpha}_{\beta} + D^{\varphi}_{\beta} \underset{(10)}{\overset{V}{L}}{}^{\alpha}_{\varphi\gamma} - D^{\varepsilon}_{\gamma}\left[\dot{\partial}_{1\varepsilon} D^{\alpha}_{\beta} + D^{\varphi}_{\beta} \underset{(11)}{\overset{V}{C}}{}^{\alpha}_{\varphi\varepsilon}\right] \\
&\quad - D^{\varepsilon}_{\alpha}\left(\underset{(00)}{\overset{H}{L}}{}^{\alpha}_{\beta\gamma} + \underset{(00)}{\Delta}{}^{\alpha}_{\beta\gamma}\right), \\
\underset{10}{\overset{\top}{D}}{}^{\mathring{H}\alpha}_{\ \beta\gamma} &= 0, \\
\underset{10}{\overset{\top}{D}}{}^{\mathring{V}\alpha}_{\ \beta\gamma} &= \underset{(10)}{\overset{V}{\Delta}}{}^{\alpha}_{\beta\gamma} + D^{\varphi}_{\gamma} \underset{(01)}{\overset{H}{C}}{}^{\alpha}_{\beta\varphi},
\end{aligned}$$

where $\underset{(00)}{\overset{H}{\Delta}}{}^{\alpha}_{\beta\gamma}$, $\underset{(10)}{\overset{V}{\Delta}}{}^{\alpha}_{\beta\gamma}$ *are given by (4.8).*

Proposition 3.5 *The torsion tensors of the N-linear connections* $\mathbb{R}D\Gamma\left(\overset{\circ}{N}\right)$, $D^{\top}\Gamma(\check{N})$

are related by:

$$\begin{aligned}
\mathring{T}\left(\check{X}^{\mathring{H}},\check{Y}^{\mathring{H}}\right) &= \overset{\top}{T}\left(\check{X}^{\mathring{H}},\check{Y}^{\mathring{H}}\right)+\overset{\top}{D}\left(\check{X}^{\mathring{H}},\check{Y}^{\mathring{H}}\right)-\overset{\top}{D}\left(\check{Y}^{\mathring{H}},\check{X}^{\mathring{H}}\right)\\
\mathring{T}\left(\check{X}^{\mathring{H}},\check{Y}^{\mathring{V}}\right) &= \overset{\top}{T}\left(\check{X}^{\mathring{H}},\check{Y}^{\mathring{V}}\right)+\overset{\top}{D}\left(\check{X}^{\mathring{H}},\check{Y}^{\mathring{V}}\right)\\
\mathring{T}\left(\check{X}^{\mathring{V}},\check{Y}^{\mathring{V}}\right) &= \overset{\top}{T}\left(\check{X}^{\mathring{V}},\check{Y}^{\mathring{V}}\right),\forall\check{X},\check{Y}\in\check{\mathcal{X}}\left(\widetilde{Osc\check{M}}\right).
\end{aligned}$$

Proposition 3.6 *The torsion d-tensors of the induced tangent connection of the Cartan metrical N-linear connection $D\Gamma(N)$ and of the intrinsic Cartan metrical N-linear connection $\mathring{D}\Gamma\left(\mathring{N}\right)$ are related by*:

$$\begin{aligned}
&\underset{(00)}{\overset{H}{\mathring{T}}}{}^{\alpha}_{\beta\gamma}=\underset{(00)}{\overset{H}{T}}{}^{\alpha}_{\beta\gamma}=0, \qquad \underset{(01)}{\overset{V}{\mathring{T}}}{}^{\alpha}_{\beta\gamma}=\underset{(01)}{\overset{V}{T}}{}^{\alpha}_{\beta\gamma}+\underset{00}{\overset{1}{D}}{}^{\alpha}_{\beta\gamma},\\
&\underset{(10)}{\overset{H}{\mathring{P}}}{}^{\alpha}_{\beta\gamma}=\underset{(10)}{\overset{H}{P}}{}^{\alpha}_{\beta\gamma}, \qquad \underset{(11)}{\overset{V}{\mathring{P}}}{}^{\alpha}_{\beta\gamma}=\underset{(11)}{\overset{V}{P}}{}^{\alpha}_{\beta\gamma}+\underset{01}{\overset{1}{D}}{}^{\alpha}_{\beta\gamma}-\underset{(10)}{\overset{V}{\Delta}}{}^{\alpha}_{\gamma\beta},\\
&\underset{(11)}{\overset{V}{\mathring{S}}}{}^{\alpha}_{\beta\gamma}=\underset{(11)}{S}{}^{\alpha}_{\beta\gamma}=0,
\end{aligned}$$

where $\underset{00}{\overset{1}{D}}{}^{\alpha}_{\beta\gamma},\underset{01}{\overset{1}{D}}{}^{\alpha}_{\beta\gamma},\underset{(10)}{\overset{V}{\Delta}}{}^{\alpha}_{\beta\gamma}$ *are given by (4.6), (4.7) and (4.8).*

Proposition 3.7 *The curvature 2-forms $\mathring{R}$ and $\overset{\top}{R}$ of the linear connections $\mathbb{R}D\Gamma\left(\mathring{N}\right)$ and $D^{\top}\Gamma\left(\check{N}\right)$ are related by:*

$$\begin{aligned}
\mathring{R}\left(\check{X}^{\mathring{H}},\check{Y}^{\mathring{H}}\right)\check{Z}^{\mathring{H}} &= \overset{\top}{R}\left(\check{X}^{\mathring{H}},\check{Y}^{\mathring{H}}\right)\check{Z}^{\mathring{H}}+\\
&+\left(\mathring{D}_{\check{X}^{\mathring{H}}}\overset{\top}{D}\right)\left(\check{Y}^{\mathring{H}},\check{Z}^{\mathring{H}}\right)-\left(\mathring{D}_{\check{Y}^{\mathring{H}}}\overset{\top}{D}\right)\left(\check{X}^{\mathring{H}},\check{Z}^{\mathring{H}}\right)\\
&+\overset{\top}{D}\left(\check{Y}^{\mathring{H}},\overset{\top}{D}\left(\check{X}^{\mathring{H}},\check{Z}^{\mathring{H}}\right)\right)-\overset{\top}{D}\left(\check{X}^{\mathring{H}},\overset{\top}{D}\left(\check{Y}^{\mathring{H}},\check{Z}^{\mathring{H}}\right)\right)\\
&+\overset{\top}{D}\left(\mathring{D}_{\check{X}^{\mathring{H}}}\check{Y}^{\mathring{H}},\check{Z}^{\mathring{H}}\right)-\overset{\top}{D}\left(\mathring{D}_{\check{Y}^{\mathring{H}}}\check{X}^{\mathring{H}},\check{Z}^{\mathring{H}}\right),
\end{aligned}$$

$$\begin{aligned}
\mathring{R}\left(\check{X}^{\mathring{H}},\check{Y}^{\mathring{H}}\right)\check{Z}^{\mathring{V}} &= \overset{\top}{R}\left(\check{X}^{\mathring{H}},\check{Y}^{\mathring{H}}\right)\check{Z}^{\mathring{V}}+ \\
&+\left(\mathring{D}_{\check{X}^{\mathring{H}}}\overset{\top}{D}\right)\left(\check{Y}^{\mathring{H}},\check{Z}^{\mathring{V}}\right)-\left(\mathring{D}_{\check{Y}^{\mathring{H}}}\overset{\top}{D}\right)\left(\check{X}^{\mathring{H}},\check{Z}^{\mathring{V}}\right) \\
&+\overset{\top}{D}\left(\check{Y}^{\mathring{H}},\overset{\top}{D}\left(\check{X}^{\mathring{H}},\check{Z}^{\mathring{V}}\right)\right)-\overset{\top}{D}\left(\check{X}^{\mathring{H}},\overset{\top}{D}\left(\check{Y}^{\mathring{H}},\check{Z}^{\mathring{V}}\right)\right) \\
&+\overset{\top}{D}\left(\mathring{D}_{\check{X}^{\mathring{H}}}\check{Y}^{\mathring{H}},\check{Z}^{\mathring{V}}\right)-\overset{\top}{D}\left(\mathring{D}_{\check{Y}^{\mathring{H}}}\check{X}^{\mathring{H}},\check{Z}^{\mathring{V}}\right), \\
\mathring{R}\left(\check{X}^{\mathring{V}},\check{Y}^{\mathring{H}}\right)\check{Z}^{\mathring{H}} &= \overset{\top}{R}\left(\check{X}^{\mathring{V}},\check{Y}^{\mathring{H}}\right)\check{Z}^{\mathring{H}}+ \\
&+\left(\mathring{D}_{\check{X}^{\mathring{V}}}\overset{\top}{D}\right)\left(\check{Y}^{\mathring{H}},\check{Z}^{\mathring{H}}\right)+\overset{\top}{D}\left(\mathring{D}_{\check{X}^{\mathring{V}}}\check{Y}^{\mathring{H}},\check{Z}^{\mathring{H}}\right), \\
\mathring{R}\left(\check{X}^{\mathring{V}},\check{Y}^{\mathring{H}}\right)\check{Z}^{\mathring{V}} &= \overset{\top}{R}\left(\check{X}^{\mathring{V}},\check{Y}^{\mathring{H}}\right)\check{Z}^{\mathring{V}}+ \\
&+\left(\mathring{D}_{\check{X}^{\mathring{V}}}\overset{\top}{D}\right)\left(\check{Y}^{\mathring{H}},\check{Z}^{\mathring{V}}\right)+\overset{\top}{D}\left(\mathring{D}_{\check{X}^{\mathring{V}}}\check{Y}^{\mathring{H}},\check{Z}^{\mathring{V}}\right), \\
\mathring{R}\left(\check{X}^{\mathring{V}},\check{Y}^{\mathring{V}}\right)\check{Z}^{\mathring{H}} &= \overset{\top}{R}\left(\check{X}^{\mathring{V}},\check{Y}^{\mathring{V}}\right)\check{Z}^{\mathring{H}}, \\
\mathring{R}\left(\check{X}^{\mathring{V}},\check{Y}^{\mathring{V}}\right)\check{Z}^{\mathring{V}} &= \overset{\top}{R}\left(\check{X}^{\mathring{V}},\check{Y}^{\mathring{V}}\right)\check{Z}^{\mathring{V}},\forall\check{X},\check{Y},\check{Z}\in\check{\mathcal{X}}\left(\widetilde{Osc\check{M}}\right).
\end{aligned}$$

Proposition 3.8 *The curvature d-tensors of the induced tangent connection of the Cartan metrical* N*-linear connection* $D\Gamma(N)$ *and of the intrinsic Cartan metrical* N*-linear connection* $\mathring{D}\Gamma\left(\mathring{N}\right)$ *of the submanifold* $\widetilde{Osc\check{M}}$ *are related by*:

$$\underset{(00)}{\overset{H}{\mathring{R}}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta}=\underset{(00)}{\overset{H}{R}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta}+\underset{(00)}{\overset{H}{\Delta}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta},\qquad \underset{(01)}{\overset{V}{\mathring{R}}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta}=\underset{(01)}{\overset{V}{R}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta}+\underset{(01)}{\overset{V}{\Delta}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta},$$

$$\underset{(10)}{\overset{H}{\mathring{P}}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta}=\underset{(10)}{\overset{H}{P}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta}+\underset{(10)}{\overset{H}{\Delta}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta},\qquad \underset{(11)}{\overset{V}{\mathring{P}}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta}=\underset{(11)}{\overset{V}{P}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta}+\underset{(11)}{\overset{V}{\Delta}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta},$$

$$\underset{(1i)}{\overset{V_i}{\mathring{S}}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta}=\underset{(1i)}{\overset{V_i}{S}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta},\ (i=0,1;V_0=H),$$

where

$$\begin{aligned}
\overset{H}{\underset{(00)}{\Delta}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} &= \delta_{\delta}\overset{H}{\underset{(00)}{\Delta}}{}^{\alpha}_{\beta\gamma} - \delta_{\gamma}\overset{H}{\underset{(00)}{\Delta}}{}^{\alpha}_{\beta\delta} - D^{\varepsilon}{}_{\delta}\dot{\partial}_{1\varepsilon}\overset{H}{\underset{(00)}{L}}{}^{\alpha}_{\beta\gamma} - D^{\varepsilon}{}_{\delta}\dot{\partial}_{1\varepsilon}\overset{H}{\underset{(00)}{\Delta}}{}^{\alpha}_{\beta\gamma} \\
&\quad + D^{\varepsilon}{}_{\gamma}\dot{\partial}_{1\varepsilon}\overset{H}{\underset{(00)}{L}}{}^{\alpha}_{\beta\delta} + D^{\varepsilon}{}_{\gamma}\dot{\partial}_{1\varepsilon}\overset{H}{\underset{(00)}{\Delta}}{}^{\alpha}_{\beta\delta} \\
&\quad + \overset{H}{\underset{(00)}{L}}{}^{\sigma}_{\beta\gamma}\overset{H}{\underset{(00)}{\Delta}}{}^{\alpha}_{\sigma\delta} + \overset{H}{\underset{(00)}{\Delta}}{}^{\sigma}_{\beta\gamma}\overset{H}{\underset{(00)}{L}}{}^{\alpha}_{\sigma\delta} - \overset{H}{\underset{(00)}{L}}{}^{\sigma}_{\beta\delta}\overset{H}{\underset{(00)}{\Delta}}{}^{\alpha}_{\sigma\gamma} - \overset{H}{\underset{(00)}{\Delta}}{}^{\sigma}_{\beta\delta}\overset{H}{\underset{(00)}{L}}{}^{\alpha}_{\sigma\gamma} \\
&\quad + \overset{H}{\underset{(01)}{C}}{}^{\alpha}_{\beta\sigma}\overset{1}{\underset{00}{D}}{}^{\sigma}_{\beta\gamma},
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
\overset{V}{\underset{(01)}{\Delta}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} &= \underset{1}{D}{}^{\varepsilon}{}_{\gamma}\dot{\partial}_{1\varepsilon}\overset{V}{\underset{(10)}{L}}{}^{\alpha}_{\beta\delta} - \underset{1}{D}{}^{\varepsilon}{}_{\delta}\dot{\partial}_{1\varepsilon}\overset{V}{\underset{(10)}{L}}{}^{\alpha}_{\beta\gamma} + \frac{1}{2}\delta_{\delta}\left(\overset{V}{\underset{(10)}{\Delta}}{}^{\alpha}_{\beta\gamma}\right) - \frac{1}{2}\delta_{\gamma}\left(\overset{V}{\underset{(10)}{\Delta}}{}^{\alpha}_{\beta\delta}\right) \\
&\quad + \frac{1}{2}\underset{1}{D}{}^{\varepsilon}{}_{\gamma}\dot{\partial}_{1\varepsilon}\left(\overset{V}{\underset{(10)}{\Delta}}{}^{\alpha}_{\beta\delta}\right) - \frac{1}{2}\underset{1}{D}{}^{\varepsilon}{}_{\delta}\dot{\partial}_{1\varepsilon}\left(\overset{V}{\underset{(10)}{\Delta}}{}^{\alpha}_{\beta\gamma}\right) + \\
&\quad \frac{1}{2}\overset{V}{\underset{(10)}{L}}{}^{\varepsilon}_{\beta\gamma}\overset{V}{\underset{(10)}{\Delta}}{}^{\alpha}_{\varepsilon\delta} - \frac{1}{2}\overset{V}{\underset{(10)}{L}}{}^{\varepsilon}_{\beta\delta}\overset{V}{\underset{(10)}{\Delta}}{}^{\alpha}_{\varepsilon\gamma} + \frac{1}{2}\overset{V}{\underset{(10)}{\Delta}}{}^{\varepsilon}_{\beta\gamma}\overset{V}{\underset{(10)}{L}}{}^{\alpha}_{\varepsilon\delta} - \frac{1}{2}\overset{V}{\underset{(10)}{\Delta}}{}^{\varepsilon}_{\beta\delta}\overset{V}{\underset{(10)}{L}}{}^{\alpha}_{\varepsilon\gamma} \\
&\quad + \frac{1}{4}\left(\overset{V}{\underset{(10)}{\Delta}}{}^{\alpha}_{\varepsilon\delta}\overset{V}{\underset{(10)}{\Delta}}{}^{\varepsilon}_{\beta\gamma} - \overset{V}{\underset{(10)}{\Delta}}{}^{\alpha}_{\varepsilon\gamma}\overset{V}{\underset{(10)}{\Delta}}{}^{\varepsilon}_{\beta\delta}\right) + \overset{V}{\underset{(11)}{C}}{}^{\alpha}_{\beta\sigma}\overset{V}{\underset{(10)}{\Delta}}{}^{\sigma}_{\gamma\delta},
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
\overset{V}{\underset{(10)}{\Delta}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} &= -\overset{H}{\underset{(00)}{\Delta}}{}^{\alpha}_{\varepsilon\gamma}\overset{H}{\underset{(01)}{C}}{}^{\varepsilon}_{\beta\delta} + \overset{H}{\underset{(00)}{\Delta}}{}^{\varepsilon}_{\beta\gamma}\overset{H}{\underset{(01)}{C}}{}^{\alpha}_{\varepsilon\delta} + \overset{H}{\underset{(01)}{C}}{}^{\alpha}_{\beta\varepsilon}\overset{1}{\underset{01}{D}}{}^{\varepsilon}_{\gamma\delta} + \dot{\partial}_{1\delta}\overset{H}{\underset{(00)}{\Delta}}{}^{\alpha}_{\beta\gamma} \\
&\quad + D^{\sigma}{}_{\gamma}\dot{\partial}_{1\sigma}\overset{H}{\underset{(01)}{C}}{}^{\alpha}_{\beta\delta},
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
\overset{V}{\underset{(11)}{\Delta}}{}_{\beta}{}^{\alpha}{}_{\gamma\delta} &= \frac{1}{2}\dot{\partial}_{1\delta}\left(\overset{V}{\underset{(10)}{\Delta}}{}^{\alpha}_{\beta\gamma}\right) - \frac{1}{2}\left(\overset{V}{\underset{(10)}{\Delta}}{}^{\alpha}_{\varepsilon\gamma}\overset{V}{\underset{(11)}{C}}{}^{\varepsilon}_{\beta\delta} - \overset{V}{\underset{(11)}{C}}{}^{\alpha}_{\varepsilon\delta}\overset{V}{\underset{(10)}{\Delta}}{}^{\varepsilon}_{\beta\gamma} - \right. \\
&\quad \left. - \overset{V}{\underset{(11)}{C}}{}^{\alpha}_{\beta\varepsilon}\left(\overset{V}{\underset{(10)}{\Delta}}{}^{\varepsilon}_{\delta\gamma} - \overset{V}{\underset{(10)}{\Delta}}{}^{\varepsilon}_{\gamma\delta}\right)\right),
\end{aligned} \tag{4.13}$$

where $\overset{H}{\underset{(00)}{\Delta}}{}^{\alpha}_{\beta\gamma}$, $\overset{V}{\underset{(10)}{\Delta}}{}^{\alpha}_{\beta\gamma}$ *are given by (4.8).*

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