Bulletin of the *Transilvania* University of Braşov • Vol 6(55), No. 1 - 2013
Series III: Mathematics, Informatics, Physics, 31-36

ON QUANTITATIVE ESTIMATION FOR THE LIMITING SEMIGROUP OF LINEAR POSITIVE OPERATORS

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Abstract

We give an improved quantitative estimation of the approximation of iterates for a sequence of positive linear operators $(L_n)_n$ to their limting C_0-semigroup $T(t)$, $0 \le t < \infty$, as given in Trotter's theorem.

2010 *Mathematics Subject Classification:* 41A36.
Key words: positive linear operators, Trotter theorem, limiting semigoup of operators, Durrmeyer type operators

1 Introduction

The study of iterates of Bernstein operators was initiated by Kelisky and Rivlin [9] and the first quantitative results were be obtained by Nagel [11] and Gonska [6]. The semigroup of operators generated by iterates of Bernstein operators was considered by da Silva [5]. More generally, as application to the Trotter's theorem [13] the semigroups generated by linear positive operators are considered in the last decades. For a general reference on semigroups of operators we cite [2] and [1]. Recently there were obtained quantitative results for Trotter's theorem in certain contexts, see: [7], [10], [3], [8].

Our aim is to give a modified method then in the paper by Gonska and Raşa [7] for a quantitative version of the Trotter's theorem in the case of a class of sequences of positive linear operators described below, which includes the sequence of Bernstein operators. We show that our method leads to estimates with better constants than in [7], when we apply this method to Bernstein operators.

Let $(L_n)_n : C[0,1] \to C^4[0,1]$ be a sequence of positive linear operators which preserves linear functions. We consider that L_n are convex of orders i, for $0 \le i \le 4$, i.e. if $f \in C^i[0,1]$ and $f^{(i)} \ge 0$ on $[0,1]$, then $(L_n(f))^{(i)} \ge 0$ on $[0,1]$. Denote

$$M_n^k(x) := L_n((t-x)^k, x) \; k \in \mathbb{N}, \; x \in [0,1].$$

We suppose that $M_n^2(x) = a_n x(1-x)$, $x \in [0,1]$, where $a_n = \mathrm{O}\left(\frac{1}{n}\right)$, $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} n(L_n(f,x) - f(x)) = \frac{a}{2} \cdot x(1-x) \cdot f''(x), \text{ uniformly, for } f \in C^2[0,1],$$

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where $a = \lim_{n\to\infty} na_n$.

We consider the differential operator $A : D(A) = C^2[0,1] \to C[0,1]$, given by $A(f)(x) = \frac{a}{2}x(1-x)f''(x)$, $f \in C^2[0,1]$, $x \in [0,1]$. Note that $D(A)$ is dense in $C[0,1]$.

In the condition above, from the general result of Trotter, we deduce that there exists a C_0-semigroup $T(t)$, such that

$$\lim_{n\to\infty} L_n^{m_n} f = T(t)f,\ \ f \in C[0,1] \tag{1}$$

if $\frac{m_n}{n} \to t$, $t \geq 0$.

2 Main results

Firstly, we present two auxiliary general results.

Lemma 1. *For every $g \in C^4[0,1]$, we have:*

$$\|L_n g - g - \frac{1}{n}Ag\| \leq \frac{1}{6}\|M_n^3\|\|g^{(3)}\| + \frac{|a|}{8}\left|\frac{a_n}{a} - \frac{1}{n}\right| \cdot \|g''\|.$$

Proof. Let $x \in [0,1]$ be fixed. For $t \in [0,1]$, with Taylor's formula, we write:

$$\left|g(t) - g(x) - (t-x)g'(x) - \frac{1}{2}(t-x)^2 g''(x)\right| \leq \frac{1}{6}|t-x|^3\|g^{(3)}\|.$$

Since L_n reproduces linear functions we obtain:

$$\|L_n g - g - \frac{a_n}{a}Ag\| \leq \frac{1}{6}\|M_n^3\|\|g^{(3)}\|,$$

then we write

$$\begin{aligned}
\|L_n g - g - \frac{1}{n}Ag\| &\leq \|L_n g - g - \frac{a_n}{a}Ag\| + \|\frac{a_n}{a}Ag - \frac{1}{n}Ag\| \\
&\leq \frac{1}{6}\|M_n^3\|\|g^{(3)}\| + \left|\frac{a_n}{a} - \frac{1}{n}\right| \cdot \|Ag\| \\
&\leq \frac{1}{6}\|M_n^3\|\|g^{(3)}\| + \frac{|a|}{8}\left|\frac{a_n}{a} - \frac{1}{n}\right| \cdot \|g''\|.
\end{aligned}$$

□

Lemma 2. *For every $g \in C^4[0,1]$, we have:*

$$\left\|T\left(\frac{1}{n}\right)g - g - \frac{1}{n}Ag\right\| \leq \frac{a^2}{128n^2}(8\|g''\| + 8\|g^{(3)}\| + \|g^{(4)}\|).$$

Proof. From

$$\|T(t)g - g - tAg\| \leq \frac{t^2}{2}\|A^2 g\|, \; t \geq 0.$$

Denoting $\Psi(x) = x(1-x)$, $x \in [0,1]$ we obtain

$$A^2(g) = A\Big(\frac{a}{2}\Psi g''\Big) = \frac{a^2}{4}\Psi\Big(\Psi g^{(4)} + 2\Psi' g^{(3)} + \Psi'' g''\Big).$$

Taking $t = \frac{1}{n}$ in the relation given above, we get

$$\|T\left(\frac{1}{n}\right) g - g - \frac{1}{n}Ag\| \leq \frac{1}{2n^2}\|A^2 g\| \leq \frac{a^2}{128n^2}(8\|g''\| + 8\|g^{(3)}\| + \|g^{(4)}\|).$$

□

Lemma 3. *Let $f \in C^k[0,1]$, $0 \leq k \leq 4$ and fix $n, j \geq 0$. The following estimation holds:*

$$\|(L_n^j f)^{(k)}\| \leq (\sigma_k)^j \|f^{(k)}\|,$$

where $\sigma_k := \frac{1}{k!}(L_n e_k)^{(k)}$.

Proof. Denote $g := \frac{1}{k!}\|f^{(k)}\| e_k \pm f$. It is clear that $g^{(k)} \geq 0$. Since L_n is convex of order k, we also have $(L_n g)^{(k)} \geq 0$, hence

$$\|L_n^{(k)} f\| \leq \frac{1}{k!}\|f^{(k)}\|(L_n e_k)^{(k)}$$

and by induction we obtain

$$\|(L_n^j f)^{(k)}\| \leq \Big(\frac{1}{k!}(L_n e_k)^{(k)}\Big)^j \|f^{(k)}\|.$$

□

Next, we present the main result of the paper.

Theorem 1. *Let $f \in C^4[0,1]$. The following estimation holds:*

$$\|L_n^m f - T\,(t)\,f\| \leq \Big[\Big(\frac{|a|}{8}\left|\frac{a_n}{a} - \frac{1}{n}\right| + \frac{a^2}{16n^2}\Big)\frac{1-(\sigma_2)^m}{1-\sigma_2} + \frac{|a|}{8}\left|\frac{m}{n} - t\right|\Big]\|f''\|$$
$$+ \Big(\frac{1}{6}\|M_n^3\| + \frac{a^2}{16n^2}\Big)\frac{1-(\sigma_3)^m}{1-\sigma_3}\|f^{(3)}\| + \frac{a^2}{128n^2}\frac{1-(\sigma_4)^m}{1-\sigma_4}\|f^{(4)}\|.$$

Proof. Firstly:

$$\begin{aligned} ||L_n^m f - T\,(t)\,f|| &\leq \left\|L_n^m f - T\left(\frac{m}{n}\right)f\right\| + \left\|T\left(\frac{m}{n}\right)f - T\,(t)\,f\right\| \\ &\leq \left\|L_n^m f - T\left(\frac{m}{n}\right)f\right\| + \left\|\int_t^{\frac{m}{n}} T(u)Af du\right\| \\ &\leq \left\|L_n^m f - T\left(\frac{m}{n}\right)f\right\| + \left|\frac{m}{n} - t\right| ||Af|| \\ &\leq \left\|L_n^m f - T\left(\frac{m}{n}\right)f\right\| + \frac{|a|}{8}\left|\frac{m}{n} - t\right| ||f''||\,. \end{aligned} \tag{2}$$

Now, using a telescopic sum and $\|T(t)\| = 1,\ t \geq 0$, we write:

$$\left\|L_n^m f - T\left(\frac{m}{n}\right) f\right\| = \left\|\sum_{j=0}^{m-1} T\left(\frac{m-1-j}{n}\right)\left(L_n - T\left(\frac{1}{n}\right)\right) L_n^j f\right\| \leq$$

$$\leq \sum_{j=0}^{m-1} \left\|\left(L_n - T\left(\frac{1}{n}\right)\right) L_n^j f\right\|. \tag{3}$$

Denote $g := L_n^j f \in C^4[0,1]$. It is convenient to write:

$$\left\|\left(L_n - T\left(\frac{1}{n}\right)\right) g\right\| \leq \left\|L_n g - g - \frac{1}{n} A g\right\| + \left\|T\left(\frac{1}{n}\right) g - g - \frac{1}{n} A g\right\|.$$

Applying Lemmas 1 and 2, we can continue with

$$\left\|\left(L_n - T\left(\frac{1}{n}\right)\right) g\right\| \leq \frac{1}{6}\|M_n^3\|\|g^{(3)}\| + \frac{|a|}{8}\left|\frac{a_n}{a} - \frac{1}{n}\right|\|g''\|$$

$$+ \frac{a^2}{128n^2}(8\|g''\| + 8\|g^{(3)}\| + \|g^{(4)}\|)$$

$$= \left(\frac{|a|}{8}\left|\frac{a_n}{a} - \frac{1}{n}\right| + \frac{a^2}{16n^2}\right)\|g''\| + \left(\frac{1}{6}\|M_n^3\| + \frac{a^2}{16n^2}\right)\|g^{(3)}\| + \frac{a^2}{128n^2}\|g^{(4)}\|.$$

Now, applying Lemma 3, we have $\|g^{(j)}\| \leq (\sigma_k)^j \|f^{(j)}\|$, for $j = 2, 3, 4$.

Then, from (3)

$$\left\|L_n^m f - T\left(\frac{m}{n}\right) f\right\| \leq \left(\frac{|a|}{8}\left|\frac{a_n}{a} - \frac{1}{n}\right| + \frac{a^2}{16n^2}\right)\frac{1-\sigma_2^m}{1-\sigma_2}\|f''\|$$

$$+ \left(\frac{1}{6}\|M_n^3\| + \frac{a^2}{16n^2}\right)\frac{1-\sigma_3^m}{1-\sigma_3}\|f^{(3)}\| + \frac{a^2}{128n^2}\frac{1-\sigma_4^m}{1-\sigma_4}\|f^{(4)}\|. \tag{4}$$

Combining (2) and (4) we now get the conclusion.

Remark 1. From the estimate given in Theorem 1 one could obtain estimates with moduli of smoothness, similarly as in [7]. See the method given there.

□

3 Applications

Denote

$$p_{nk}(x) := \binom{n}{k} x^k (1-x)^{n-k},\ x \in [0,1].$$

I. Consider the sequence of Bernstein operators $(B_n)_n$ given by

$$B_n(f,x) = \sum_{k=0}^{n} p_{nk}(x) f\left(\frac{k}{n}\right),\ f \in C[0,1].$$

For these operators, we have $a = 1$, $a_n = \frac{1}{n}$, $\|M_n^3\| = \frac{1}{4n^2}$ and, from [4] we deduce:

$$\begin{aligned}\sigma_2 &= \frac{n-1}{n}\\ \sigma_3 &= \frac{(n-1)(n-2)}{n^2}\\ \sigma_4 &= \frac{(n-1)(n-2)(n-3)}{n^3}.\end{aligned}$$

Hence, we find the following estimation:

$$\begin{aligned}\|B_n^m f - T(t)f\| \leq & \left(\frac{1-(\sigma_2)^m}{16n} + \frac{1}{8}\left|\frac{m}{n} - t\right|\right)\|f''\| + \frac{5(1-(\sigma_2)^m)}{48(3n-2)}\|f^{(3)}\|\\ & + \frac{n(1-(\sigma_4)^m)}{128(6n^2-11n+6)}\|f^{(4)}\|.\end{aligned}$$

As consequence, we obtain:

$$\begin{aligned}\|B_n^m f - T(t)f\| \leq & \left(\frac{1}{16n} + \frac{1}{8}\left|\frac{m}{n} - t\right|\right)\|f''\| + \frac{5}{48(3n-2)}\|f^{(3)}\|\\ & + \frac{n}{128(6n^2-11n+6)}\|f^{(4)}\|.\end{aligned}$$

These estimate improves the following estimate given in [7] - formula (12):

$$\|B_n^m f - T(t)f\| \leq \frac{1}{8}\left(\left|\frac{m}{n} - t\right| + \frac{1}{2n-1}\right)\|f''\| + \frac{5}{96(2n-1)}\left(4\|f^{(3)}\| + \|f^{(4)}\|\right).$$

II. For the sequence of genuine Durrmeyer operators $(U_n)_n$, given by

$$U_n(f,x) = (1-x)^n f(0) + x^n f(1) + (n-1)\sum_{k=1}^{n} p_{nk}(x)\int_0^1 p_{n-2,k-1}(t)f(t)dt$$

we find $a = 1$, $a_n = \frac{2}{2n+1}$, $\|M_n^3\| = \frac{3}{2(2n+1)(2n+2)}$ and from [12], pag. 149 we deduce:

$$\begin{aligned}\sigma_2 &= \frac{n-1}{n+1}\\ \sigma_3 &= \frac{(n-1)(n-2)}{(n+1)(n+2)}\\ \sigma_4 &= \frac{(n-1)(n-2)(n-3)}{(n+1)(n+2)(n+3)}.\end{aligned}$$

Thus, we can write the following estimation:

$$\begin{aligned}\|U_n^m f - T(t)f\| \leq & \left[(n+1)\left(\frac{1}{16n(2n+1)} + \frac{1}{32n^2}\right)(1-\sigma_2^m) + \frac{1}{8}\left|\frac{m}{n} - t\right|\right]\|f''\|\\ & + \left(\frac{1}{4(2n+1)(2n+2)} + \frac{1}{16n^2}\right)\frac{n^2+3n+2}{6n}(1-\sigma_3^m)\|f^{(3)}\|\\ & + \frac{n^3+6n^2+11n+6}{1536n^2(n^2+1)}(1-\sigma_4^m)\|f^{(4)}\|.\end{aligned}$$

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