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AN INDUCTIVE GENERALIZATION OF 1-DIFFERENTIABLE COHOMOLOGY

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Abstract

In this note we give an inductive generalization of 1-differentiable cohomology and Betti numbers of a smooth manifold endowed with a non-closed 1–form (for instance a contact manifold). Some cohomology classes in such a generalized cohomology are studied and also, an inductive generalization of coeffective 1-differentiable cohomology is considered and studied in the paper.

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1 Introduction

The 1-differentiable cohomology was introduced and intensively studied by A. Lichnerowicz in [10, 6] in the context of symplectic and contact manifolds and in [11, 12] in the context of Poisson or Jacobi manifolds. We also notice that 1-differentiable cohomology is isomorphic with the following cohomology: for every non-closed 1-form η on a smooth manifold M we define a coboundary operator d_1 on the complex $\Omega_1^\bullet(M) = \Omega^\bullet(M) \oplus \Omega^{\bullet-1}(M)$ by $d_1(\varphi, \psi) = (d\varphi - d\eta \wedge \psi, -d\psi)$, where $\Omega^\bullet(M) = \oplus_{p\geq 0}\Omega^p(M)$; $\Omega^p(M)$ is the space of p-forms on M. The resulting cohomology is named here d_1-cohomology of M. On the other hand, the coeffective cohomology was introduced by T. Bouché [2] for symplectic manifolds. Further signifiant developments are given in [3, 4, 5].

The purpose of this note is to present an inductive generalization of 1-differentiable cohomology (or d_1-cohomology) and Betti numbers of a smooth manifold endowed with a non-closed 1–form (for instance a contact manifold). Some cohomology classes in a such generalized cohomology are studied and also, an inductive generalization of coeffective 1-differentiable cohomology is considered and studied in the paper.

The paper is organized as follows. In Section 2 we construct an inductive generalization of 1-differentiable cohomology and Betti numbers of a smooth manifold endowed with a non-closed 1–form. In Section 3 we construct some cohomology classes in our generalized cohomology. In Section 4 we present an inductve generalization of coeffective 1–differentiable cohomology and the case of an almost contact manifold is studied.

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2 A generalization of 1-differentiable cohomology and Betti numbers

Let M be a smooth manifold of dimension n endowed with a non-closed 1–form η. We consider the field $\Omega^0(M) = \mathcal{F}(M)$ of smooth real valued functions defined on M. For each $p = 1, \dots, n = \dim M$ denote by $\Omega^p(M)$ the module of p-forms on M and by $\Omega(M) = \oplus_{p\geq 0}\Omega^p(M)$ the exterior algebra of M.

We denote by $\Omega_0^p(M) = \Omega^p(M)$ and

$$\Omega_1^p(M) = \Omega^p(M) \oplus \Omega^{p-1}(M) = \{\varphi_1 = (\varphi, \psi) | \varphi \in \Omega^p(M), \psi \in \Omega^{p-1}(M)\}.$$

After a terminology used in [10] we call $\varphi_1 = (\varphi, \psi)$ an 1-*differentiable p-form.*

In this manner, for any integer $k > 1$ we define

$$\begin{aligned}\Omega_k^p(M) &= \Omega_{k-1}^p(M) \oplus \Omega_{k-1}^{p-1}(M) \\ &= \{\varphi_k = (\varphi_{k-1}, \psi_{k-1}) | \varphi_{k-1} \in \Omega_{k-1}^p(M), \psi_{k-1} \in \Omega_{k-1}^{p-1}(M)\}\end{aligned}$$

and its elements are called *generalized* 1-*differentiable p-forms of order k.*

Now we give an inductive generalization of some classical operators on smooth manifolds. For any p-form φ we have the operators:

$$e_\eta : \Omega^p(M) \to \Omega^{p+1}(M)\,,\ e_\eta\varphi = \eta \wedge \varphi,$$

$$L : \Omega^p(M) \to \Omega^{p+2}(M)\,,\ L\varphi = d\eta \wedge \varphi,$$

where d is the exterior derivative on M and η is a non-closed 1–form on M.

Considering $e_{0,\eta} = e_\eta$, $L_0 = L$ and $d_0 = d$, for any integer $k > 0$ we define

$$e_{k,\eta} : \Omega_k^p(M) \to \Omega_k^{p+1}(M),\ e_{k,\eta}\varphi_k = (e_{k-1,\eta}\varphi_{k-1}, -e_{k-1,\eta}\psi_{k-1}),$$

$$L_k : \Omega_k^p(M) \to \Omega_k^{p+2}(M),\ L_k\varphi_k = (L_{k-1}\varphi_{k-1}, L_{k-1}\psi_{k-1})$$

and

$$d_k : \Omega_k^p(M) \to \Omega_k^{p+1}(M),\ d_k\varphi_k = (d_{k-1}\varphi_{k-1} - L_{k-1}\psi_{k-1}, -d_{k-1}\psi_{k-1}).$$

We also denote $0_1 = (0, 0)$ and $0_k = (0_{k-1}, 0_{k-1})$ for any integer $k > 1$.

Proposition 2.1. *For any integer $k > 0$ we have*

$$L_k e_{k,\eta} = e_{k,\eta} L_k,\ d_k e_{k,\eta} + e_{k,\eta} d_k = L_k,\ d_k L_k = L_k d_k,\ d_k^2 = 0_k. \tag{2.1}$$

Proof. It follows by induction after k. For $k = 1$ we have

$$L_1 e_{1,\eta}\varphi_1 = L_1(e_\eta\varphi, -e_\eta\psi) = (Le_\eta\varphi, -Le_\eta\psi)$$

and

$$e_{1,\eta}L_1\varphi_1 = e_{1,\eta}(L\varphi, L\psi) = (e_\eta L\varphi, -e_\eta L\psi).$$

Using the relation $Le_\eta = e_\eta L$ we obtain the first relation of (2.1) for $k = 1$.

Next we suppose that this relation is valid for $k = j$ and we prove it for $k = j + 1$.

$$L_{j+1}e_{j+1,\eta}\varphi_{j+1} = L_{j+1}(e_{j,\eta}\varphi_j, -e_{j,\eta}\psi_j) = (L_je_{j,\eta}\varphi_j, -L_je_{j,\eta}\psi_j),$$

$$e_{j+1,\eta}L_{j+1}\varphi_{j+1} = e_{j+1,\eta}(L_j\varphi_j, L_j\psi_j) = (e_{j,\eta}L_j\varphi_j, -e_{j,\eta}L_j\psi_j)$$

and so $L_ke_{k,\eta} = e_{k,\eta}L_k$ for any integer $k > 0$. The other three relations of (2.1) follow in a similar manner by induction after k, using the following known relations:

$$de_\eta + e_\eta d = L,\ dL = Ld,\ d^2 = 0.$$

□

Denote by $H_k^\bullet(M)$ the cohomology groups of the differential complex $(\Omega_k^\bullet(M), d_k)$. If these cohomologies are finite dimensional, then we denote $b_{k,p}(M) = \dim H_k^p(M)$ and we call it the *generalized p-Betti numbers of order k of M*.

Theorem 2.1. *Let M be a smooth manifold endowed with a non-closed 1–form η. Then*

$$b_{k,p}(M) = \sum_{t=0}^{k} C_k^t b_{p-t}(M) \tag{2.2}$$

where $b_p(M)$ are the classical p-Betti numbers of M and $C_k^t = \begin{pmatrix} k \\ t \end{pmatrix}$ are the binomial numbers.

Proof. It follows by induction after k.

For $k = 1$ we consider the mappings:

$$\alpha_0 : \Omega_0^p(M) \to \Omega_1^p(M),\ \alpha_0(\varphi_0) = (\varphi_0, 0),$$

$$\beta_1 : \Omega_1^p(M) \to \Omega_0^{p-1}(M),\ \beta_1(\varphi_0, \psi_0) = \psi_0,$$

for all $\varphi_0 \in \Omega_0^p(M)$ and $\psi_0 \in \Omega_0^{p-1}(M)$. Then we can relate the cohomology $H_1^\bullet(M)$ with the de Rham cohomology $H^\bullet(M) = H_{dR}^\bullet(M)$ in the following manner:

The mappings α_0 and β_1 induce an exact sequence of complexes

$$0 \to (\Omega^\bullet(M), d) \xrightarrow{\alpha_0} (\Omega_1^\bullet(M), d_1) \xrightarrow{\beta_1} (\Omega^{\bullet-1}(M), -d) \to 0.$$

This exact sequence induces a long exact cohomology sequence

$$\ldots \to H_{dR}^p(M) \xrightarrow{\alpha_0^*} H_1^p(M) \xrightarrow{\beta_1^*} H_{dR}^{p-1}(M) \xrightarrow{\delta_{1,p-1}^*} H_{dR}^{p+1}(M) \to \ldots$$

where α_0^* and β_1^* are the homomorphisms induced by α_0^* and β_1^*, respectively and $\delta_{1,p-1}^*$ is the connecting homomorphism defined by

$$\delta_{1,p-1}^*[\psi_0] = [-L\psi_0],\ \forall\,[\psi_0] \in H_{dR}^{p-1}(M).$$

Since $L = e_\eta d + de_\eta$ it follows that $\delta^*_{1,p-1} = 0$ and then we have the isomorphism

$$H^p_1(M) \approx H^p_{dR}(M) \oplus H^{p-1}_{dR}(M)$$

and consequently $b_{1,p}(M) = b_p(M) + b_{p-1}(M)$. Thus the theorem is proved for $k = 1$.

Now we suppose that relation (2.2) is valid for $k = j$ and we prove it for $k = j + 1$.

In a similar manner with the previous reasoning, we consider the mappings:

$$\alpha_j : \Omega^p_j(M) \to \Omega^p_{j+1}(M),\ \alpha_j(\varphi_j) = (\varphi_j, 0_j),$$

$$\beta_{j+1} : \Omega^p_{j+1}(M) \to \Omega^{p-1}_j(M),\ \beta_{j+1}(\varphi_j, \psi_j) = \psi_j,$$

for all $\varphi_j \in \Omega^p_j(M)$ and $\varphi_{j+1} = (\varphi_j, \psi_j) \in \Omega^p_{j+1}(M)$. Then, since $L_j = e_{j,\eta} d_j + d_j e_{j,\eta}$ we obtain that the connecting homomorphism $\delta^*_{j+1,p-1}$ defined by $\delta^*_{j+1,p-1}[\psi_j] = [-L_j \psi_j]$ vanishes and hence the following isomorphism holds:

$$H^p_{j+1}(M) \approx H^p_j(M) \oplus H^{p-1}_j(M)$$

and consequently $b_{j+1,p}(M) = b_{j,p}(M) + b_{j,p-1}(M)$. By induction hypothesis we have

$$b_{j,p}(M) = \sum_{t=0}^{j} C^t_j b_{p-t}(M),\ b_{j,p-1}(M) = \sum_{t=0}^{j} C^t_j b_{p-t-1}(M)$$

and so

$$b_{j+1,p}(M) = \sum_{t=0}^{j+1} C^t_{j+1} b_{p-t}(M)$$

which ends the proof. □

3 Some d_k–cohomology classes

It is easy to see that in the case $k = 1$ the 1–differentiable form $(\eta, 1)$ is d_1–closed and its d_1–cohomology class $[(\eta, 1)]$ in $H^1_1(M)$ is non-vanishing, see [8].

In the following we intend to find similar non-vanishing cohomology classes in d_k–cohomology $H^1_k(M)$, for any integer $k > 1$. For this reason we define

$$\eta_k = -(\eta_{k-1}, 2_{k-1}) \in \Omega^1_k(M),$$

where $\eta_0 = \eta$ and $2_k = (2_{k-1}, 0_{k-1})$ with $2_0 = 2$.

Proposition 3.1. *For every $k \geq 2$, the generalized 1–differentiable 1-forms of order k defined by $(\eta_{k-1}, 1_{k-1})$, where $1_k = (1_{k-1}, 0_{k-1})$ with $1_1 = (1, 0)$, are d_k–closed and their associated cohomolgy classes $[(\eta_{k-1}, 1_{k-1})]$ in $H^1_k(M)$ are non-vanishing.*

Proof. It follows by induction after k. For $k = 2$, we have

$$\begin{aligned} d_2(\eta_1, 1_1) &= d_2(-(\eta, 2), (1, 0)) \\ &= (-d_1(\eta, 2) - L_1(1, 0), -d_1(1, 0)) \\ &= ((-d\eta + 2d\eta, 0) - (d\eta, 0), (0, 0)) \\ &= (0_1, 0_1) = 0_2. \end{aligned}$$

Now we suppose that $(\eta_{k-1}, 1_{k-1})$ is d_k–closed and we prove that this implies that $(\eta_k, 1_k)$ is d_{k+1}–closed. Indeed, we have

$$\begin{aligned} d_{k+1}(\eta_k, 1_k) &= (d_k\eta_k - L_k 1_k, 0_k) \\ &= (-d_k(\eta_{k-1}, 2_{k-1}) - (L_{k-1}1_{k-1}, 0_{k-1}), 0_k) \\ &= (-d_k(\eta_{k-1}, 1_{k-1}) - d_k(0_{k-1}, 1_{k-1}) - (L_{k-1}1_{k-1}, 0_{k-1}), 0_k) \\ &= (0_k - (-L_{k-1}1_{k-1}, 0_{k-1}) - (L_{k-1}1_{k-1}, 0_{k-1}), 0_k) \\ &= 0_{k+1}, \end{aligned}$$

where we have used the induction hypothesis and $2_k = 1_k + 1_k$. This means that $(\eta_{k-1}, 1_{k-1})$ are d_k-closed and gives some d_k–cohomology classes in $H^1_k(M)$.

Now, suppose that the associated d_k–cohomology classes $[(\eta_{k-1}, 1_{k-1})]$ vanish. This means that there is a generalized 1–differentiable function $f_k = (f_{k-1}, 0_{k-1})$ of order k such that $(\eta_{k-1}, 1_{k-1}) = d_k f_k$ that is imposible. So, $[(\eta_{k-1}, 1_{k-1})]$ are non-vanishing for every integer $k \geq 2$. □

In order to obtain another d_k–cohomology classes, we define an inductive generalization of an exterior product in the space $\Omega^\bullet_k(M)$ of generalized 1–differentiable forms of order k, by

$$\varphi_k \wedge_k \varphi_k^{'} = (\varphi_{k-1} \wedge_{k-1} \varphi_{k-1}^{'}, (-1)^p \varphi_{k-1} \wedge_{k-1} \psi_{k-1}^{'} + \psi_{k-1} \wedge_{k-1} \varphi_{k-1}^{'})$$

for any $\varphi_k = (\varphi_{k-1}, \psi_{k-1}) \in \Omega^p_k(M)$ and $\varphi_k^{'} = (\varphi_{k-1}^{'}, \psi_{k-1}^{'}) \in \Omega^{p^{'}}_k(M)$. Note that the above product is anticommutative, associative and distributive with respect to sum, i.e. $(\Omega_k(M), \wedge_k)$ is a graded algebra on M and d_k is an antiderivation with respect to this product, namely

$$d_k(\varphi_k \wedge_k \varphi_k^{'}) = d_k\varphi_k \wedge_k \varphi_k^{'} + (-1)^p \varphi_k \wedge_k d_k\varphi_k^{'}.$$

We also notice that for $k = 1$ we have that $\wedge_0 = \wedge$ is the classical exterior product and the exterior product $\wedge_1$ is just the product defined by Lichnerowicz (see the relation (5.5) from [11]).

In the case $k = 1$ it is easy to see that $(\eta \wedge (d\eta)^p, (d\eta)^p)$ is d_1–closed and its cohomology class $[(\eta \wedge (d\eta)^p, (d\eta)^p)] \in H^{2p+1}_1(M)$ is zero for every $p = 0, 1, \ldots, \left[\frac{n}{2}\right]$, see also [7]. Using the generalized 1–differentiable 1–forms η_k defined above, one gets

Proposition 3.2. *For every $k \geq 2$ the generalized 1–differentiable $2p+1$–forms of order k given by $(\eta_k \wedge_k (d_k\eta_k)^p, (d_k\eta_k)^p)$ are d_{k+1}–closed for every $p = 0, 1, \ldots, \left[\frac{n}{2}\right]$ and their associated d_{k+1}–cohomology classes vanish.*

Proof. Firstly, we notice that we can easily obtain

$$(d_{k-1}\eta_{k-1}, 0_{k-1})^p := \underbrace{(d_{k-1}\eta_{k-1}, 0_{k-1}) \wedge_k \ldots \wedge_k (d_{k-1}\eta_{k-1}, 0_{k-1})}_{p} = (d_k\eta_k)^p. \tag{3.1}$$

Using (3.1) and mathematical induction after k we obtain

$$L_k(\eta_k \wedge_k (d_k\eta_k)^{p-1}) = \eta_k \wedge_k (d_k\eta_k)^p. \tag{3.2}$$

Indeed, for $k = 0$ we have $L(\eta \wedge (d\eta)^{p-1}) = d\eta \wedge \eta \wedge (d\eta)^{p-1} = \eta \wedge (d\eta)^p$. Suppose now that the relation (3.2) is valid for $k = j - 1$ and we prove it for $k = j$. We have

$$\begin{aligned}
L_j(\eta_j \wedge_j (d_j\eta_j)^{p-1}) &= L_j(-(\eta_{j-1}, 2_{j-1}) \wedge_j ((d_{j-1}\eta_{j-1})^{p-1}, 0_j)) \\
&= L_j(-(\eta_{j-1} \wedge_{j-1} (d_{j-1}\eta_{j-1})^{p-1}, 2_{j-1}(d_{j-1}\eta_{j-1})^{p-1})) \\
&= -(L_{j-1}(\eta_{j-1} \wedge_{j-1} (d_{j-1}\eta_{j-1})^{p-1}), 2_{j-1}L_{j-1}(d_{j-1}\eta_{j-1})^{p-1}) \\
&= -(\eta_{j-1} \wedge_{j-1} (d_{j-1}\eta_{j-1}\eta_{j-1})^p, 2_j(d_{j-1}\eta_{j-1})^p) \\
&= (-(\eta_{j-1}, 2_{j-1}) \wedge_j (d_{j-1}\eta_{j-1})^p, 0_{j-1}) \\
&= \eta_j \wedge_j (d_j\eta_j)^p,
\end{aligned}$$

where we have used

$$L_k(d_k\eta_k)^p = (d_k\eta_k)^{p+1}$$

which also follows by mathematical induction after k.

Finally, using (3.2) we obtain

$$d_{k+1}((d_k\eta_k)^p, -\eta_k \wedge_k (d_k\eta_k)^{p-1}) = (\eta_k \wedge_k (d_k\eta_k)^p, (d_k\eta_k)^p)$$

which ends the proof. □

4 A generalization of coeffective 1-differentiable cohomology and coeffective numbers

In this section we consider an inductive generalization of coeffective 1-differentiable cohomology and coeffective numbers of a smooth manifold. In the case of an almost contact manifold of finite type we prove that the generalized coeffective numbers are bounded by generalized Betti numbers of the manifold.

Let us consider

$$\mathcal{A}_k^p(M) = \ker\{L_k : \Omega_k^p(M) \to \Omega_k^{p+2}(M)\}$$

called the coeffective space of 1-differentiable p–forms of order k.

It is easy to see that

$$\mathcal{A}_k^p(M) = \mathcal{A}_{k-1}^p(M) \oplus \mathcal{A}_{k-1}^{p-1}(M).$$

Indeed, we have that $L_k\varphi_k = 0_k$ if and only if $L_{k-1}\varphi_{k-1} = L_{k-1}\psi_{k-1} = 0_{k-1}$, for any $\varphi_k = (\varphi_{k-1}, \psi_{k-1}) \in \Omega_k^p(M)$.

Since $L_k d_k = d_k L_k$, we obtain that $(\mathcal{A}_k^\bullet(M), d_k)$ is a differential subcomplex of $(\Omega_k^\bullet(M), d_k)$ and consider the associated cohomology $H_k^\bullet(\mathcal{A}(M))$ called *generalized coeffective* 1*-differentiable cohomology of order* k of M. If this cohomology is finite, we define the *generalized* p*-coeffective numbers of order* k by $c_{k,p}(M) = \dim H_k^p(\mathcal{A}(M))$.

Using the same technique as in Section 2 we can prove the following isomorphism:

$$H_k^p(\mathcal{A}(M)) \approx H_{k-1}^p(\mathcal{A}(M)) \oplus H_{k-1}^{p-1}(\mathcal{A}(M))$$

and consequently

$$c_{k,p}(M) = \sum_{t=0}^{k} C_k^t c_{p-t}(M), \tag{4.1}$$

where $c_p(M) = \dim H^p(\mathcal{A}(M))$ are the classical coeffective numbers of M.

According to [4, 5], if (M, F, ξ, η) is an almost contact manifold of dimension $2n+1$ of finite order then

$$b_p(M) - b_{p+2}(M) \leq c_p(M) \leq b_p(M) + b_{p+1}(M) \tag{4.2}$$

for any $p \geq n+2$ and if M is a contact manifold of dimension $2n+1$ of finite order then

$$c_p(M) = b_p(M) + b_{p+1}(M) \tag{4.3}$$

for any $p \geq n+2$.

Now, using (2.2), (4.1), (4.2) and (4.3), we obtain

Theorem 4.1. *i) If* (M, F, ξ, η) *is an almost contact manifold of dimension* $2n+1$ *then*

$$b_{k,p}(M) - b_{k,p+2}(M) \leq c_{k,p}(M) \leq b_{k,p}(M) + b_{k,p+1}(M) \tag{4.4}$$

for any $p \geq k+n+2$.

ii) If (M, η) *is a contact manifold then*

$$c_{k,p}(M) = b_{k,p}(M) + b_{k,p+1}(M) \tag{4.5}$$

for any $p \geq k+n+2$.

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