CONHARMONIC CURVATURE TENSOR ON KENMOTSU MANIFOLDS

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Abstract

The object of the present paper is to characterize Kenmotsu manifolds satisfying certain curvature conditions on the conharmonic curvature tensor. Next we study 3-dimensional Kenmotsu manifolds admitting a non-null concircular vector field. Also we study 3-dimensional locally ϕ-conharmonically symmetric Kenmotsu manifolds. Finally, we give an example of a locally ϕ-conharmonically symmetric Kenmotsu manifolds.

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1 Introduction

The product of an almost contact manifold M and the real line $\mathbb{R}$ carries a natural almost complex structure. However, if one takes M to be an almost contact metric manifold and suppose that the product metric G on $M \times \mathbb{R}$ is Kaehlerian, then the structure on M is cosymplectic [10] and not Sasakian. On the other hand, Oubina [14] pointed out that if the conformally related metric $e^{2t}G$, t being the coordinates on $\mathbb{R}$, is Kaehlerian, then M is Sasakian and conversely.
In [20] S. Tanno classified almost contact metric manifolds whose automorphism groups possesses the maximum dimension. For such a manifold M, the sectional curvature of a plane section containing ξ is a constant, say c. If $c > 0$, M is a homogeneous Sasakian manifold of constant sectional curvature. If $c = 0$, M is the product of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature. If $c < 0$, M is a warped product space $\mathbb{R} \times f^{C^n}$. In 1972 , K. Kenmotsu [13] abstracted the differential geometric properties of the third case. We call it Kenmotsu manifold. Kenmotsu manifolds have been studied by several

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authors such as G.Pitis ([18],[19]), Jun, De and Pathak [12], De and Pathak [7], De and De ([5],[6]) Binh, Tamassy, De and Tarafdar [2], Ozgur ([16],[17]) and many others.

It is known that a harmonic function is defined as a function whose Laplacian vanishes. In general a harmonic function is not transformed into a harmonic function. The conditions under which a harmonic function remains invariant have been studied by Ishii [11] who introduced the conharmonic transformation as a subgroup of the conformal transformation (1.1) satisfying the condition

$$\sigma^i_{,i} + \sigma_{,i}\sigma^i_{,} = 0, \tag{1.1}$$

where *comma* denotes the covariant differentiation with respect to the metric g.

A rank four tensor $\overline{C}$ that remains invariant under conharmonic transformation for an $2n + 1$-dimensional Riemannian manifold M^{2n+1}, is given by

$$
\begin{aligned}
\overline{C}(X, Y, Z, W) \;=\; & \overline{R}(X, Y, Z, W) \\
& -\frac{1}{2n-1}[g(Y, Z)S(X, W) - g(X, Z)S(Y, W) \\
& +S(Y, Z)g(X, W) - S(X, Z)g(Y, W)],
\end{aligned} \tag{1.2}
$$

where $\overline{R}$ denotes the Riemannian curvature tensor of type $(0, 4)$ and $\overline{C}$ denotes the Conharmonic curvature tensor of type $(0, 4)$ defined by

$$\overline{R}(X, Y, Z, W) = g(R(X, Y)Z, W),$$

$$\overline{C}(X, Y, Z, W) = g(\tilde{C}(X, Y)Z, W),$$

where R is the Riemannian curvature tensor of type $(1, 3)$, $\tilde{C}$ is the conharmonic curvature tensor of type $(1, 3)$ and S denotes the Ricci tensor of type $(0, 2)$.

The curvature tensor defined by equation (1.2) is known as conharmonic curvature tensor. A manifold whose conharmonic curvature vanishes at every point of the manifold is called conharmonically flat manifold. Thus this tensor represents the deviation of the manifold from conharmonic flatness. It satisfies all the symmetric properties of the Riemannian curvature tensor R. There are many physical applications of tensor $\tilde{C}$. For example, in [1], Abdussattar showed that the sufficient condition for a space-time to be conharmonic to a flat space-time is that tensor $\tilde{C}$ vanishes identically. A conharmonically flat space-time is either empty, in which case it is flat, or filled with a distribution represented by energy momentum tensor T possessing the algebraic structure of an electromagnetic field and conformal to a flat space-time [1]. Also, he described the gravitational field due to a distribution of pure radiation in presence of disordered radiation by means of spherically symmetric conharmonically flat space-time. Conharmonic curvature tensors have been

studied by Ghosh, De and Taleshian [9], *Özgür* [15] and many others. Motivated by the above studies we would like to study the properties of conharmonic curvature tensor in a Kenmotsu manifold.

Again a Kenmotsu manifold is called Einstein if the Ricci tensor S is of the form $S = \lambda g$, where λ is a constant and η- Einstein if the Ricci tensor S is of the form $S = ag + b\eta \otimes \eta$, where a, b are smooth functions on M.

The paper is organized as follows:

After preliminaries in Section 3 and 4, we consider conharmonically flat and ϕ-conharmonically flat Kenmotsu manifold and prove that the manifold is an Einstein manifold and a η- Einstein manifold. In the next section we study a 3-dimensional Kenmotsu manifold admitting a non-null concircular vector field. Section 6, deals with the study of 3-dimensional locally ϕ-conharmonically symmetric Kenmotsu manifolds. We prove that a 3-dimensional Kenmotsu manifold is locally ϕ-conharmonically symmetric if and only if it is locally ϕ-symmetric. Finally, we cited an example of ϕ-conharmonically symmetric Kenmotsu manifold.

2 Preliminaries

Let M be a $(2n + 1)$- dimensional connected almost contact metric manifold with an almost contact metric structure (ϕ,ξ,η,g),that is, ϕ is an (1,1) tensor field, ξ is a vector field ,η is a 1 - form and g is a compatible Riemannian metric such that

$$\phi^2(X) = -X + \eta(X)\xi, \eta(\xi) = 1, \phi\xi = 0, \eta\phi = 0 \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.2}$$

$$g(X,\xi) = \eta(X) \tag{2.3}$$

for all $X, Y \, \epsilon \, T(M)$([3],[4], [19]).

If an almost contact metric manifold satisfies

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.4}$$

then M is called a Kenmotsu manifold [13], where ∇ is the Levi-Civita connection of g. From the above equation it follows that

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.5}$$

and

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y). \tag{2.6}$$

Moreover, the curvature tensor R and the Ricci tensor S satisfy

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X \tag{2.7}$$

and
$$S(X, \xi) = -2n\eta(X). \tag{2.8}$$

From [7] we know that for a 3-dimensional Kenmotsu manifold

$$\begin{aligned}
R(X,Y)Z &= (\frac{r+4}{2})[g(Y,Z)X - g(X,Z)Y] \\
&\quad -(\frac{r+6}{2})[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\
&\quad +\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y],
\end{aligned} \tag{2.9}$$

$$S(X,Y) = \frac{1}{2}[(r+2)g(X,Y) - (r+6)\eta(X)\eta(Y)], \tag{2.10}$$

where S is the Ricci tensor of type (0,2), R is the curvature tensor of type (1,3) and r is the scalar curvature of the manifold M.

In a $(2n+1)$-dimensional almost contact metric manifold, if $\{e_1, ..., e_{2n}, \xi\}$ is a local orthonormal basis of vector fields, then $\{\phi e_1, ..., \phi e_{2n}, \xi\}$ is also a local orthonormal basis. It is easy to verify that

$$\sum_{i=1}^{2n} g(e_i, e_i) = \sum_{i=1}^{2n} g(\phi e_i, \phi e_i) = 2n. \tag{2.11}$$

$$\begin{aligned}
\sum_{i=1}^{2n} g(e_i, Z)S(Y, e_i) &= \sum_{i=1}^{2n} g(\phi e_i, Z)S(Y, \phi e_i) \tag{2.12}\\
&= S(Y, Z) - S(Y, \xi)\eta(Z),
\end{aligned}$$

for $Y, Z \in T(M)$. In particular in view of $\eta \circ \phi = 0$, we get

$$\sum_{i=1}^{2n} g(e_i, \phi Z)S(Y, e_i) = \sum_{i=1}^{2n} g(\phi e_i, \phi Z)S(Y, \phi e_i) = S(Y, \phi Z), \tag{2.13}$$

for $Y, Z \in T(M)$. If M is a Kenmotsu manifold then it is known that

$$R(X, \xi)\xi = \eta(X)\xi - X, \quad X \in T(M) \tag{2.14}$$

and

$$S(\xi, \xi) = -2n. \tag{2.15}$$

From (2.15) we get

$$\sum_{i=1}^{2n} S(e_i, e_i) = \sum_{i=1}^{2n} S(\phi e_i, \phi e_i) = r + 2n, \tag{2.16}$$

where r is the scalar curvature. In a Kenmotsu manifold we also have

$$\overline{R}(\xi, Y, Z, \xi) = -g(\phi Y, \phi Z), \quad Y, Z \in T(M). \tag{2.17}$$

Consequently

$$\sum_{i=1}^{2n} \overline{R}(e_i, Y, Z, e_i) = \sum_{i=1}^{2n} \tilde{R}(\phi e_i, Y, Z, \phi e_i) = S(Y, Z) + g(\phi Y, \phi Z). \qquad (2.18)$$

Now we state the following Lemmas:

Lemma 2.1 [7] :A 3-dimensional Kenmotsu manifold is a manifold of constant negative curvature if and only if the scalar curvature $r = -6$.

Lemma 2.2 [7]: A 3-dimensional Kenmotsu manifold is locally ϕ-symmetric if and only if the scalar curvature r is constant.

Lemma 2.3 [12]: Any η-Einstein Kenmotsu manifold of dimension ≥ 5 with $b = $ constant is Einstein.

3   Conharmonically flat Kenmotsu manifold

In this section we study conharmonically flat Kenmotsu manifold.

Definition 3.1. *A Kenmotsu manifold is said to be conharmonically flat if*

$$g(\tilde{C}(X, Y)Z, W) = 0. \qquad (3.1)$$

Let a $(2n+1)$-dimensional Kenmotsu manifold M be conharmonically flat. Then using (3.1) in (1.2) we have

$$\begin{aligned} R(X, Y)Z &= \frac{1}{2n-1}[S(Y, Z)X - S(X, Z)Y \\ &\quad + g(Y, Z)QX - g(X, Z)QY]. \end{aligned} \qquad (3.2)$$

Taking $Z = \xi$ and using (2.7) and (2.8) we have

$$\begin{aligned} \eta(X)Y - \eta(Y)X &= \frac{1}{2n-1}[2n\{\eta(X)Y - \eta(Y)X\} \\ &\quad - \eta(X)QY + \eta(Y)QX].. \end{aligned} \qquad (3.3)$$

Again putting $Y = \xi$ in (3.3) we get

$$\begin{aligned} \eta(X)\xi - X &= \frac{1}{2n-1}[2n\{\eta(X)\xi - X\} \\ &\quad - \eta(X)Q\xi + QX] \end{aligned} \qquad (3.4)$$

and after simplification the above equation reduces to

$$S(X, Y) = g(X, Y) - (2n+1)\eta(X)\eta(Y). \qquad (3.5)$$

So in view of (3.5) and Lemma 2.3 we state the following:

Theorem 3.1. *A conharmonically flat Kenmotsu manifold is an Einstein manifold*
.

Now we consider conharmonically flat Kenmotsu manifolds. Then using Theorem 3.1 in equation (3.2) we obtain the following :

Corollary 1. *A conharmonically flat Kenmotsu manifold is a manifold of constant curvature.*

4 ϕ-Conharmonically flat Kenmotsu manifold

In this section we study ϕ-conharmonically flat Kenmotsu manifolds.

Definition 4.1. *A Kenmotsu manifold is said to be ϕ-conharmonically flat if*

$$g(\tilde{C}(\phi X, \phi Y)\phi Z, \phi W) = 0, \tag{4.1}$$

where $X, Y, Z, W \in T(M)$.

The notion of ϕ-conformally flat for K-contact manifolds was first introduced by G. Zhen [23]. In a recent paper [15] Chian Ozgur studied ϕ-conformally flat Lorentzian Para-Sasakian Manifold.

Let a $(2n + 1)$-dimensional Kenmotsu manifold M be ϕ-conharmonically flat. Then using (4.1) in (1.2) we have

$$
\begin{aligned}
\overline{R}(\phi X, \phi Y, \phi Z, \phi W) &= \frac{1}{2n-1}[S(\phi Y, \phi Z)g(\phi X, \phi W) \\
&\quad -S(\phi X, \phi Z)g(\phi Y, \phi W) \\
&\quad +S(\phi X, \phi W)g(\phi Y, \phi Z) \\
&\quad -S(\phi Y, \phi W)g(\phi X, \phi Z)].
\end{aligned}
\tag{4.2}
$$

Let $\{e_1, ..., e_{2n}, \xi\}$ be a local orthonormal basis of vector fields in M. Putting $X = W = e_i$ in (4.2) and summing up from 1 to $2n$ we have

$$
\begin{aligned}
\sum_1^{2n} \tilde{R}(\phi e_i, \phi Y, \phi Z, \phi e_i) &= \frac{1}{2n-1}\sum_1^{2n}[S(\phi Y, \phi Z)g(\phi e_i, \phi e_i) \\
&\quad -S(\phi e_i, \phi Z)g(\phi Y, \phi e_i) \\
&\quad +S(\phi e_i, \phi e_i)g(\phi Y, \phi Z) \\
&\quad -S(\phi Y, \phi e_i)g(\phi e_i, \phi Z)].
\end{aligned}
\tag{4.3}
$$

Using (2.11), (2.12), (2.16) and (2.18) in (4.3) we get

$$
\begin{aligned}
S(\phi Y, \phi Z) + g(\phi^2 Y, \phi^2 Z) &= \frac{2n-2}{2n-1}S(\phi Y, \phi Z) \\
&\quad +\frac{r+2n}{2n-1}g(\phi Y, \phi Z).
\end{aligned}
\tag{4.4}
$$

i.e.,

$$S(\phi Y, \phi Z) = (r + 1)g(\phi Y, \phi Z). \tag{4.5}$$

Substituting Y by ϕY and Z by ϕZ in (4.5) we have

$$S(\phi^2 Y, \phi^2 Z) = (r + 1)g(\phi Y, \phi Z). \tag{4.6}$$

Using (2.1), (2.2) and (2.8) in (4.6) we get

$$S(Y, Z) = (r + 1)g(Y, Z) - (2n + 1 + r)\eta(Y)\eta(Z). \tag{4.7}$$

Contacting (4.7) we have

$$r = 0. \tag{4.8}$$

In view of (4.7) and (4.8) we have the following :

Theorem 4.1. *A ϕ-conharmonically flat Kenmotsu manifold is an η-Einstein manifold with vanishing scalar curvature.*

5   3-dimensional Kenmotsu manifold admitting a non-null concircular vector field

Definition 5.1. *A vector field V on a Riemannian manifold is said to be a concircular vector field [22] if it satisfies an equation of the form*

$$\nabla_X V = \rho X \tag{5.1}$$

for all X, where ρ is a scalar function. In particular if $\rho = 0$, then V is parallel.

We suppose that a 3-dimensional Kenmotsu manifold admits a non-null concircular vector field. Then differentiating (5.1) covariantly we get

$$\nabla_Y \nabla_X V = \rho \nabla_Y X + d\rho(Y)X. \tag{5.2}$$

From (5.2) it follows that (since the torsion tensor $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0$)

$$\nabla_Y \nabla_X V - \nabla_X \nabla_Y V - \nabla_{[X,Y]} V = d\rho XY - d\rho(Y)X. \tag{5.3}$$

Hence by Ricci identity we obtain from (5.3)

$$R(X, Y)V = d\rho(X)Y - d\rho(Y)X, \tag{5.4}$$

which implies that

$$\overline{R}(X, Y, V, Z) = d\rho(X)g(Y, Z) - d\rho(Y)g(X, Z), \tag{5.5}$$

where $\overline{R}(X, Y, V, Z) = g(R(X, Y)V, Z)$.
Replacing Z by ξ in (5.5) we get

$$\eta(R(X, Y)V) = d\rho(X)\eta(Y) - d\rho(Y)\eta(X). \tag{5.6}$$

Again

$$\eta(R(X,Y)V) = \eta(Y)g(X,V) - \eta(X)g(Y,V). \tag{5.7}$$

From (5.6) and (5.7) we have

$$d\rho(X)\eta(Y) - d\rho(Y)\eta(X) = \eta(Y)g(X,V) - \eta(X)g(Y,V). \tag{5.8}$$

Putting $X = \phi X$ and $Y = \xi$ in (5.8), we get

$$d\rho(\phi X) = g(\phi X, V). \tag{5.9}$$

Substituting X by ϕX in (5.9), we obtain

$$d\rho(X) - d\rho(\xi)\eta(X) = g(X,V) - \eta(X)\eta(V). \tag{5.10}$$

Here $g(X,V) \neq 0$ for all X. For, if $g(X,V) = 0$ for all X, then $g(V,V) = 0$ which means that V is a null vector field. This is contradicting our assumption. Hence multiplying both sides of (5.10) by $g(X,V)$ we get

$$d\rho(X)g(X,V) - d\rho(\xi)g(X,V)\eta(X) = g(X,V)[g(X,V) - \eta(X)\eta(V)]. \tag{5.11}$$

Also putting $Z = V$ in (5.5), we get

$$d\rho(X)g(Y,V) = d\rho(Y)g(X,V). \tag{5.12}$$

For $Y = \xi$, we obtain from (5.12) that

$$d\rho(X)\eta(V) = d\rho(\xi)g(X,V). \tag{5.13}$$

Since $\eta(X) \neq 0$ for all X, multiplying both sides of (5.13) by $\eta(X)$, we have

$$d\rho(X)\eta(V)\eta(X) = d\rho(\xi)\eta(X)g(X,V). \tag{5.14}$$

By virtue of (5.11) and (5.14) we get

$$[d\rho(X) - g(X,V)][g(X,V) - \eta(X)\eta(V)] = 0. \tag{5.15}$$

Hence it follows from (5.15) that

$$either \quad d\rho(X) = g(X,V) \quad for\ all\ X \tag{5.16}$$

$$or, \quad g(X,V) - \eta(X)\eta(V) = 0 \quad for\ all\ X. \tag{5.17}$$

First we consider the case of (5.16). Then we obtain from (5.5)

$$\overline{R}(X,Y,V,Z) = g(X,V)g(Y,Z) - g(Y,V)g(X,Z). \tag{5.18}$$

Then putting $X = Z = e_i,\ i = 1,2,3$ in (5.18) and taking summation over $1 \le i \le 3$, we get

$$S(Y,V) = -2g(Y,V). \tag{5.19}$$

By virtue of (2.10) and (5.19) we obtain

$$(r + 6)[g(Y, V) - \eta(Y)\eta(V)] = 0. \tag{5.20}$$

Since in this case $g(Y, V) - \eta(Y)\eta(V) \neq 0$, it follows from (5.20) that

$$r = -6. \tag{5.21}$$

Next, we consider case (5.17). Differentiating (5.17) covariantly along Z, we get

$$(\nabla_Z \eta)(X)\eta(V) + (\nabla_Z \eta)(V)\eta(X) = 0. \tag{5.22}$$

Using (2.6) in (5.22), we obtain

$$g(X, Z)\eta(V) + g(V, Z)\eta(X) - 2\eta(X)\eta(Z)\eta(V) = 0. \tag{5.23}$$

Then putting $X = Z = e_i$, $i = 1, 2, 3$ in (5.23) and taking summation over $1 \leq i \leq 3$, we get $\eta(V) = 0$, which contradicts our assumption.
Therefore, by virtue of (5.21) and Lemma 2.1 we can state the following:

Theorem 5.1. *If a 3-dimensional Kenmotsu manifold admits a non-null concircular vector field, then the manifold is a manifold of constant negative curvature.*

6 Locally ϕ- conharmonically symmetric Three dimensional Kenmotsu manifolds

The notion of locally ϕ-symmetry was first introduced by Takahashi [21] on a Sasakian manifold. In a recent paper [8] De and Sarkar introduced the notion of locally ϕ-Ricci symmetric Sasakian manifolds again. In this paper we consider a locally ϕ- conharmonically symmetric 3- dimensional Kenmotsu manifolds.

Definition 6.1. *A three- dimensional Kenmotsu manifold is said to be locally ϕ-conharmonically symmetric if the conharmonic curvature tensor $\tilde{C}$ satisfies*

$$\phi^2(\nabla_W \tilde{C})(X, Y)Z = 0, \tag{6.1}$$

where X, Y and Z are horizontal vector fields.

Using (2.10) in (1.2),in a 3-dimensional Kenmotsu manifold the conharmonic curvature tensor is given by

$$\begin{aligned}
\tilde{C}(X, Y)Z &= (\frac{r}{2})[g(X, Z)Y - g(Y, Z)X] \\
&\quad -(\frac{r + 6}{2})[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \\
&\quad +\eta(Y)\xi - \eta(X)\xi].
\end{aligned} \tag{6.2}$$

Taking the covariant differentiation to both sides of equation (6.2), we have

$$
\begin{aligned}
(\nabla_W \tilde{C})(X,Y)Z &= \frac{dr(W)}{2}[g(X,Z)Y - g(Y,Z)X] \\
&\quad - \frac{dr(W)}{2}[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\
&\quad + \eta(Y)\xi - \eta(X)\xi] \\
&\quad - (\frac{r+6}{2})[g(Y,Z)(\nabla_W \eta)(X)\xi - g(X,Z)(\nabla_W \eta)(Y)\xi \\
&\quad + g(Y,Z)\eta(X)\nabla_W \xi - g(X,Z)\eta(Y)\nabla_W \xi \\
&\quad + (\nabla_W \eta)(Y)\xi + \eta(Y)(\nabla_W \xi \\
&\quad - (\nabla_W \eta)(X)\xi - \eta(X)(\nabla_W \xi].
\end{aligned}
\tag{6.3}
$$

Now assume that X, Y and Z are horizontal vector fields. So equation (6.3) becomes

$$
\begin{aligned}
(\nabla_W \tilde{C})(X,Y)Z &= \frac{dr(W)}{2}[g(X,Z)Y - g(Y,Z)X] \\
&\quad - (\frac{r+6}{2})[g(Y,Z)(\nabla_W \eta)(X)\xi - g(X,Z)(\nabla_W \eta)(Y)\xi \\
&\quad - (\nabla_W \eta)(X)\xi - \eta(X)(\nabla_W \xi].
\end{aligned}
\tag{6.4}
$$

From (6.4) it follows that

$$
\phi^2((\nabla_W \tilde{C})(X,Y)Z) = \frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y].
\tag{6.5}
$$

Hence we can state the following :

Theorem 6.1. *A 3-dimensional Kenmotsu manifold is locally ϕ-conharmonically symmetric if and only if the scalar curvature r is constant.*

Using Lemma 2.2, we can state the following theorem:

Theorem 6.2. *A 3-dimensional Kenmotsu manifold is locally ϕ-conharmonically symmetric if and only if it is locally ϕ-symmetric.*

7   Examples of a 3-dimensional Kenmotsu manifold

We consider the 3-dimensional manifold $M = \{(x,y,z)\varepsilon\mathbb{R}^3, z \neq 0\}$, where (x,y,z) are standard coordinates of $\mathbb{R}^3$.

The vector fields

$$
e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = -z\frac{\partial}{\partial z}
$$

are linearly independent at each point of M.

Let g be the Riemannian metric defined by

$$
g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,
$$

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z\varepsilon\chi(M)$.
Let ϕ be the $(1,1)$ tensor field defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g, we have

$$\eta(e_3) = 1,$$

$$\phi^2 Z = -Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W\varepsilon\chi(M)$.

Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to metric g. Then we have

$$
\begin{aligned}
[e_1, e_3] &= e_1 e_3 - e_3 e_1 \\
&= z\frac{\partial}{\partial x}\left(-z\frac{\partial}{\partial z}\right) - \left(-z\frac{\partial}{\partial z}\right)\left(z\frac{\partial}{\partial x}\right) \\
&= -z^2\frac{\partial^2}{\partial x\partial z} + z^2\frac{\partial^2}{\partial z\partial x} + z\frac{\partial}{\partial x} \\
&= e_1.
\end{aligned}
$$

Similarly

$$[e_1, e_2] = 0 \quad and \quad [e_2, e_3] = e_2.$$

The Riemannian connection ∇ of the metric g is given by

$$
\begin{aligned}
2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\
&- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]), \qquad (7.1)
\end{aligned}
$$

which is known as Koszul's formula.

Using (7.1) we have

$$
\begin{aligned}
2g(\nabla_{e_1} e_3, e_1) &= -2g(e_1, -e_1) \\
&= 2g(e_1, e_1). \qquad (7.2)
\end{aligned}
$$

Again by (7.1)

$$2g(\nabla_{e_1} e_3, e_2) = 0 = 2g(e_1, e_2) \qquad (7.3)$$

and

$$2g(\nabla_{e_1} e_3, e_3) = 0 = 2g(e_1, e_3). \qquad (7.4)$$

From (7.2), (7.3) and (7.4) we obtain

$$2g(\nabla_{e_1} e_3, X) = 2g(e_1, X),$$

for all $X\varepsilon\chi(M)$.

Thus
$$\nabla_{e_1}e_3 = e_1.$$

Therefore, (7.1) further yields

$$\nabla_{e_1}e_3 = e_1, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_1 = -e_3,$$

$$\nabla_{e_2}e_3 = e_2, \quad \nabla_{e_2}e_2 = e_3, \quad \nabla_{e_2}e_1 = 0,$$

$$\nabla_{e_3}e_3 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_1 = 0. \tag{7.5}$$

From the above it follows that the manifold satisfies $\nabla_X\xi = X - \eta(X)\xi$, for $\xi = e_3$. Hence the manifold is a Kenmotsu manifold. It is known that

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{7.6}$$

With the help of the above results and using (7.6), it can be easily verified that

$$R(e_1,e_2)e_3 = 0, \quad R(e_2,e_3)e_3 = -e_2, \quad R(e_1,e_3)e_3 = -e_1,$$

$$R(e_1,e_2)e_2 = -e_1, \quad R(e_2,e_3)e_2 = e_3, \quad R(e_1,e_3)e_2 = 0,$$

$$R(e_1,e_2)e_1 = e_2, \quad R(e_2,e_3)e_1 = 0, \quad R(e_1,e_3)e_1 = e_3.$$

From the above expressions of the curvature tensor R we obtain

$$\begin{aligned} S(e_1,e_1) &= g(R(e_1,e_2)e_2,e_1) + g(R(e_1,e_3)e_3,e_1)\\ &= -2. \end{aligned}$$

Similarly, we have
$$S(e_2,e_2) = S(e_3,e_3) = -2.$$

Therefore,
$$r = S(e_1,e_1) + S(e_2,e_2) + S(e_3,e_3) = -6.$$

We note that here r is constant. Thus Theorem 6.1 is verified.

References

[1] Abdussattar, D. B., *On conharmonic transformations in general relativity*, Bull Calcutta Math. Soc. **41** (1966), 409-416.

[2] Binh, T. Q., Tamassy, L., De, U.C. and Tarafdar,M., *Some remarks on almost Kenmotsu manifolds,* Math. Pannonica **13** (2002), 31-39.

[3] Blair, D. E., *Contact manifolds in Riemannian geometry*, Lecture Note in Mathematics, **509**, Springer-Verlag, Berlin-New York, 1976.

[4] Blair, D. E., *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics **203**, Birkhauser Boston, Inc., Boston, 2002.

[5] De, U.C. and De, K., *On three dimensional Kenmotsu manifolds admitting a quater- symmetric metric connection*, Azerbaijan J. Math. **1** (2011), 132-142.

[6] De, U.C. and De, K., *On φ-concircularly symmetric Kenmotsu manifolds*, Thai J. Math. **10**, (2012) 1-11.

[7] De, U. C. and Pathak, G., *On 3-dimensional Kenmotsu manifold*, Indian J. Pure Appl. Math. **35** (2004), 159-165.

[8] De, U. C. and Sarkar, A., *On φ- Ricci symmetric Sasakian manifolds*, Proceeding of the Jangjeon Mathematical Society **11** (2008), 47-52.

[9] Ghosh, S., De, U.C. and Taleshian, A., *Conharmonic curvature tensor on N(k)-contact metric manifolds*, ISRN Geometry, 2011, Art. ID 423798, 11 pages.

[10] Ianuş, S. and Smaranda, D., *Some remarkable structures on the products of an almost contact metric manifold with the real line*, Papers from the National Coll. on Geometry and Topology, Univ. Timişoara, 1997, 107-110.

[11] Ishii, Y., *Conharmonic transformations*, Tensor (N.S.) **7** (1957), 73-80.

[12] Jun, J.B., De, U. C. and Pathak, G., *On Kenmotsu manifold*, J. Korean Math. Soc. **42** (2005), 435-445.

[13] Kenmotsu, K.,*A class of almost contact Riemannian manifolds*, Tohoku Math. J. **24** (1972), no. 2, 93-103.

[14] Oubina, J. A.,*New classes of almost contact metric structures*, Publ. Math. Debrecen **32**, no. 3-4,(1985), 187-193.

[15] Özgür, C., *φ-conformally flat Lorentzian para-Sasakian manifolds*, Radovi Matemaicki, **12** (2003), 99-106.

[16] Özgür, C., *On weakly symmetric Kenmotsu manifolds*, Differ. Geom. Dyn. Syt. **8** (2006), 277-290.

[17] Özgür, C., *On generalized recurrent Kenmotsu manifolds*, World, Applied Sciences Journal **2** (2007), 29-33.

[18] Pitiş, G., *A remark on Kenmotsu manifolds.* Bul. Univ. Braşov Ser. C **30** (1988), 31-32.

[19] Pitiş, G., *Geometry of Kenmotsu manifolds.* Publishing House of Transilvania Braşov, 2007.

[20] Tanno, S., *The automorphism groups of almost contact Riemannian manifolds*, Tohoku Math. J. **21** (1969), 221-228.

[21] Takahashi, T., *Sasakian φ-symmetric spaces*, Tôhoku Math. J. (2) **29** (1977), 91-113.

[22] Yano, K., *On the torse forming direction in Riemannian spaces*, Proc. Imp. Acad. Tokyo, **20** (1940), 340-345.

[23] Zhen, G., *On conformal symmetric K-contact manifolds*, Chinese Quart. J. of Math., **7** (1992), 5-10.