Bulletin of the *Transilvania* University of Braşov • Vol 6(55), No. 1 - 2013
Series III: Mathematics, Informatics, Physics, 1-8

A NEW LINEAR POSITIVE OPERATOR OF DURRMEYER TYPE ASSOCIATED WITH BLEIMANN - BUTZER - HAHN OPERATOR

Cristina Sanda CISMAŞIU[1]

Abstract

In this paper we define a new linear positive operator associated with Bleimann - Butzer - Hahn operator and study some of its approximation properties.

2010 *Mathematics Subject Classification:* 41A36.
Key words: positive linear operators, Bleimann - Butzer - Hahn operator, approximation property.

1 Indroduction

Consider the following spaces of the real valued functions on $[0, +\infty)$:
$\mathbf{B}[0, +\infty)$ the space of bounded functions;
$\mathbf{C}[0, +\infty)$ the space of continuous functions;
$\mathbf{C_B}[0, +\infty) = \mathbf{C}[0, +\infty) \cap \mathbf{B}[0, +\infty)$;
$\mathbf{C_0}[0, +\infty) = \left\{ f \in \mathbf{C}[0, +\infty) \mid (\exists) \left| \lim_{x \to +\infty} f(x) \right| < \infty \right\}$.

Spaces $\mathbf{C_0}[0, +\infty)$ and $\mathbf{C_B}[0, +\infty)$ are Banach spaces regard with the sup - norm $\|f\| = \sup_{x \in [0,+\infty)} |f(x)|$ and $\mathbf{C_0}[0, +\infty)$ is a linear subspace of $\mathbf{C_B}[0, +\infty)$.

Denote by e_i, $i \in \mathbb{N} \cup \{0\}$, the monomial functions $e_i(x) = x^i$, $x \in [0, +\infty)$ and by $\omega(f, \cdot)$ the first modulus of continuity of f.

For $n \in \mathbb{N}$, let $L_n : \mathbf{C_B}[0, +\infty) \longrightarrow \mathbf{C_B}[0, +\infty)$ be the Bleimann - Butzer - Hahn operator [7] defined as:

$$L_n(f; x) = (1 + x)^{-n} \sum_{k=0}^{n} \binom{n}{k} x^k f\left(\frac{k}{n - k + 1}\right), \; f \in \mathbf{C_B}[0, +\infty), \; x \geq 0, \; n \in \mathbb{N}. \quad (1)$$

The approximation properties of this operator are well-known and were studied by many authors (G. Bleimann, P. L. Butzer, L. Hahn [7]; U. Abel, M. Ivan [9]; O. Agratini [5]; F. Altomare, M. Campiti [6], etc) and the most important of them were included in the next lemma.

[1]Faculty of Mathematics and Informatics, *Transilvania* University of Braşov, Romania, e-mail: c.cismasiu@unitbv.ro

Lemma 1. *For $n \in \mathbb{N}$ and $x \geq 0$ operator (1) has the following properties:*

1. $L_n(e_0; x) = 1$;
2. $L_n(e_1; x) = x - x\left(\dfrac{x}{1+x}\right)^n$;
3. *If* $n \geq 24(1+x)$ *then* $\left|L_n(e_2; x) - x^2\right| \leq \dfrac{2x(1+x)^2}{n+2}$;
4. $|L_n(f; x)| \leq \|f\|$;
5. $\lim\limits_{n\to\infty} L_n(f; x) = f(x)$ *uniformly in x on each compact* $[0, b]$, $b > 0$;
6. *For* $(\forall)\delta > 0$, $(\forall)x \geq 0$, $(\forall)n \geq 24(1+x)$
$$|L_n(f; x) - x| \leq \left(1 + \frac{1}{\delta}\sqrt{\frac{4x(1+x)^2}{n+2}}\right)\omega(f; \delta);$$
7. *If f has derivative on* $[0, +\infty)$ *and* $f' \in \mathbf{C_B}[0, +\infty)$ *then for* $(\forall)n \geq 24(1+x)$, $n \in \mathbb{N}$, $x \geq 0$
$$|L_n(f; x) - f(x)| \leq \frac{(x+1)^2}{n+2}\left|f'(x)\right| + 2\delta_n(x)\omega\left(f'; \delta_n(x)\right),$$
with $\delta_n = 2(1+x)\sqrt{\dfrac{x}{n+2}}$;
8. $L_n(F; y) = \sum\limits_{k=0}^{n} \binom{n}{k} y^k (1-y)^{n-k} F\left(\dfrac{k}{n+1}\right)$, *with* $F(y) = f\left(\dfrac{y}{1-y}\right)$, $y \in [0, 1)$ *(the Bernstein - Stancu operator, see Stancu [11] respectively* $L_n = V \circ B_{n+1} \circ U$, $n \in \mathbb{N}$ *with*
$$U : \mathbb{R}^{[0,+\infty)} \longrightarrow \mathbb{R}^{[0,1)},\ U(f; t) = \begin{cases} (1-t)f\left(\dfrac{t}{1-t}\right), & t \in [0, 1) \\ 0, & t = 1 \end{cases}$$
and
$$V : \mathbb{R}^{[0,1)} \longrightarrow \mathbb{R}^{[0,+\infty)},\ V(g; x) = (1+x)g\left(\frac{x}{1+x}\right),\ x \geq 0,$$
$B_n f$ *being the n-th Bernstein operator, see J. A. Adell, F. G. Badia, J. De la Cal [2] and M. Ivan [9]);*
9. *If* $f \in \mathbb{R}^{[0,+\infty)}$, $f(x) = O(1+x)\ (x \to +\infty)$ *and f has derivative on a neighbourhood of the point* $x_0 \in [0, +\infty)$ *and* $f''(x_0)$ *exists, then*
$$\lim_{n\to\infty} n\left(L_n(f; x_0) - f(x_0)\right) = \frac{x_0(1+x_o)^2}{2} f''(x_0);$$

10. *If* $f \in \mathbf{C}[0,+\infty)$, $f(x) = o(x)\,(x \to +\infty)$, f *is convex function or* $f \in \mathbb{R}^{[0,+\infty)}$ *is nonincreasing and convex function, then* $L_n f$ *is convex and* $f \leq L_{n+1}f \leq L_n f$, $n \in \mathbb{N}$;

11. $f \in Lip_M\alpha \Longleftrightarrow L_n f \in Lip_M\alpha$, $\alpha \in (0,1]$.

Using a general method of construction of linear positive operators, which means associating to operator $H_n : \mathbf{C_B}(I) \longrightarrow \mathbf{C_B}(I)$, $H_n(f;x) = \sum\limits_{k=0}^{n} h_{n,k}(x) f(x_{n,k})$, $f \in \mathbf{C_B}(I)$, $x \in I$, a linear positive operator defined as follows:

$$I_n(f;x) = \sum_{k=0}^{n} h_{n,k}(x) F_{n,k}(f),\ f \in \mathbf{C_B}(I),\ x \in I, \tag{2}$$

(with $h_{n,k} \in \mathbf{C_B}(I)$, $h_{n,k} \geq 0$, under condition that $x_{n,k} \in I$ exists, the barycenter of a $\mu_{n,k}$ probability Borel measure on I, $n \geq 1$, $k = \overline{0,n}$ i.e. $x_{n,k} = \int\limits_I t d\mu_{n,k}(t)$ and $F_{n,k}(f) = \int\limits_I f(t) d\mu_{n,k}(t)$), we study in the next section a new operator which is associated to Bleimann - Butzer - Hahn operator.

2 A new linear positive operator

Let $F_{n,k}(f) = \begin{cases} f(0), & k = 0 \\ \int\limits_0^\infty f(t)\rho_{n,k}(t)dt, & 1 \leq k \leq n-1 \\ f(n), & k = n \end{cases}$, $n > 1$ be, for each k and n, a linear positive functional, defined for $\rho_{n,k}(t) = \begin{cases} 0, & t \leq 0 \\ \dfrac{1}{B(k,n-k+2)} \cdot \dfrac{t^{k-1}}{(1+t)^{n+2}}, & t > 0,\ 1 \leq k \leq n-1 \end{cases}$, $n > 1$ an Inverse - Beta probability density function, with $B(a,b) = \int\limits_0^\infty \dfrac{t^{a-1}}{(1+t)^{a+b}} dt$, $a > 0$, $b > 0$ the Inverse - Beta function.

We obtain the knots $x_{n,k} = \begin{cases} 0, & k = 0 \\ \dfrac{k}{n-k+1} = \int\limits_0^\infty t\rho_{n,k}(t)dt = \dfrac{B(k+1,n-k+1)}{B(k,n-k+2)}, & 1 \leq k \leq n \end{cases}$, and we consider a new linear positive operator $C_n : \mathbf{C_B}[0,+\infty) \longrightarrow \mathbf{C_B}[0,+\infty)$ defined by (2) as follows:

$$\begin{aligned} C_n(f;x) &= \frac{1}{(1+x)^n} f(0) \\ &+ \sum_{k=1}^{n-1} \binom{n}{k} \frac{x^k}{(1+x)^n} \cdot \frac{1}{B(k,n-k+2)} \int_0^\infty f(t) \frac{t^{k-1}}{(1+t)^{n+2}} dt \\ &+ \left(\frac{x}{1+x}\right)^n f(n), \end{aligned} \tag{3}$$

$x \geq 0$, $f \in \mathbf{C_B}[0, +\infty)$.

3 Some approximation properties

It is easy to see, that

$$C_n(e_0; x) = 1, \tag{4}$$

$$\begin{aligned}
C_n(e_1; x) &= \sum_{k=1}^{n-1} \binom{n}{k} \frac{x^k}{(1+x)^n} \cdot \frac{k}{n-k+1} + n\left(\frac{x}{1+x}\right)^n \\
&= \sum_{k=1}^{n-1} \binom{n}{k-1} \frac{x^k}{(1+x)^n} + n\left(\frac{x}{1+x}\right)^n = \sum_{j=0}^{n-2} \binom{n}{j} \frac{x^{j+1}}{(1+x)^n} + n\left(\frac{x}{1+x}\right)^n \\
&= \frac{x}{(1+x)^n}\left[(1+x)^n - nx^{n-1} - x^n\right] + n\left(\frac{x}{1+x}\right)^n = x - x\left(\frac{x}{1+x}\right)^n.
\end{aligned} \tag{5}$$

For the monomial function $e_2(x)$ we have

$$\begin{aligned}
C_n(e_2; x) &= \sum_{k=1}^{n-1} \binom{n}{k} \frac{x^k}{(1+x)^n} \cdot \frac{B(k+2, n-k)}{B(k, n-k+2)} + n^2\left(\frac{x}{1+x}\right)^n \\
&= \sum_{k=1}^{n-1} \binom{n}{k} \frac{x^k}{(1+x)^n} \cdot \frac{k(k+1)}{(n-k)(n-k+1)} + n^2\left(\frac{x}{1+x}\right)^n \\
&= \sum_{k=1}^{n-1} \binom{n}{k-1} \left(\frac{n+1}{n-k} - 1\right) \frac{x^k}{(1+x)^n} + n^2\left(\frac{x}{1+x}\right)^n \\
&= \sum_{k=1}^{n-1} \binom{n+1}{k-1} \frac{n-k+2}{n-k} \cdot \frac{x^k}{(1+x)^n} - \sum_{j=0}^{n-2} \binom{n}{j} \frac{x^{j+1}}{(1+x)^n} + n^2\left(\frac{x}{1+x}\right)^n \\
&= \sum_{k=1}^{n-1} \binom{n+1}{k-1} \frac{x^k}{(1+x)^n} + 2\sum_{k=1}^{n-1} \binom{n+1}{k-1} \frac{1}{n-k} \cdot \frac{x^k}{(1+x)^n} - x \\
&\quad + n\left(\frac{x}{1+x}\right)^n + x\left(\frac{x}{1+x}\right)^n + n^2\left(\frac{x}{1+x}\right)^n \\
&= \sum_{j=0}^{n-2} \binom{n+1}{j} \frac{x^{j+1}}{(1+x)^n} + 2\sum_{j=0}^{n-2} \binom{n+1}{j} \frac{1}{n-j-1} \cdot \frac{x^{j+1}}{(1+x)^n} - x \\
&\quad + n\left(\frac{x}{1+x}\right)^n + x\left(\frac{x}{1+x}\right)^n + n^2\left(\frac{x}{1+x}\right)^n \\
&= \frac{x}{(1+x)^n}\left[(1+x)^{n+1} - \frac{n(n+1)}{2} x^{n-1} - (n+1)x^n - x^{n+1}\right] \\
&\quad + 2\sum_{j=0}^{n-2} \binom{n+1}{j} \frac{1}{n-j-1} \cdot \frac{x^{j+1}}{(1+x)^n} - x + n\left(\frac{x}{1+x}\right)^n + x\left(\frac{x}{1+x}\right)^n + n^2\left(\frac{x}{1+x}\right)^n
\end{aligned}$$

$$= x^2 + \frac{n(n+1)}{2}\left(\frac{x}{1+x}\right)^n - nx\left(\frac{x}{1+x}\right)^n - \frac{x^{n+2}}{(1+x)^n} + R$$

with $R = 2\sum_{j=0}^{n-2}\binom{n+1}{j}\frac{1}{n-j-1}\cdot\frac{x^{j+1}}{(1+x)^n}$.

Since $\frac{1}{n-j-1} \le \frac{4}{n-j+2}$, $0 \le j \le n-2$, $n \ge 2$ we obtain

$$\begin{aligned} R &\le 8\sum_{j=0}^{n-2}\binom{n+1}{j}\frac{1}{n-j+2}\cdot\frac{x^{j+1}}{(1+x)^n} \\ &< 8x\sum_{j=0}^{n+1}\binom{n+1}{j}\frac{1}{n-j+2}\cdot\frac{x^j}{(1+x)^j}\left(1-\frac{x}{1+x}\right)^{n-j}. \end{aligned}$$

Considering the random variable $\frac{1}{n+2-U}$, with U having Bernoulli distribution with parameters $n+1$ and $p = \frac{x}{x+1}$, using the mean value, we have

$$\begin{aligned} R &< 8x(1+x)\sum_{j=0}^{n+1}\binom{n+1}{j}\frac{1}{n-j+2}\cdot\frac{x^j}{(1+x)^j}\left(1-\frac{x}{1+x}\right)^{n+1-j} \\ &= 8x(1+x)E\left[\frac{1}{n+2-U}\right]. \end{aligned}$$

Using the caracteristic function it is easy to see that, the random variable $n+1-U$ has a Bernoulli distribution with parameters $n+1$ and $q = 1-p = \frac{1}{x+1}$.

Together with a result of Chao and Strawdermann [8, (3.4)] we obtain for the mean value

$$E\left[\frac{1}{n+2-U}\right] = E\left[\frac{1}{1+(n+1-U)}\right] = \frac{1-p^{n+2}}{(n+2)q} < \frac{1}{(n+2)q} = \frac{1+x}{n+2}$$

and so, $R \le \frac{8x(1+x)^2}{n+2}$. It results

$$\begin{aligned} C_n(e_2;x) &\le x^2 + \frac{n(n+1)}{2}\left(\frac{x}{1+x}\right)^n - nx\left(\frac{x}{1+x}\right)^n - \frac{x^{n+2}}{(1+x)^n} + 8\frac{x(1+x)^2}{n+2} \\ &\le x^2 + \frac{n(n+1)}{2}\left(\frac{x}{1+x}\right)^n + 8\frac{x(1+x)^2}{n+2} \\ &\le x^2 + 11\frac{x(1+x)^2}{n+2}. \end{aligned} \tag{6}$$

Indeed, because $(1+x)^{n+2} \ge \binom{n+2}{3}x^{n-1}$, $x \ge 0$ it results $\frac{n(n+1)}{2}\left(\frac{x}{1+x}\right)^n \le \frac{3x(1+x)^2}{n+2}$ and so, $\left|C_n(e_2;x) - x^2\right| \le \frac{11x(1+x)^2}{n+2}$, $x \ge 0$.

Theorem 1. *For each function* $f \in \mathbf{C_B}[0,\infty)$, $x \geq 0$ *follows*

1.

$$\begin{cases} C_n(e_0;x) = 1 \\ C_n(e_1;x) = x - x\left(\dfrac{x}{1+x}\right)^n \\ C_n(e_2;x) \leq x^2 + \dfrac{11x(1+x)^2}{n+2},\ n \geq 2 \end{cases} \tag{7}$$

2.

$$C_n(f) \Longrightarrow f \ \text{on}\ [0,b],\ b>0 \ \text{(the symbol} \Longrightarrow \text{means the uniform convergence)}; \tag{8}$$

3. For $\delta > 0$, $x \geq 0$, $n \geq 2$

$$|C_n(f;x) - f(x)| \leq \left(1 + \frac{1}{\delta}\sqrt{\frac{13x(1+x)^2}{n+2}}\right)\omega(f,\delta); \tag{9}$$

4. If f *is differentiable and* $f' \in \mathbf{C_B}[0,\infty)$ *then* $(\forall)\delta > 0$, $x \geq 0$, $n \geq 2$

$$|C_n(f;x) - f(x)| \leq \frac{(1+x)^2}{n+2}\left|f'(x)\right| + 2\sqrt{\frac{13x(1+x)^2}{n+2}}\,\omega\left(f', \sqrt{\frac{13x(1+x)^2}{n+2}}\right). \tag{10}$$

Proof. **1.** It results from (4), (5) and (6).

2. Since operators C_n are positive, it is enough to prove the relation (8) for three test functions $e_i(x) = x^i$, $i = 0,1,2$, $x \geq 0$. Using the theorem Popoviciu - Bohmann - Korovkin we obtain with (4), (5), (6) the uniform convergence (8).

3. - 4. Let $\varphi_x(t) = |t-x|$, $t \geq 0$, be a function for each fixed $x \geq 0$. We have

$$\begin{aligned} C_n(\varphi_x;x) &= C_n(e_2;x) - 2xC_n(e_1;x) + x^2 \\ &\leq x^2 + \frac{11x(1+x)^2}{n+2} - 2x\left(x - x\left(\frac{x}{1+x}\right)^n\right) + x^2 \\ &\leq \frac{11x(1+x)^2}{n+2} + 2x^2\left(\frac{x}{1+x}\right)^n. \end{aligned}$$

Since, $\binom{n+2}{n+1}x^{n+1} \leq (1+x)^{n+2}$ we obtain $\left(\dfrac{x}{1+x}\right)^n < \dfrac{(1+x)^2}{x(n+2)}$ and

$$C_n(\varphi_x;x) < \frac{13x(1+x)^2}{n+2}. \tag{11}$$

Using a well-known result of O. Shisha , B. Mond [12, $Th.1$],with (7) and (11) we have (9) and (10).

□

References

[1] Abel, U., Ivan, M., *Some identities for the operators of Bleimann, Butzer and Hahn involving divided differences*, Calcolo **36** (1999), 143-160.

[2] Adell, J. A., Badia, F. G., De la Cal, J., *On the Iterates of Some Bernstein - Type Operators*, Journal of Math. Analyssis and Application **209** (1997), 529-541.

[3] Adell, J. A., De la Cal, J., *Limiting properties of Inverse Beta and generalized Bleimann - Butzer - Hahn operators*, Math. Proc. Camb. Phil. Soc. **114** (1993), 489-498.

[4] Adell, J. A., De la Cal, J., San Miguel, M., *Inverse Beta and Generalized Bleimann Butzer - Hahn Operators*, J. Approx. Theory **76** (1994), 54-64.

[5] Agratini, O., *Approximation properties of a generalization of Bleimann, Butzer and Hahn operators*, Mathematica Pannonica **9**, no. 2, (1998), 165-171.

[6] Altomare, F., Campiti, M., *Korovkin-type Approximation Theory and its Applications*, De Gruyter Series Studies in Mathematics, vol. 17, Walter de Gruyter & Co., Berlin, New York, 1994.

[7] Bleimann, G., Butzer, P. L., Hahn, L. A., *Bernstein - type operator approximating continuous functions on the semi-axis*, Indag. Math. **42** (1980), 255-262.

[8] Chao, M. T., Strawdermann, W. E., *Negative moments of positive random variables*, J. Amer. Statist. Assoc. **67** (1972), 429-431.

[9] Ivan, M., *A note on the Bleimann - Butzer - Hahn operator*, Automat. Comput. Appl. Math. **6** (1997), 11-15.

[10] De Vore, R. A., *The approximation of continuous functions by positive linear operators*, Lecture Notes in Math. 293, Springer - Verlag - Heidelberg - Berlin - New York, 1972.

[11] Stancu, D. D., *Asupra unei generalizri a polinoamelor lui Bernstein*, Studia Univ. Babe - Bolyai **14**, no. 2, (1969), 31-45.

[12] Shisha, O., Mond, B., *The degree of convergence of linear positive operators*, Proc. Nat. Acad. Sci. U.S.A. **60** (1968), 1196-1200.