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STRONGLY DIAGONAL DOMINANT MATRICES

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Abstract

An $n \cdot n$ matrix A is said to be strongly diagonally dominant if for every row of the matrix, the magnitude of the diagonal entry in a row is larger than or equal to the sum of the magnitudes of all the other (non-diagonal) entries in that row and the magnitude of the diagonal entry in a column is larger than or equal to the sum of the magnitudes of all the other (non-diagonal) entries in that column. Given a matrix A we show how a strongly diagonally dominant matrix B can be computed in $O(n^2)$ time so that the distance between A and B is minimum (the distance is measured using l_1 norm), i.e., the sum of differences between A and B is minimum.

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1 Introduction

In mathematics, a matrix $A = (a_{ij})_{,i,j \in \{1,2,....,n\}}$ is said to be diagonally dominant [1], [4], [5] if for every row of the matrix, the magnitude of the diagonal entry in a row is larger than or equal to the sum of the magnitudes of all the other (non-diagonal) entries in that row. More precisely, the matrix A is diagonally dominant if

$$|a_{ii}| \geq \sum_{j \in \{1,2,....,n\}, j \neq i} |a_{ij}|, i \in \{1, 2, ..., n\}. \quad (1)$$

We recall some applications and properties of the diagonally dominant matrices.

A strictly diagonally dominant matrix is non-singular. This result is known as the Levy-Desplanques theorem. This can be proved, for strictly diagonal dominant matrices, using the Gershgorin circle theorem [1][3].

No (partial) pivoting is necessary for a strictly column diagonally dominant matrix when performing Gaussian elimination (LU factorization).

The Jacobi and GaussSeidel methods for solving a linear system converge if the matrix is strictly (or irreducibly) diagonally dominant.

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Many matrices that arise in finite element methods are diagonally dominant.

A slight variation on the idea of diagonal dominance is used to prove that the pairing on diagrams without loops in the TemperleyLieb algebra is nondegenerate.[2] For a matrix with polynomial entries, one sensible definition of diagonal dominance is if the highest power of q appearing in each row appears only on the diagonal. (The evaluations of such a matrix at large values of q are diagonally dominant in the above sense.) [1]

As we can see above, there are applications of the diagonally dominancy on rows and on columns. In this paper we introduce the more restrictive diagonally dominancy property for a matrix (both, on rows and on columns).

Definition 1. *A matrix $A = (a_{ij})_{,i,j \in \{1,2,....,n\}}$ is said to be strongly diagonally dominant if for every row of the matrix, the magnitude of the diagonal entry in a row is larger than or equal to the sum of the magnitudes of all the other (non-diagonal) entries in that row and the magnitude of the diagonal entry in a column is larger than or equal to the sum of the magnitudes of all the other (non-diagonal) entries in that column, i.e.:*

$$|a_{ii}| \geq \sum_{k \in \{1,2,....,n\}, k \neq i} |a_{ik}|, i \in \{1, 2, ..., n\} \tag{2}$$

and

$$|a_{ii}| \geq \sum_{k \in \{1,2,....,n\}, k \neq i} |a_{ki}|, i \in \{1, 2, ..., n\}. \tag{3}$$

A matrix can be tested in $O(n^2)$ time if it is or not (strongly) diagonally dominant.

2 Transformation of a given matrix into a strongly diagonally dominant matrix

If a given matrix A is not (strongly) diagonally dominant we intend to find a method to construct a (strongly) diagonally dominant matrix B by modifying A as little as possible.

The problem of transforming a given matrix A into a diagonally dominant one so that the sum of modifications applied to A is minimum (denoted TDDM), is not difficult to solve, since the dominancy is only on each row of the matrix. It can be solved by computing the sum of all non-diagonal elements in absolute value on each row i:

$$s_i^r = \sum_{k \in \{1,2,....,n\}, k \neq i} |a_{ik}| \tag{4}$$

An optimum solution (denoted B) of TDDM can be easily constructed as follows:

$$b_{ij} = \begin{cases} sgn(a_{ii}) \cdot s_i^r, \text{if } i = j \text{ and } s_i^r > |a_{ii}| \\ a_{ij}, \text{otherwise} \end{cases} \tag{5}$$

The transformation of a matrix into a strongly diagonal matrix is not a trivial task since the diagonally dominance must have effect on both, rows and columns.

We shall present now a method that can be used to transform a given matrix $A = (a_{ij})_{,i,j\in\{1,2,\ldots,n\}}$ into a matrix $B = (b_{ij})_{,i,j\in\{1,2,\ldots,n\}}$ that is strongly diagonally dominant and the distance between A and B measured using l_1 norm is minimum, i.e., the sum of modifications applied to A is minimum:

$$min\{|A - B||B \text{ is a strongly diagonally dominant matrix}\} \tag{6}$$

where

$$|A - B| = |A - B|_1 = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij} - b_{ij}| \tag{7}$$

We denote this problem as TSDDM.

We cannot apply the similar technique of TDDM for TSDDM (modification of A only on diagonal). Here is an example:

$$A = \begin{pmatrix} -5 & 2 & 5 \\ 3 & 2 & -1 \\ -3 & -4 & 4 \end{pmatrix} \tag{8}$$

If we modify only the diagonal elements so that the resulting matrix is strongly diagonally dominant, we obtain:

$$B = \begin{pmatrix} -7 & 2 & 5 \\ 3 & 6 & -1 \\ -3 & -4 & 7 \end{pmatrix} \tag{9}$$

The sum of modifications is:

$$|A - B| = 2 + 4 + 3 = 9 \tag{10}$$

The following matrix is strongly diagonally dominant but the sum of modifications is lower:

$$B' = \begin{pmatrix} -5 & 0 & 5 \\ 2 & 2 & 0 \\ -3 & -2 & 5 \end{pmatrix} \tag{11}$$

The distance between A and B' is

$$|A - B'| = 7 < 9 \tag{12}$$

Moreover, B' is the optimum solution for TSDDM.

Let us see now how such a solution can be obtained.

It is clear that in order to solve TSDDM, if modified, the values on diagonal are increased in absolute value and if modified, the non-diagonal values are decreased in absolute value.

Let us consider a pair of indices i and j, $i \neq j$. We make the following notations:

$$s_i^r = \sum_{k \in \{1,2,....,n\}, k \neq i} |a_{ik}| \tag{13}$$

and

$$s_j^c = \sum_{k \in \{1,2,....,n\}, k \neq j} |a_{kj}| \tag{14}$$

The following algorithm solves TSDDM:

ATSDDM:

$B = A$;
if A is not strongly diagonally dominant **then**
  Compute the vectors S^r and S^c using formulas 13 and 14;
  for i=1 **to** n do
    for j=1 **to** n do
      if $i \neq j$ **and** $|a_{ii}| < s_i^r$ **and** $|a_{jj}| < s_j^c$ **and** $|a_{ij}| > 0$ **then**
        $m = min\{s_i^r - |a_{ii}|, s_j^c - |a_{jj}|, |a_{ij}|\}$;
        $b_{ij} = sgn(a_{ij}) \cdot (|aij| - m)$;
      end if;
    end for;
  end for;
  Compute the vectors S^r and S^c;
  for i=1 **to** n do
    if $|b_{ii}| < s_i^r$ **or** $|b_{ii}| < s_i^c$ **then**
      $b_{ii} = sgn(a_{ii}) \cdot max\{s_i^r, s_i^c\}$;
    end if;
  end for;
end if;
B is optimum solution of TSDDM.

Theorem 1. *ATSDDM ends in a finite number of steps and the matrix B is optimum solution of TSDDM.*

Proof

The algorithm ends in finite number of steps since we have only finite *for* statements.

We get closer to the optimum solution of TSDDM if we increase in absolute value (where needed) the values on the diagonal of the matrix or if we decrease in absolute value (where needed) the values of non-diagonal elements of the matrix.

If we decrease the value of $|a_{ij}|$ by $min\{s_i^r - |a_{ii}|, s_j^c - |a_{jj}|, |a_{ij}|\}$ we get closer to a strongly diagonally dominant matrix and we (partially) solve the problem for two diagonal elements, a_{ii} and a_{jj}. If we do this for each pair of indices i and j the resulting matrix may still not be strongly diagonally dominant. If this happens, we then can increase the value of diagonal elements in absolute value (the last part of the algorithm).

Theorem 2. *The time complexity of ATSDDM is $O(n^2)$.*

Proof

It is easy to see that the time complexity of ATSDDM is $O(n^2)$, since the verification of A being strongly diagonally dominant and the computation of S^r and S^c take $O(n^2)$ time. Besides that we have two for statements, both from 1 to n and one more for statement from 1 to n at the end of the algorithm.

So, the time complexity of ATSDDM is $O(n^2)$.

If we apply ATSDDM to the matrix A in the above example we obtain B' (see 11) as the optimum solution.

3 Conclusion and future work

In this paper we presented a method in $O(n^2)$ time to transform a given matrix into a strongly diagonally dominant one so that the distance measured using l_1 norm between the two matrices is minimum.

As a future work this problem can be studied under other norms or distance formulas such as weighted l_1 norm, l_2 norm, Hamming bottleneck distance or sum type distance, etc.

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