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ON THE UNIVALENCE OF CERTAIN INTEGRALS

Horiana TUDOR[1]

Abstract

We obtain sufficient conditions for the analyticity and the univalence of some integral operators if involved functions belong to subclasses of univalent functions.

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1 Introduction

Let $\mathcal{A}$ be the class of analytic functions f in the unit disk $\mathcal{U} = \{ z \in \mathbb{C} : |z| < 1\}$ with $f(0) = 0,\ f'(0) = 1$. Let $\mathcal{S}$ denote the class of functions $f \in \mathcal{A}$, f univalent in $\mathcal{U}$.

An important problem in the theory of univalent functions is to find integral operators which preserve the class of univalent functions. We mention the well known integral operators due to Kim and Merkes [2], Pfaltzgraff [5], Moldoveanu and Pascu [4] and the recently generalization of these results obtained by the author in [6].

Theorem 1. ([6]). *Let $f,\ g \in \mathcal{S}$ and $\alpha,\ \beta,\ \gamma \in \mathbb{C}$, $n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$. If*

$$|\alpha - 1| + |\beta| + |\gamma| \le \frac{1}{4n} \tag{1}$$

then the function

$$G_{n,\alpha}(z) = \left[(n(\alpha-1)+1) \int_0^z g^{\alpha-1}(u^n) \left(\frac{f(u^n)}{u^n} \right)^{\beta} \left(f'(u^n) \right)^{\gamma} du \right]^{\frac{1}{n(\alpha-1)+1}} , \tag{2}$$

is analytic and univalent in $\mathcal{U}$. Also the functions $G_{n-k,\alpha}(z)$, $k \in \mathbb{N}$, $1 \le k \le n-1$ defined by (2) are analytic and univalent in $\mathcal{U}$.

[1]Faculty of Mathematics and Informatics, *Transilvania* University of Braşov, Romania,
e-mail: htudor@unitbv.ro

The usual subclasses of the class $\mathcal{S}$ consisting of starlike and convex functions will be denoted by $\mathcal{S}^*$, respectively $\mathcal{C}V$. Also we consider the subclasses of φ-spiral functions of order ρ and convex φ-spiral functions of order ρ defined as follows:

$$\mathcal{S}^*(\varphi,\rho)=\left\{ f\in S:\ \ Re\left(e^{i\varphi}\frac{zf'(z)}{f(z)}\right) > \rho\cos\varphi, \quad z\in\mathcal{U}\right\}$$

and

$$\mathcal{C}(\varphi,\rho)=\left\{ f\in S:\ \ Re\left[e^{i\varphi}\left(1+\frac{zf''(z)}{f'(z)}\right)\right] > \rho\cos\varphi,\ \ z\in\mathcal{U}\right\},$$

where $\varphi\in(-\pi/2,\ \pi/2)$, $\rho\in[0,1)$.

We observe that $\mathcal{S}^*=\mathcal{S}^*(0,0)$ and $\mathcal{C}V=\mathcal{C}(0,0)$.

2 Preliminaries

We first recall here some results which will be used in the sequel.

Theorem 2. ([3]). *If $f\in\mathcal{S}^*(\varphi,\rho)$ and a is a fixed point from the unit disk $\mathcal{U}$, then the function h,*

$$h(z)=\frac{a\cdot z}{f(a)(z+a)(1+\bar{a}z)^{\psi}}\cdot f\left(\frac{z+a}{1+\bar{a}z}\right) \tag{3}$$

where

$$\psi=e^{-2i\varphi}-2\rho e^{-i\varphi}\cos\varphi \tag{4}$$

is a function of the class $\mathcal{S}^*(\varphi,\rho)$.

The new results will be proved by using the following univalence criteria:

Theorem 3. ([6]). *Let α, c be complex numbers, n natural number, $n\geq 1$, such that*

$$\left|\alpha-2+\frac{1}{n}\right|<1 \qquad \text{and} \qquad |\,c\,|<1. \tag{5}$$

For $h\in\mathcal{A}$, if the inequality

$$\left| c|z|^{2n}+(1-|z|^{2n})\left[(\alpha-1)\frac{z^nh'(z^n)}{h(z^n)}+\frac{1-n}{n}\right]\right|\leq 1 \tag{6}$$

is true for all $z\in\mathcal{U}$, then the function

$$H_{n,\alpha}(z)=\left[(n(\alpha-1)+1)\int_0^z h^{\alpha-1}(u^n)du\right]^{\frac{1}{n(\alpha-1)+1}}, \tag{7}$$

is analytic and univalent in $\mathcal{U}$, where the principal branch is considered.

3 Main result

Next we consider the case when functions f and g belong to some subsets of $\mathcal{S}$ and we expect that the hypothesis (1) of the Theorem 1 becomes larger.

Theorem 4. *Let $f,\ g \in \mathcal{S}^*(\varphi,\rho)$, $\alpha,\ \beta \in \mathbb{C}$, $n \in \mathbb{N}^*$. If*

$$(\ 1 + 2(1-\rho)\cos\varphi\ )|\alpha - 1| + 2(1-\rho)\cos\varphi \cdot |\beta| < \frac{1}{n} \tag{8}$$

then the function

$$G_{n,\alpha}(z) = \left[(n(\alpha-1)+1) \int_0^z g^{\alpha-1}(u^n) \left(\frac{f(u^n)}{u^n} \right)^{\beta} du \right]^{\frac{1}{n(\alpha-1)+1}} , \tag{9}$$

is analytic and univalent in $\mathcal{U}$. Also all the functions $G_{n-k,\alpha}(z)$, $k \in \mathbb{N}$, $1 \leq k \leq n-1$ defined by (9) are analytic and univalent in $\mathcal{U}$.

Proof. Let $f,\ g \in \mathcal{S}^*(\varphi,\rho)$. Since f is univalent in $\mathcal{U}$ we can choose the analytic branch of $\left(\frac{f(z)}{z}\right)^{\beta}$ equal to 1 at the origin. Let now a function $h \in \mathcal{A}$, $h(z) \neq 0$ for all $z \in \mathcal{U} \setminus \{0\}$. Since g is univalent in $\mathcal{U}$ we can choose the analytic branch of $\left(\frac{h(z)}{g(z)}\right)^{\alpha-1}$ equal to 1 at the origin. We consider that function h for which we have

$$\left(\frac{h(z)}{g(z)}\right)^{\alpha-1} = \left(\frac{f(z)}{z}\right)^{\beta}$$

or

$$h^{\alpha-1}(u^n) = g^{\alpha-1}(u^n) \left(\frac{f(u^n)}{u^n} \right)^{\beta} \tag{10}$$

For this function h we shall establish if inequality (6) from Theorem 3 is true. We mention that (6) assures that h does not vanish in $\mathcal{U} \setminus \{0\}$. It is known that for $f \in \mathcal{S}^*(\varphi,\rho)$, $f(z) = z + a_2 z^2 + \ldots$, we have (see [1]).

$$|\ a_2\ | \leq 2(1-\rho)\cos\varphi. \tag{11}$$

Also, for $f \in \mathcal{S}^*(\varphi,\rho)$, let h be the function defined by (3), $h(z) = z + a_2 z^2 + \ldots$. We have

$$a_2 = \frac{h''(0)}{2} = (1-|a|^2)\frac{f'(a)}{f(a)} - \frac{1+\psi|a|^2}{a} ,$$

where ψ is given by (4). It follows that

$$\frac{af'(a)}{f(a)} = \frac{1 + a \cdot a_2 + \psi|a|^2}{1-|a|^2} \tag{12}$$

According to (12) and (10) we have

$$c|z|^{2n} + (1-|z|^{2n}) \left[\ (\alpha-1)\frac{z^n h'(z^n)}{h(z^n)} + \frac{1-n}{n}\ \right] \tag{13}$$

$$= c|z|^{2n} + (1-|z|^{2n})\left[\ (\alpha-1)\frac{z^n g'(z^n)}{g(z^n)} + \beta\left(\frac{z^n f'(z^n)}{f(z^n)} - 1\right) + \frac{1-n}{n}\ \right]$$

$$= c|z|^{2n} + (\alpha-1)(1+a_2z^n+\psi|z|^{2n}) + \beta(b_2z^n+\psi|z|^{2n}+|z|^{2n}) + \frac{1-n}{n}(1-|z|^{2n})$$

$$= |z|^{2n}\left[\ c+(\alpha-1)\psi+\beta(\psi+1)+\frac{n-1}{n}\right] + (\alpha-1)(1+a_2z^n) + b_2\beta z^n + \frac{1-n}{n}.$$

If $c = -[\ (\alpha-1)\psi+\beta(\psi+1)+\frac{n-1}{n}\ ]$, then

$$|\ c\ | = \left|(\alpha-1)(\psi+1-1)+\beta(\psi+1)+\frac{n-1}{n}\ \right| \le |\alpha-1|(|\psi+1|+1)+|\beta||\psi+1|+\frac{n-1}{n}$$

and since $|\psi+1| = 2(1-\rho)\cos\varphi$, in view of (8), it is clear that $|c|<1$. It is easy to check that inequality (8) implies also

$$|\ \alpha-2+\frac{1}{n}\ | \le |\alpha-1| + \frac{n-1}{n} \le \frac{1}{n(1+2(1-\rho)\cos\varphi)} + \frac{n-1}{n} < 1.$$

Taking into account (8) and (11), relation (13) reduces to

$$\left|c|z|^{2n} + (1-|z|^{2n})\left[(\alpha-1)\frac{z^n h'(z^n)}{h(z^n)} + \frac{1-n}{n}\right]\right| \le |\alpha-1|(1+|a_2|)+|b_2||\beta|+\frac{n-1}{n} < 1$$

The conditions of Theorem 3 are verified. We can conclude that the function $H_{n,\alpha}(z)$ defined by (7) is analytic and univalent in $\mathcal{U}$,

$$H_{n,\alpha}(z) = \left[(n(\alpha-1)+1)\int_0^z h^{\alpha-1}(u^n)du\right]^{\frac{1}{n(\alpha-1)+1}}$$

$$= \left[(n(\alpha-1)+1)\int_0^z g^{\alpha-1}(u^n)\left(\frac{f(u^n)}{u^n}\right)^{\beta} du\right]^{\frac{1}{n(\alpha-1)+1}}$$

Therefore, the function $G_{n,\alpha}(z)$ defined by (9) is analytic and univalent in $\mathcal{U}$. Since $\frac{1}{n} < \frac{1}{n-k}$, for k natural number, $1 \le k \le n-1$, inequality (8) implies

$$(\ 1+2(1-\rho)\cos\varphi\ )|\alpha-1| + 2(1-\rho)\cos\varphi\cdot|\beta| \le \frac{1}{n-k},$$

and then all the functions $G_{n-k,\alpha}(z)$ defined by (9) are analytic and univalent in $\mathcal{U}$. $\square$

Corollary 1. *Let* $f,\ g \in \mathcal{S}^*,\ \alpha,\ \beta \in \mathbb{C}\ ,\ n \in \mathbb{N}^*$. *If*

$$3|\alpha-1| + 2|\beta|\ <\ \frac{1}{n} \tag{14}$$

then the function $G_{n,\alpha}(z)$ *defined by (9) and all the functions* $G_{n-k,\alpha}(z)$, $k \in \mathbb{N}$, $1 \le k \le n-1$ *are analytic and univalent in* $\mathcal{U}$.

Theorem 5. *Let* $g \in \mathcal{S}^*(\varphi, \rho)$, $f \in \mathcal{C}(\varphi, \rho)$, α, $\gamma \in \mathbb{C}$, $n \in \mathbb{N}^*$. *If*

$$(1 + 2(1 - \rho) \cos \varphi)|\alpha - 1| + 2(1 - \rho) \cos \varphi \cdot |\gamma| < \frac{1}{n} \tag{15}$$

then the function

$$G_{n,\alpha}(z) = \left[(n(\alpha - 1) + 1) \int_0^z g^{\alpha - 1}(u^n) \left(f'(u^n) \right)^{\gamma} du \right]^{\frac{1}{n(\alpha-1)+1}} , \tag{16}$$

is analytic and univalent in $\mathcal{U}$. *Also all the functions* $G_{n-k,\alpha}(z)$, $k \in \mathbb{N}$, $1 \leq k \leq n - 1$ *defined by (16) are analytic and univalent in* $\mathcal{U}$.

Proof. The proof of Theorem 5 is analogous to that of Theorem 4 and it uses the relationship between the classes $S^*(\varphi, \rho)$ and $C(\varphi, \rho)$: if $f \in C(\varphi, \rho)$ then $h \in S^*(\varphi, \rho)$, where $h(z) = zf'(z)$. □

Corollary 2. *Let* $g \in \mathcal{S}^*$, $f \in \mathcal{C}V$ *and* α, $\gamma \in \mathbb{C}$, $n \in \mathbb{N}^*$. *If*

$$3|\alpha - 1| + 2|\gamma| < \frac{1}{n} \tag{17}$$

then the function $G_{n,\alpha}(z)$ *defined by (16) and all the functions* $G_{n-k,\alpha}(z)$, $k \in \mathbb{N}$, $1 \leq k \leq n - 1$ *are analytic and univalent in* $\mathcal{U}$.

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