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MATHEMATICAL MODEL FOR THE 0.85 BILLION YEAR SUN

Emil TATOMIR[1]

Abstract

An algorithm for constructing evolutionary tracks for a star with the mass equal to one solar mass is given. The model presented can be applied to the stars belonging to the inferior main sequence, which have the proton-proton reaction as energy source and present a radiative core and a convective shell. This paper presents an original way of resolving the system of equations corresponding to the radiative nucleus by using Taylor's series in close vicinity to the center of the Sun. It also presents the numerical integration and the results for a 0.85 billion year solar model.

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1 Basic formulae for the evolutive model

Consider that for the radiative core of the Sun the equations of hydrostatic equilibrium, mass distribution, luminosity and temperature are valid (see, e.g., Menzel and others, 1963; Aller and McLaughlin, 1965; Cox and Giuli, 1968), given by:

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}, \tag{1}$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r), \tag{2}$$

$$\frac{dL(r)}{dr} = 4\pi^2 \rho(r)\varepsilon(r), \tag{3}$$

$$\frac{dT(r)}{dr} = -\frac{3}{4ac}\frac{\kappa(r)\rho(r)}{T^3(r)}\frac{L(r)}{4\pi r^2}, \tag{4}$$

[1]Faculty of Mathematics and Informatics, *Transilvania* University of Braşov, Romania, e-mail: emil.tatomir@gmail.com

where $P(r), M(r), L(r)$ and $T(r)$ represent the values of the pressure, the mass, the luminosity and the temperature in a point placed at the distance r from the center of the star. By using Schwarzschild's (1958) transformations:

$$\begin{aligned} P(r) &= \frac{pGM^2}{4\pi R^4}, \\ M(r) &= qM, \\ L(r) &= fL, \\ T(r) &= t\frac{\mu H}{k}\frac{GM}{R}, \\ r &= xR, \end{aligned} \tag{5}$$

the dimensionless variables p, q, f, t, x are introduced. With these variables, the system $(1)-(4)$ becomes:

$$\begin{aligned} \frac{dp}{dx} &= -\frac{pq}{tx^2}, \\ \frac{dq}{dx} &= \frac{px^2}{t}, \\ \frac{df}{dx} &= C\frac{px^2}{t}, \\ \frac{dt}{dx} &= -D\frac{pf}{t^4x^2}, \end{aligned} \tag{6}$$

where we have denoted:

$$\begin{aligned} C &= \frac{M}{L}\left(\varepsilon_{pp} + \varepsilon_{CN}\right), \\ D &= \frac{3Lk^4}{64\pi^2 ac\mu^4 H^4 G^4 M^4}\kappa, \\ A &= \frac{3Lk^4}{64\pi^2 acH^4 G^4 M^4}. \end{aligned} \tag{7}$$

The production of energy per gram-mass and per second due to the proton-proton reaction is given by the relation:

$$\begin{aligned} \varepsilon_{pp} = {} & \varepsilon_0\left(1 + 0.25\rho^{\frac{1}{2}}T_6^{-\frac{3}{2}}\right)\left(1 + 0.012T_6^{\frac{1}{3}} + 0.008T_6^{\frac{2}{3}} + 0.00065T_6\right) \\ & \rho X^2 10^6 T_6^{-\frac{2}{3}} e^{-33.804T_6^{-\frac{1}{3}}}, \end{aligned} \tag{8}$$

where $\varepsilon_0 = 2.625, \rho$ is the matter density expressed in $\frac{g}{cm^3}, T_6$ is the temperature expressed in 10^6K, while X denotes the hydrogen abundance.

Since the production of energy due to the carbon-nitrogen cycle will also be considered, this one is given by the relation:

$$\varepsilon_{CN} = 7.94 \cdot 10^{27} \left(1 + 0.027 T_6^{\frac{1}{3}} - 0.0037 T_6^{\frac{2}{3}} - 0.00007 T_6\right) \left(1 + 1.75 \rho^{\frac{1}{2}} T_6^{-\frac{3}{2}}\right) 0.00445 \rho \cdot X^2 e^{-152.313 T_6^{-\frac{1}{3}}} T_6^{-\frac{2}{3}}. \quad (9)$$

The formulae (8) and (9) are in (Aller and McLaughlin, 1965). The total production of energy per gram-mass and per second is given by the formula:

$$\varepsilon = \varepsilon_{pp} + \varepsilon_{CN}. \quad (10)$$

The opacity of stellar matter is considered to be given by the following analytical expressions:

- for $0.1 \leq T_6 < 10$:

$$\kappa = 0.19\,(1 + X) + a_1 \rho\,(1 + X)\,; \quad (11)$$

- for $10 \leq T_6 \leq 20$:

$$\kappa = 0.19\,(1 + X) + a_1 \rho\,(1 + X) + \rho\,(1 - 0.1 T_6)\,((1 + X)\,a_1 - b_1)\,; \quad (12)$$

- for $T_6 > 20$:

$$\kappa = 0.19\,(1 + X) + b_1 \rho; \quad (13)$$

where a_1 and b_1 have respectively the expressions:

$$a_1 = 6.5 \cdot 10^4 \frac{Z\,(1 - 0.05 T_6)}{T_6^2 + 2.5 T_6^4} e^{-7.75 \rho \frac{1+X}{T_6^3}} + 4.15 \cdot 10^4 \left(\frac{X}{250 T_6^4 - T_6^2} + \frac{1 - X - Z}{250 T_6^4} \left(1 + 5.5 e^{-\left|\frac{2}{3} - 2.873 T_6\right|}\right)\right) e^{-2\rho \frac{1+X}{T_6^3}} \quad (14)$$

and

$$b_1 = \frac{35 + 75X + 190Z}{T_6^{3.5}} \quad (15)$$

in which Z features the metals.

For the convective shell of the Sun, equations (1) and (2) , and the adiabatic equation:

$$P\,(r) = K \rho^{\gamma}\,(r)\,, \quad \gamma = \frac{5}{3}, \quad (16)$$

are considered to be valid. The ideal gas law is considered to be valid for the whole star.

Introducing the parameter:

$$(n+1)_{rad} = \frac{d\log P}{d\log T} = \frac{1}{D}\frac{qt}{pf}, \tag{17}$$

the radiative equilibrium is ensured by the condition:

$$(n+1)_{rad} > 2.5. \tag{18}$$

The system (1) − (4) is to be integrated with the boundary conditions (at center):

$$x = 0, \quad f = 0, \quad q = 0, \quad t = t_c, \quad p = p_c, \tag{19}$$

where t_c and p_c denote the dimensionless values of temperature and pressure, respectively, at the Sun's center.

2 Numerical solution of the model

For the evolutive model which will be presented, we shall use the numerical results obtained by the author (Tatomir, 1986), which provide: pressure P, temperature T, dimensionless mass q and dimensionless luminosity f values in the points of a division:

$$\begin{aligned} x_1 < x_2 < ... < x_{155}&, \\ x_1 = 0&, \\ x_i = (i-1)\,h&, \end{aligned} \tag{20}$$

where the integration step was taken $h = 0.0058$.

The following values (corresponding to the parameters of the homogeneous model and the constants appearing in calculations) are used:

$$\begin{array}{ll} L_o^* = 3.12E+33\left(\frac{erg}{s}\right), & c = 2.9978E+10\left(\frac{cm}{s}\right), \\ R_o = 6.96E+10\,(cm), & G = 6.672E-08\left(\frac{cm^3}{gs^2}\right), \\ M_o = 1.99E+33\,(g), & Q_{pp}^* = 6.3E+18\left(\frac{erg}{s}\right), \\ k = 1.379E-16\,(erg), & Q_{CN}^* = 6.0E+18\left(\frac{erg}{s}\right), \\ H = 1.672E-24\,(g), & X = 0.709, \\ a = 7.55E-15\left(\frac{dyne}{cm^2}\right), & Z = 0.021, \end{array} \tag{21}$$

System (6) has a singularity in $x_1 = 0$; for calculating the values of p, q, f, t in the points $x_1, x_2, ..., x_7$, we have used in (14) the expanding in series method, obtaining:

$$p\,(x) = p_0 - \frac{1}{6}\frac{p_0^2}{t_0^2}x^2 + \left(\frac{1}{45}\frac{p_0^3}{t_0^4} - \frac{DC}{45}\frac{p_0^4}{t_0^8}\right)x^4 + 0\cdot x^5 + ..., \tag{22}$$

$$q(x) = \frac{1}{3}\frac{p_0}{t_0}x^3 + \left(\frac{DC}{30}\frac{p_0^3}{t_0^7} - \frac{1}{30}\frac{p_0^2}{t_0^3}\right)x^5 + 0 \cdot x^6 + ..., \tag{23}$$

$$f(x) = \frac{C}{3}\frac{p_0}{t_0}x^3 + C\left(\frac{DC}{30}\frac{p_0^3}{t_0^7} - \frac{1}{30}\frac{p_0^2}{t_0^3}\right)x^5 + 0 \cdot x^6 + ..., \tag{24}$$

$$t(x) = t_0 - \frac{DC}{6}\frac{p_0^2}{t_0^5}x^3 + \left(\frac{DC}{45}\frac{p_0^3}{t_0^7} - \frac{23D^2C^2}{360}\frac{p_0^4}{t_0^{11}}\right)x^4 + ..., \tag{25}$$

where we have denoted $p_0 = p_c$ and $t_0 = t_c$.

By means of the values $X(x_i, 0), \rho(x_i, 0), T(x_i, 0)$ from the homogeneous model, one calculates the production of energy $\varepsilon_{pp}(x_i, 0) + \varepsilon_{CN}(x_i, 0)$ and the opacity $\kappa(x_i, 0)$, using the expressions $(8) - (13)$.

As time step, we have chosen $\tau = 0.1 \cdot 10^9$ years; with this, the variation of chemical composition with time due to the nuclear reactions is given by:

$$X(x_i, \tau) = X(x_i, 0) - \left(\frac{\varepsilon_{pp}(x_i, 0)}{Q^*_{pp}} + \frac{\varepsilon_{CN}(x_i, 0)}{Q^*_{CN}}\right)\tau \tag{26}$$

At first for each integration point x_i and for the epoch τ the molecular weight μ and the values of the coefficients C, D are calculated:

$$\mu(x_i, \tau) = \frac{4}{3 + 5X(x_i, \tau) - Z}, \tag{27}$$

$$C(x_i, \tau) = \frac{1.99}{3.12}\left(\varepsilon_{pp}(x_i, 0) + \varepsilon_{CN}(x_i, 0)\right), \tag{28}$$

$$D(x_i, \tau) = A\frac{\kappa(x_i, 0)}{\mu^4(x_i, \tau)}, \tag{29}$$

where A is a numerical constant known from (7). In order to obtain the central values of density (ρ) and temperature (T) at instant τ, the nonlinear system:

$$\varepsilon_{pp}(0,0) + \varepsilon_{CN}(0,0) = \varepsilon_{pp}(\rho, T, X(0, \tau)) + \varepsilon_{CN}(\rho, T, X(0, \tau)), \tag{30}$$

$$\kappa(0,0) = \kappa(\rho, T, X(0, \tau)), \tag{31}$$

is solved by means of the Newton-Kantorovici method.

Next we will show how the Newton-Kantorovici method is used for the evolutive solar model.

Introducing the next notations:

$$\begin{aligned} f(\rho, T) \quad = \quad & \varepsilon_{pp}(\rho, T, X(0, \tau)) + \varepsilon_{CN}(\rho, T, X(0, \tau)) - \\ & \varepsilon_{pp}(0,0) - \varepsilon_{CN}(0,0), \end{aligned} \tag{32}$$

$$g(\rho,T)=\kappa(\rho,T,X(0,\tau))-\kappa(0,0) \tag{33}$$

system (30) receives the form:

$$\begin{aligned} f(\rho,T) &= 0 \\ g(\rho,T) &= 0 \end{aligned} \tag{34}$$

And now

$$H : R^2 \to R^2, \quad H(\rho,T)=\begin{pmatrix} f(\rho,T) \\ g(\rho,T) \end{pmatrix} \tag{35}$$

where f and g are some functions of density ρ and temperature T.

Then system (33) can be written under the form:

$$H(\rho,T)=0 \tag{36}$$

To obtain the solution of system (35) we start with an initial value of $\begin{pmatrix} \rho_k \\ T_k \end{pmatrix}$ which, in our case, is just the central value from the homogeneous model, $\begin{pmatrix} \rho_k \\ T_c \end{pmatrix}$. A better approximation of the solution of system (33) can be obtained from the next formula:

$$\begin{pmatrix} \rho_{k+1} \\ T_{k+1} \end{pmatrix} = \begin{pmatrix} \rho_k \\ T_k \end{pmatrix} - \begin{pmatrix} \frac{\partial f(\rho_k,T_k)}{\partial \rho} & \frac{\partial f(\rho_k,T_k)}{\partial T} \\ \frac{\partial g(\rho_k,T_k)}{\partial \rho} & \frac{\partial g(\rho_k,T_k)}{\partial T} \end{pmatrix}^{-1} \cdot \begin{pmatrix} f(\rho_k,T_k) \\ g(\rho_k,T_k) \end{pmatrix} \tag{37}$$

We note with ρ_c^1 and T_c^1 the central values for the model of the type τ; they are ρ^{k+1} and T^{k+1} in formula (36) . Starting from ρ_c^1 and T_c^1 we obtain t_c^1 and p_c^1 from Schwarzschild's transformations (5) and from the law of gases

$$P(r)=\frac{1}{\mu}\frac{k}{H}\rho(r)\,T(r) \tag{38}$$

With $t_0=t_c^1$ and $p_0=p_c^1$ and by help of the series $(22)-(25)$, we obtain the values for p,q,f,t in six points near the origin: $x_2,x_3,...,x_7$. System (6) is integrated using the Adams-Bashforth method of the sixth order (Moszynski, 1973)

$$V_{k+1}=V_k+\frac{h}{1440}(4277f_k-7923f_{k-1}+9982f_{k-2}-7298f_{k-3}+ \\ 2877f_{k-4}-475f_{k-5})$$

To improve the numerical results which have been obtained with the help of formula (38) , the Adams-Moulton corrector method of the sixth order is used (Moszynski, 1973):

$$V_k = V_{k-1} + \frac{h}{1440}\left(475 f_k + 1427 f_{k-1} - 798 f_{k-2} + 482 f_{k-3} - 173 f_{k-4} + 27 f_{k-5}\right) \tag{39}$$

In this way we obtain the values for $t^1(x_i,\tau), p^1(x_i,\tau), f^1(x_i,\tau), q^1(x_i,\tau), \rho^1(x_i,\tau), T^1(x_i,\tau)$. The integration of system(6) continues as long as $(n+1)_{rad} \leq 2.5$. The integration of the model at the moment τ is repeated iteratively and we consider that for iteration n we have the next values:

$$X^n(x_i,\tau), \rho^n(x_i,\tau), T^n(x_i,\tau), p^n(x_i,\tau), f^n(x_i,\tau), q^n(x_i,\tau), \tag{40}$$

The passing from iteration n to iteration $n+1$ of the model at moment τ is done in this way:

$$X^{n+1}(x_i,\tau) = X(x_i,0) - \frac{1}{2}\tau\left(\frac{\varepsilon_{pp}(x_i,0) + \varepsilon_{pp}^n(x_i,\tau)}{Q_{pp}^*} + \frac{\varepsilon_{CN}(x_i,0) + \varepsilon_{CN}^n(x_i,\tau)}{Q_{CN}^*}\right)$$

$$\mu^{n+1}(x_i,\tau) = \frac{4}{3 + 5X^n(x_i,\tau) - Z} \tag{41}$$

$$C^{n+1}(x_i,\tau) = \frac{1.9891}{3.826}\left(\varepsilon_{pp}^n(x_i,\tau) + \varepsilon_{CN}^n(x_i,\tau)\right) \tag{42}$$

$$D^{n+1}(x_i,\tau) = A\frac{\kappa^n(x_i,\tau)}{\left(\mu^n(x_i,\tau)\right)^4} \tag{43}$$

The central values of the model ρ_c^{n+1}, T_c^{n+1} at moment τ and at iteration $n+1$ are obtained from the system:

$$\frac{1}{2}\left(\varepsilon_{pp}(0,0) + \varepsilon_{pp}^n(0,\tau) + \varepsilon_{CN}(0,0) + \varepsilon_{CN}^n(0,\tau)\right) = \varepsilon_{pp}\left(\rho, T, X^{n+1}(0,\tau)\right) + \varepsilon_{CN}\left(\rho, T, X^{n+1}(0,\tau)\right)$$

$$\frac{1}{2}\left(\kappa(0,0) + \kappa^n(0,\tau)\right) = \kappa\left(\rho, T, X^{n+1}(0,\tau)\right)$$

The conditions to stop the iterations of the models at moment τ are:

$$\left|\rho_c^n - \rho_c^{n+1}\right| < \varepsilon_1 \tag{44}$$

$$\left|T_c^n - T_c^{n+1}\right| < \varepsilon_2 \tag{45}$$

When conditions (46) and (47) are accomplished it is considered that the model from iteration n is good and this model will be considered as being the one at moment τ.

The passing from a model at moment $m \cdot \tau$ to a model at moment $(m+1) \cdot \tau$ is done in the identical way as the passing from the model at moment $\tau = 0$ to the model at moment τ.

3 Numerical results

Table 1 lists the numerical values featuring the solar model which correspond to the age $\tau = 0.85 \cdot 10^9$ years.

Table 1

x	P	q	f	T	ρ	X
0.0	0.1847	0.0	0.0	14.1830	99.200	0.6782
0.063	0.1610	0.0171	0.1517	13.5520	89.5230	0.6858
0.116	0.1260	0.0898	0.5659	12.2421	71.4260	0.6967
0.162	0.761E-01	0.2068	0.8881	10.7821	52.5380	0.7036
0.208	0.468E-01	0.3564	1.0460	9.3291	35.7474	0.7071
0.307	0.118E-01	0.6515	1.1082	6.8160	12.9451	0.7088
0.406	0.284E-02	0.8463	1.1102	4.7905	4.2140	0.7089
0.510	0.543E-03	0.9510	1.1102	3.2888	1.1970	0.7090
0.609	0.102E-03	0.9946	1.1102	2.1978	0.3493	0.7090
0.707	0.174E-04	0.9989	1.1102	1.4455	0.859E-01	0.7090
0.806	0.198E-05	0.9989	1.1102	0.9731	0.151E-01	0.7090
0.893	0.258E-06	0.9989	1.1102	0.8052	0.238E-02	0.7090

The quantities appearing in Table 1 are:

- P - pressure (expressed in $10^{18} \frac{dyne}{cm^2}$);
- T - temperature (expressed in $10^6 K$);
- ρ - density (expressed in $\frac{g}{cm^3}$);
- X - hydrogen abundance;
- x - non-dimensional radius;
- q - non-dimensional mass;
- f - non-dimensional luminosity.

4 Personal contributions to the elaboration of an evolutive model

System (6), which has to be integrated on condition (19) , presents an indetermination under the form of $\frac{0}{0}$. For the elimination of the indetermination we obtained the series $(22) - (25)$.

We have shown how the Newton-Kantorovicz method can be used for the evolutive solar models and we have integrated system (6) using the method of succesive approximations.

In conclusion, the original way of numerical solving, which is presented in section 3, can be used by all the evolutive models which have a radiative nucleus and a convective cover.

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