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GENERALIZED COMPLEX LAGRANGE SPACE WITH BEIL METRIC

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Abstract

In this paper we introduce the generalized complex Lagrange metrics of Beil type, which arise from a fundamental metric tensor of a complex Finsler space, both on the same complex manifold. Also, we determine the conditions under which, the complex non-linear connection, related only to a generalized complex Lagrange metric of Beil type, can be obtained.

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1 Indroduction

The real Beil metrics were first introduced by. R.G. Beil in order to develop his unified theory in [6]. These were called Beil metrics on a real Finsler space (M, F) with the metric tensor $g_{ij}(x, y)$, having the formula

$$^*g_{ij}(x, y) = g_{ij}(x, y) + \sigma(x, y)B_i(x, y)B_j(x, y), \tag{1}$$

were $B_i(x, y) = g_{ij}(x, y)B^j(x, y)$, for $B^j(x, y)$ a given Finsler vector field. R.G. Beil tells about his choice of metric (1): "Since in my unified theory the quantity k which corresponds to your σ is related to the gravitational constant, this means that a possible physical interpretation of your theory with a $y-$dependent σ is that gravitation is itself velocity dependent". The major use of the real Beil metrics is also pointed in [3, 7, 8, 10, 11, 14], etc. Our aim is to give a systematic

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description of generalized complex Lagrange spaces, endowed with complex Beil metric on a complex manifold M,

$$^{*}g_{i\bar{j}}(z,\eta) = g_{i\bar{j}}(z,\eta) + \sigma(z,\eta)B_i(z,\eta)B_{\bar{j}}(z,\eta), \tag{2}$$

with $g_{i\bar{j}}(z,\eta)$ as the fundamental metric tensor of a complex Finsler space (M, F), and $B_i(z,\eta) = g_{i\bar{j}}(z,\eta)\overline{B^j}(z,\eta)$, for $B^j(z,\eta)$ a given complex Finsler vector field. This space is most interesting for its applications in the theoretical physics, having as basis the geometry of the complex Finsler space.

The particular form of (2) achieved for $\sigma = a(F^2)$, $a : \mathbb{R} \to \mathbb{R}$, and $B_i = \eta_i$ was studied by the author in [15]. In this case the metric tensor $^{*}g_{i\bar{j}}$ will be non-degenerate, Hermitian and homogeneous in η, and so $(M,^{*} g_{i\bar{j}})$ becomes a complex Finsler space.

In this paper, after some preliminaries in Section 2, we introduce the complex Beil metric, i.e., a complex metric which contains two parts: the first is a fundamental metric tensor of a complex Finsler space and the second one is the product of a real valued function with two complex vector fields, all defined on an underlying complex manifold. In Theorem 1. we show that the above constructed metric is a generalized complex Lagrange metric under some conditions, calculating meanwhile its inverse and its determinant (Proposition 2.).

Further on, we point out cases when the generalized complex Lagrange metric becomes a weakly regular metric, (Proposition 3.). In this case we are able to determine the complex non-linear connection, briefly (*c.n.c.*), of it. A further customization of this space, is the generalized complex Lagrange space with regular metric. This case is discussed in Proposition 4.

Another special case of generalized complex Lagrange spaces is given by the generalized complex local Minkowski spaces. They are treated in Proposition 6.

After this, we find exactly the form of the complex Lagrangian of a complex Beil metric.

2 Preliminaries

Let M be an $n-$dimensional complex manifold. The complexified of the real tangent bundle $T_C M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$. The bundle $T'M$ is in its turn a complex manifold. The local coordinates in a chart will be denoted by $u = (z^k, \eta^k),\ k = 1, \ldots, n$, which are changed by the rules: $z'^k = z'^k(z),\ \eta'^k = \frac{\partial z'^k}{\partial z^j}\eta^j$. The complexified tangent bundle of $T'M$ is decomposed in the direct sum of $T'(T'M)$ and $T''(T'M)$ respectively. A natural local frame for $T'_u(T'M)$ is $\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k}\}$, and it changes by the rules:

$$\begin{aligned} \frac{\partial}{\partial z^k} &= \frac{\partial z'k}{\partial z^h}\frac{\partial}{\partial z'^k} + \frac{\partial^2 z'k}{\partial z^j \partial z^h}\eta^j \frac{\partial}{\partial \eta'^k}; \\ \frac{\partial}{\partial \eta^k} &= \frac{\partial z'k}{\partial z^h}\frac{\partial}{\partial \eta'^k}. \end{aligned} \tag{3}$$

Let $V(T'M) = \mathrm{Ker}(\pi^*) \subset T'(T'M)$ be the vertical bundle, spanned locally by $\frac{\partial}{\partial \eta^k}$. A complex non-linear connection, briefly (*c.n.c.*), determines a supplementary complex subbundle to $V(T'M)$, i.e. $T'(T'M) = V(T'M) \oplus H(T'M)$. It determines an adapted frame $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z,\eta)$ are the coefficients of the (*c.n.c.*). These functions have a special rule of change obtained by (3). Then $\{\delta_k := \frac{\delta}{\delta z^k}, \dot{\partial}_k := \frac{\partial}{\partial \eta^k}\}$ is an adapted basis of $H(T'M)$. For more details you can see [8, 9]. Moreover, the pair (M,F) is called a *complex Finsler space,* where $F : T'M \to \mathbb{R}^+$ is a continuous function which satisfies:

i) $L := F^2$ is smooth on $\widetilde{T'M} := T'M \setminus \{0\}$;

ii) $F(z,\eta) \geq 0$, the equality holds if and only if $\eta = 0$;

iii) $F(z, \lambda\eta) = |\lambda| F(z,\eta)$ for $\lambda \in \mathbb{C}$;

iv) the following Hermitian matrix $(g_{i,\bar{j}}(z,\eta))$, with $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$, is positive definite on $\widetilde{T'M}$, and it is called *the fundamental metric tensor.*

If the iv)-th assumption is satisfied, then the Finsler metric F is strongly pseudoconvex, this means that the complex indicatrix $I_{F,z} = \{\eta \in T'_zM \mid F(z,\eta) < 1\}$ is strongly pseudoconvex.

Certainly, a main problem in this geometry is to determine a (*c.n.c.*) related only to the fundamental metric tensor $g_{i\bar{j}}$ of the complex Finsler space (M,F).

A Hermitian connection D on the sections of $T_C(T'M)$, of $(1,0)-$type, which satisfies in addition $D_{JX}Y = JD_XY$, for X horizontal vectors and J the natural complex structure of the manifold, is the Chern-Finsler connection (see [12]). This connection is locally given by the following coefficients:

$$\overset{CF}{N_j^i} = g^{\bar{m}i} \frac{\partial g_{l\bar{m}}}{\partial z^j} \eta^l; \quad L^i_{jk} = g^{\bar{m}i} \delta_k g_{j\bar{m}} = \dot{\partial}_j N^i_k; \quad C^i_{jk} = g^{\bar{m}i} \dot{\partial}_k g_{j\bar{m}}, \tag{4}$$

and $L^i_{j\bar{k}} = C^i_{j\bar{k}} = 0$, where here and subsequently δ_k is the adapted frame of the Chern-Finsler (*c.n.c.*) and $D_{\delta_k}\delta_j = L^i_{jk}\delta_i,\ D_{\dot{\partial}_k}\dot{\partial}_j = C^i_{jk}\dot{\partial}_i$, etc.

Let N^i_j be the local coefficients of a fixed (*c.n.c.*).

Definition 1. *A generalized Lagrange metric on M is a $d-$tensor field $\tilde{g}_{i\bar{j}}(z,\eta)$ of $\binom{0\ \bar{0}}{1\ \bar{1}}$-type, non-degenerated and Hermitian $\tilde{g}_{i\bar{j}} = \overline{\tilde{g}_{j\bar{i}}}$. The pair $(M, \tilde{g}_{i\bar{j}})$ is said to be a generalized complex Lagrange space.*

In particular, when $\tilde{g}_{i\bar{j}}$ derives from a complex Lagrangian function $\tilde{L} : T'M \to \mathbb{R}$, i.e. $\tilde{g}_{i\bar{j}} = \frac{\partial^2 \tilde{L}}{\partial \eta^i \partial \bar{\eta}^j}$, then the pair $(M, \tilde{g}_{i\bar{j}})$ is called a *complex Lagrange space* and a study of such structure is made in [13].

A complex Lagrange space for which the Lagrange function $\tilde{L}(z,\eta)$ is homogeneous with respect to η, i.e. $\tilde{L}(z,\lambda\eta) = |\lambda|^2 \tilde{L}(z,\eta)$, for $\lambda \in \mathbb{C}$, is a complex Finsler space. For detailed analysis, [12, 13] can be consulted.

3 The complex Beil metric

Following the ideas from real case, [3, 4, 7], we shall introduce a new class of complex metrics. Let (M, F) be a an $n-$dimensional complex Finsler space, and $g_{j\bar{k}}$ its fundamental metric tensor. Assume that (M, F) is endowed with a complex Finsler vector field $B = B^k(z,\eta)\dot{\partial}_k$ and let $B_k(z,\eta)dz^k$ be a differential $(1,0)-$form with $B_k = g_{k\bar{m}}B^{\bar{m}}$, where $B^{\bar{m}} := \overline{B^m}$. The lowering and rising of indices will be done with $(g_{i\bar{j}})$ and $(g^{\bar{j}k})$, where $g_{i\bar{j}}g^{\bar{j}k} = \delta^i_k$, respectively.

Also, we consider $\sigma : T'M \to \mathbb{R}$, a real valued function, on $T'M$. By these objects we set

$$^*g_{i\bar{j}}(z,\eta) = g_{i\bar{j}}(z,\eta) + \sigma(z,\eta)B_i(z,\eta)B_{\bar{j}}(z,\eta) \tag{5}$$

Our aim is to prove, that the matrix $(^*g_{i\bar{j}})$, defined above, is non-degenerate and $^*g_{i\bar{j}}$ is a Hermitian $d-$tensor of $\binom{0\ \bar{0}}{1\ \bar{1}}$-type.

It is clear that $(^*g_{i\bar{j}})$ are the components of a Hermitian matrix, and they satisfy the transformation law $^*g'_{i\bar{j}} = \frac{\partial z'^k}{\partial z^i}\frac{\partial \bar{z}'^l}{\partial \bar{z}^j}{}^*g_{k\bar{l}}$. To find the formulas for the inverse and the determinant of the matrix $(^*g_{i\bar{j}})$, we use the Proposition 2.2 from [2] for an arbitrary non-singular Hermitian matrix $(Q_{i\bar{j}})$:

Proposition 1. *[2] Suppose:*

- $(Q_{i\bar{j}})$ *is a non-singular* $n \times n$ *complex matrix with inverse* $(Q^{\bar{j}i})$;
- C_i *and* $C_{\bar{i}} := \overline{C_i}$, $i = 1, \dots, n$ *are complex numbers;*
- $C^i := Q^{\bar{j}i}C_{\bar{j}}$ *and its conjugates;* $C^2 := C^iC_i = C^{\bar{i}}C_{\bar{i}}$; $H_{i\bar{j}} := Q_{i\bar{j}} \pm C_iC_{\bar{j}}$.

Then

i) $\det(H_{i\bar{j}}) = (1 \pm C^2)\det(Q_{i\bar{j}})$;

ii) Whenever $1 \pm C^2 \neq 0$, *the matrix* $(H_{i\bar{j}})$ *is invertible and in this case its inverse is* $H^{\bar{j}i} = Q^{\bar{j}i} \mp \frac{1}{1\pm C^2}C^iC^{\bar{j}}$.

Proposition 2. *For the* $d-$*tensor* $^*g_{i\bar{j}}$ *from (5) we have,*

i) $\det(^*g_{i\bar{j}}) = (1 + \sigma B^2)\det(g_{i\bar{j}})$;

ii) If $1 + \sigma B^2 \neq 0$, *the* $d-$*tensor* $g_{i\bar{j}}$ *is non-degenerate, and his inverse has the following expression* $^*g^{\bar{j}i} = g^{\bar{j}i} - \frac{\sigma}{1+\sigma B^2}B^iB^{\bar{j}}$,

where $B^2 = B_iB^i = g_{i\bar{j}}B^iB^{\bar{j}}$ *(the length of* B *with respect to* $g_{i\bar{j}}$*).*

Proof. To prove the affirmations we apply the above Proposition. First we set

$$Q_{i\bar{j}} := g_{i\bar{j}}, \quad C_i = \sqrt{\sigma}B_i, \quad C^i = g^{\bar{j}i}C_{\bar{j}} = \sqrt{\sigma}B^i,$$
$$C^2 = C^iC_i = \sqrt{\sigma}B^i \cdot \sqrt{\sigma}B_i = \sigma B^2,$$
$$H_{i\bar{j}} = g_{i\bar{j}} + C_iC_{\bar{j}} = g_{i\bar{j}} + \sigma B_iB_{\bar{j}} =^* g_{i\bar{j}}.$$

Applying *i*) from the Proposition 1. we obtain,

$$\det(H_{i\bar{j}}) = \det({}^*g_{i\bar{j}}) = (1 + C^2)\det(Q_{i\bar{j}}) = (1 + \sigma B^2)\det(g_{i\bar{j}}) \qquad (6)$$

To show, that $({}^*g_{i\bar{j}})$ is invertible, we use *ii*) from the Proposition 1. From this we obtain, that if $1 + C^2 = 1 + \sigma B^2 \neq 0$, and then the matrix $(H_{i\bar{j}}) = ({}^*g_{i\bar{j}})$, is invertible and its inverse has the following expression

$${}^*g^{\bar{j}i} = H^{\bar{j}i} = Q^{\bar{j}i} - \frac{1}{1 + C^2} C^i C^{\bar{j}} = g^{\bar{j}i} - \frac{\sigma}{1 + \sigma B^2} B^i B^{\bar{j}}. \qquad (7)$$

□

So we have proven:

Theorem 1. *The pair* $(M, {}^* g_{i\bar{j}})$ *is a generalized Lagrange space, briefly* $(g.c.L)$ *space, if and only if* $1 + \sigma B^2 \neq 0$.

Therefore, $({}^*g_{i\bar{j}})$ from (5) define a $(g.c.L)$ metric, if $1 + \sigma B^2 \neq 0$, which we call *the complex Beil metric,* by analogy with the real case, [3].

From the expression of $\det({}^*g_{i\bar{j}})$, we can affirm the followig.

Lemma 1. *The Beil metric* ${}^*g_{i\bar{j}}(z, \eta)$ *from (5) is a positive definite (g.c.L) metric if and only if* $1 + \sigma B^2 > 0$.

Of course, the most tangible examples of generalized complex Lagrange metric are those which are obtained from a Lagrange or a Finsler function on $T'M$.

We shall consider some proper subclasses of $(g.c.L)$ spaces, for which a $(c.n.c.)$ on $T'M$ can be obtained from the metric tensor ${}^*g_{i\bar{j}}$.

It is said that ${}^*g_{i\bar{j}}$ is a *weakly regular* metric if $(M, \mathcal{E} = {}^* g_{i\bar{j}}\eta^i\bar{\eta}^j)$ is a complex Lagrange space. This means that $\tilde{g}_{i\bar{j}} := \frac{\partial^2 \mathcal{E}}{\partial\eta^i\partial\bar{\eta}^j}$ is a Hermitian non-degenerated metric. In our case, the complex energy $\mathcal{E}$ has the formula:

$$\mathcal{E}(z, \eta) = {}^* g_{i\bar{j}}(z, \eta)\eta^i\bar{\eta}^j = L(z, \eta) + \sigma(z, \eta)B_i B_{\bar{j}}\eta^i\bar{\eta}^j = L + \sigma|\beta|^2, \qquad (8)$$

where $\beta = B_i(z, \eta)\eta^i$, and $\bar{\beta}$ is its conjugate.

By a direct computation, we deduce

$$\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + \sigma_{i\bar{j}}|\beta|^2 + \sigma_i(\beta_{\bar{j}}\bar{\beta} + \beta\bar{\beta}_{\bar{j}}) + \sigma_{\bar{j}}(\beta_i\bar{\beta} + \beta\bar{\beta}_i) + \sigma(\beta_i\bar{\beta}_{\bar{j}} + \beta_{\bar{j}}\bar{\beta}_{\bar{i}} + \beta_{i\bar{j}}\bar{\beta} + \beta\bar{\beta}_{i\bar{j}}), \quad (9)$$

where $\beta_i := \dot{\partial}_i\beta,\ \bar{\beta}_i := \dot{\partial}_i\bar{\beta},\ \beta_{i\bar{j}} := \dot{\partial}_i\dot{\partial}_{\bar{j}}\beta,\ \sigma_i := \dot{\partial}_i\sigma,\ \sigma_{\bar{j}} := \dot{\partial}_{\bar{j}}\sigma$ and their conjugates.

It is hopeless to decide if $\tilde{g}_{i\bar{j}}$ is invertible or not. However we can study some particular cases.

Proposition 3. *i) If the complex Liouville vector field* $\Gamma = \eta^k \frac{\partial}{\partial \eta^k}$ *is orthogonal to* B, *then* ${}^*g_{i\bar{j}}$ *is a weakly regular metric, and* $\tilde{g}_{i\bar{j}} = g_{i\bar{j}}$.

ii) If $B_i = B_i(z)$ *and* $\sigma(z,\eta) = f(|\beta|^2)$, *for* $f : \mathbb{R} \to \mathbb{R}^+$ *a smooth function, then* ${}^*g_{i\bar{j}}$ *is a weakly regular metric if and only if* $1 + \varphi B^2 \neq 0$, *where* $\varphi(|\beta|^2) = f''|\beta|^4 + 3f'|\beta|^2 + f$, $f' := \frac{\mathrm{d}f}{\mathrm{d}|\beta^2|}$, $f'' := \frac{\mathrm{d}^2 f}{(\mathrm{d}|\beta^2|)^2}$, *and also we have*

$$\tilde{g}_{i\bar{j}} = g_{i\bar{j}} + \varphi(z,\eta) B_i(z) B_{\bar{j}}(z). \tag{10}$$

Proof. i) The condition Γ orthogonal to B is equivalent with $\beta = 0$. In this case $\mathcal{E} = L$, and so $(M, \mathcal{E})$ becomes a complex Finsler space, which is also a complex Lagrange space. Hence (9) is reduced to $\tilde{g}_{i\bar{j}} = g_{i\bar{j}}$.

ii) A direct computation leads to (10). With σ replaced by φ, the forms of $\tilde{g}_{i\bar{j}}$ coincide with the form of it ${}^*g_{i\bar{j}}$. It follows, that the metric $\tilde{g}_{i\bar{j}}$ became a non-degenerate Hermitian metric. So, $(M, \mathcal{E})$ becomes a complex Lagrange space. □

Remark 1. *Condition* $\beta = 0$ *and* $B_i = B_i(z)$ *are incompatible, because they imply that* $B = 0$.

It is well known, that the complex energy $\mathcal{E}$ is uniquely determined by the metric of the $(g.c.L)$ space $(M, {}^* g_{i\bar{j}})$. Then it follows, that also the Chern-Lagrange $(c.n.c.)$ depends only on ${}^*g_{i\bar{j}}$. From (5.1.8) [12] we have ${}^*\overset{CL}{N_j^k} = {}^* g^{\bar{\imath}k} \frac{\partial^2 \mathcal{E}}{\partial z^j \partial \bar{\eta}^i}$, from here we obtain.

Lemma 2. *The Chern-Lagrange* $(c.n.c.)$ *of the* $(g.c.L)$ *space* $(M, {}^* g_{i\bar{j}})$ *with weakly regular metric given by the Proposition 3. case ii) has the following expression:*

$$\begin{aligned} {}^*\overset{CL}{N_j^k} \;=\; & \overset{CF}{N_j^k} + \tilde{g}^{\bar{\imath}k} (f B_{\bar{\imath}} B_p)_{|j} \eta^p + B^k B_m \overset{CF}{N_j^m} \left[(1 - {}^* \varphi B^2)(f'|\beta|^2 + f) - \varphi \right] \\ & + \tilde{g}^{\bar{\imath}k} (\partial_j \dot{\partial}_{\bar{\imath}} f) |\beta|^2, \end{aligned} \tag{11}$$

where $\varphi(|\beta|^2) = f''|\beta|^4 + 3f'|\beta|^2 + f$, ${}^*\varphi := \frac{\varphi}{\varphi B^2 + 1}$ *and "*$_{|j}$*" denotes the horizontal covariant derivative of the Chern-Finsler* $(c.n.c.)$ *of the complex Finsler space* (M, F).

Another subclass of the $(g.c.L)$ spaces, are $(g.c.L)$ spaces with regular metric. To determine when the complex Beil metric ${}^*g_{i\bar{j}}$ is a regular metric we use the Proposition 5.2.2 from [12], p.110 for an arbitrary $(g.c.L)$ space $(M, g_{i\bar{j}})$:

Proposition 4. *[12] The pair* $(M, g_{i\bar{j}})$ *is a* $(g.c.L)$ *space with regular metric if and only if the following conditions are satisfied:*

1) $(M, g_{i\bar{j}})$ *is a* $(g.c.L)$ *space with weakly regular metric;*

2) $\frac{\partial \mathcal{E}}{\partial \eta^i} \eta^i = \mathcal{E}$.

Proposition 5. *i) If the complex Liouville vector field* $\Gamma = \eta^k \frac{\partial}{\partial \eta^k}$ *is orthogonal to* B, *then* ${}^*g_{i\bar{j}}$ *is a regular metric, and* $\tilde{g}_{i\bar{j}} = g_{i\bar{j}}$.

ii) If $B_i = B_i(z)$ and $\sigma(z, \eta) = f(|\beta|^2)$, for $f : \mathbb{R} \to \mathbb{R}^+$ a smooth function, then ${}^*g_{i\bar{j}}$ is a regular metric if and only if $f = c \in \mathbb{R}$ and $1 + cB^2 \neq 0$, and also we have

$$\tilde{g}_{i\bar{j}} =^* g_{i\bar{j}} = g_{i\bar{j}} + cB_i(z)B_{\bar{j}}(z). \tag{12}$$

Proof. i) From Proposition 3. we know that, in this case, $(M,^* g_{i\bar{j}})$ is a $(g.c.L)$ space with weakly regular metric, and $\mathcal{E} = L$. Because L is a complex Finsler metric, then the condition 2) from the above Proposition is also satisfied. The form of $\tilde{g}_{i\bar{j}}$ results likewise from Proposition 1.

ii) The second condition from the above presented Proposition asks that the complex energy tensor $\mathcal{E}$ of the $(g.c.L)$ space be homogeneous in η.

$$\frac{\partial \mathcal{E}}{\partial \eta^i}\eta^i = \dot{\partial}_i(L + f|\beta^2|)\eta^i = (\dot{\partial}_i L)\eta^i + f'(\dot{\partial}_i|\beta|^2)\eta^i + f(\dot{\partial}_i|\beta|^2)\eta^i = L + (f + f'^2)|\beta|^2 \tag{13}$$

The equality $\frac{\partial \mathcal{E}}{\partial \eta^i}\eta^i = \mathcal{E}$ holds if and only if $f' = 0 \;\Leftrightarrow\; f = c \in \mathbb{R}$. In this case, $\varphi = c$, and so the condition $1 + \varphi B^2 \neq 0$, will be equivalent with $1 + cB^2 \neq 0$. The formula of $\tilde{g}_{i\bar{j}}$ instantly results.

□

An immediate conclusion of this Proposition and of Lemma 2 is

Lemma 3. *The Chern-Lagrange (c.n.c.) of the (g.c.L) space $(M,^* g_{i\bar{j}})$ with regular metric given by the Proposition 5. case ii) has the following expression:*

$$^*\overset{CL}{N_j^k} = \overset{CF}{N_j^k} + c^* g^{\bar{l}k}(B_{\bar{l}}B_p)_{|j}\eta^p - \frac{c^2B^2}{cB^2 + 1}B^k B_m \overset{CF}{N_j^m} \tag{14}$$

where ${}^*g_{i\bar{j}}$ is expressed in (12), and "$_{|j}$" denotes the horizontal covariant derivative of the Chern-Finsler (c.n.c.) of the complex Finsler space (M, F).

A sufficient condition for the $(g.c.L)$ space $(M,^* g_{i\bar{j}})$ to be a locally Minkowski space, (briefly, $(g.c.l.M.)$), is given by the following.

Proposition 6. *Let $(M,^* g_{i\bar{j}})$ be the $(g.c.L)$, with the complex Beil metric (5). If the complex Finsler metric $g_{i\bar{j}}$ is locally Minkowski. (i.e., there exists the local chart on $T'M$ in which $g_{i\bar{j}}$ depends only on η), and $\partial_k(\sigma B_i B_{\bar{m}}) = 0$, then $(M,^* g_{i\bar{j}})$ is a $(g.l.c.M)$.*

Proof. Indeed, under our assumptions, the derivative with respect to z of (5) gives $\partial_k({}^*g_{i\bar{j}}) = 0$, which means that ${}^*g_{i\bar{j}}$ is locally Minkowski. □

In the following, our aim is to construct the Lagrangian corresponding to the complex Beil metric (5). We keep the hypothesis $B_i = B_i(z)$ and $\sigma = f(|\beta|^2)$, $\beta \neq 0$. From (10) we can deduce that

$$\tilde{g}_{i\bar{j}} =^* g_{i\bar{j}} + (f''|\beta|^4 + 3f'|\beta|^2)B_i B_{\bar{j}}. \tag{15}$$

From here, we have $\tilde{g}_{i\bar{j}} =^* g_{i\bar{j}}$ if and only if f is a solution of the differential equation

$$f''|\beta|^4 + 3f'|\beta|^2 = 0 \;\Leftrightarrow\; f(|\beta|^2) = -\frac{a}{2|\beta|^4} + b,\; a, b \in \mathbb{R}. \tag{16}$$

In this case we know that ${}^*g_{i\bar{j}}$ is a complex Lagrange metric. From the condition $\tilde{g}_{i\bar{j}} =^* g_{i\bar{j}}$ we can deduce the form of ${}^*L$:

$${}^*L(z,\eta) = \mathcal{E}(z,\eta) + A_i(z)\eta^i + \overline{A_j(z)}\bar{\eta}^j + \Psi(z), \tag{17}$$

where $A_i(z)$ is a covector, and Ψ is a real valued function. Plugging in (17) the expresion of $\mathcal{E}$ from (8), we obtain

$${}^*L(z,\eta) = L(z,\eta) - \frac{a}{2B_iB_{\bar{j}}\eta^i\bar{\eta}^j} + bB_iB_{\bar{j}}\eta^i\bar{\eta}^j + A_i(z)\eta^i + \overline{A_j(z)}\bar{\eta}^j + \Psi(z),\; a, b \in \mathbb{R}. \tag{18}$$

Consequently, we found that for the complex Lagrange metric ${}^*g_{i\bar{j}}(z,\eta) = g_{i\bar{j}}(z,\eta) + \left(b - \frac{a}{2(B_iB_j\eta^i\bar{\eta}^j)^2}\right)B_i(z)B_{\bar{j}}(z)$, the corresponding complex Lagrangian is ${}^*L$ from (18).

Remark 2. *If we take $a = 0$, $A_i = 0$, $\Psi = 0$ and $b \geq 0$ in the expression of ${}^*L$ from (18), then ${}^*F^2 :=^* L(z,\eta)$ will be homogeneous in η, and so the complex Lagrange space $(M,^* F)$ becomes a complex Finsler space.*

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References

[1] Abate, M., Patrizio, G., *Finsler metrics - A global approach*, Lecture Notes in Math., **1591**, Springer-Verlag, 1994.

[2] Aldea, N. and Munteanu, G., *On complex Finsler spaces with Randers metrics*, Journal Korean Math. Soc., **46** , no. 5 (2009), 949-966.

[3] Anastasiei, M. and Shimada, H., *The Beil metrics associated to a Finsler space*, Balkan J. Geom. Appl. **3** (1998), no. 2, 1-16.

[4] Anastasiei, M. and Shimada, H., *Deformation of Finslerian Metrics,* Kluwer Acad. Publ. Dordrecht., Fund. Theories Phys., **109** 2000.

[5] Bao, D., Chern, S. S. and Shen, Z., *An introduction to Riemannian Finsler Geom.,* Graduate Texts in Math., **200,** Springer-Verlag, 2000.

[6] Beil, R., *Finsler geometry and a unified field theory in Finsler geometry*, Contemporary Math., **196,** AMS (1993), 265-272.

[7] Balan, V, Stavrinos, P. and Trenčevski, K., *Weak gravitational models based on Beil metrics*, Proceedings of The Conference of Applied Differential Geometry - General Relativity and The Workshop on Global Analysis, Differential Geometry and Lie Algebras, 7-22, 2000.

[8] Miron, R. and Anastasiei, M., *The geometry of Lagrange spaces: theory and applications,* Fundamental Theories of Physics, Vol. 59, Kluwer Academic Publishers, 1994.

[9] Miron, R., *Lagrange geometry,* Comput. Modeling, **20** (1994), no. 4-5, 25-40.

[10] Miron, R., *The geometry of higher order Lagrange spaces. Applications to mechanics and physics,* Kluwer Academic Publishers, Dordrecht, 1997.

[11] Miron, R., Hrimiuc, D., Shimada, H. and Sabău, V. S. *The geometry of Hamilton and Lagrange spaces,* Fund. Theories Phys., **118**, Kluwer Academic Publishers, Dordrecht, 2001.

[12] Munteanu, G., *Complex spaces in Finsler, Lagrange and Hamilton geometries,* Kluwer Acad. Publ., **141**, FTPH, 2004.

[13] Muntenau, G., *Generalized Complex Lagrange Spaces*, Finslerian Geom., Kluwert Acad. Publ., **109** 2000, 209-222.

[14] Sandovici, A. and Blănuţă, V., *Generalized Beil metric on Finsler space,* Libertas Mathematica, **27** (2007).

[15] Szász, A., *Generalized quaternionic structures on the total space of a complex Finsler space,* Bull. of the *Transilvania* Univ. Braşov, Ser. III, **5(54)** (2012), no. 1, 85-96.