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NILPOTENT QUANTUM THEORY: A REVIEW

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Abstract

Nilpotent quantum mechanics has developed over a twenty year period into a uniquely powerful method of relativistic quantum mechanics and quantum field theory, which offers many new results relating to particle physics and other fundamental studies. A review of the developments shows that they produce a coherent and integrated approach to a number of fundamental questions.

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1 Indroduction: Fundamental Symmetries

Nilpotent quantum theory first appeared in the literature twenty years ago, [1][2], and provides an exceptionally streamlined and powerful route to quantum mechanics, quantum field theory and particle physics. It can be derived in a completely formal way using hypercomplex algebra in place of the usual matrix formalisms associated with these subjects. Its origin, however, can be placed twenty years earlier in a much more physically-inspired theory involving symmetries between the fundamental physical parameters. Because of the additional information contained within these symmetries, in addition to providing a formalism for reproducing the known results of relativistic quantum mechanics and the Standard Model of particle physics in an integrated and systematic way, nilpotent quantum theory also generates many new ones which are not accessible by any other known method. In effect, the formalisms which are used routinely in these areas of physics are not there purely for mathematical convenience, but also contain coded physical information which can be extracted if we can find a more fundamental way of expressing them. The formalisms generated from the hypercomplex algebra and the related symmetries also create further formalisms

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with additional physical information, which connect with areas such as Finsler geometry, creating even further layers of physical meaning.

As a review, this will highlight significant developments rather than concentrating on detail. Physics, as we know it today, manifests a broken symmetry, in the Standard Model. My view is that larger broken symmetries tend to be the result of combining more perfect smaller ones. Nature does not create larger symmetries and then find a mechanism for breaking them. If we want to understand complex mathematical developments in physics (e.g. Finsler geometry) we should start by finding their simpler components. The developments I am going to outline started from some very basic ideas about physical symmetries in physics many decades ago.[3][4] Over the years the mathematics grew organically with the physics. The basic idea was that, if physics had a starting point it had to be ultimately simple, and that the best bet was not in laws of physics, or equations, or in particles or other structures. It had to be the fundamental parameters, the ideas through which everything else is observed or constructed. Space and time were obvious candidates, but what else? At the time, it seemed likely to be the sources of the 4 interactions: mass(-energy) and 3 types of charge. I have never seen the need to revise this list.

The next stage was to examine the properties of these quantities in as much detail as possible and see how this might lead to physics as we know it. Quickly it became apparent that there were 3 basic dualities, suggesting an overall group structure. At first, this required one or two assumptions about how a broken symmetry might emerge, but, after testing the model to destruction over nearly forty years, I am now convinced that it is exact. Initial difficulties have proved to be sources of further discovery.

Conserved	*Nonconserved*
Mass	Space
Charge	Time

Conserved	*Imaginary*
Mass	Charge
Space	Time

Anticommutative	*Commutative*
Space	Mass
Charge	Time

The symmetry involved can be identified by arranging the dual properties in a table:

mass	conserved	real	commutative
time	nonconserved	imaginary	commutative
charge	conserved	imaginary	anticommutative
space	nonconserved	real	commutative

mass	x	y	z
time	$-x$	$-y$	z
charge	x	$-y$	$-z$
space	$-x$	y	$-z$

This is a finite noncyclic group of order 4: Klein-4 or D_2. We can produce this group by devising a binary operation of the form:

$$x * x = x * x = x \tag{1.1}$$

$$x * -x = -x * x = x \tag{1.2}$$

$$x * y = y * x = 0 \tag{1.3}$$

and similarly for y and z. Effectively, any combination of a single property or antiproperty with itself gives the property; but a combination of a property with its antiproperty gives the antiproperty; while the combination of any property with any other property or antiproperty vanishes. This gives us the group table with mass as identity element:

*	mass	charge	time	space
mass	mass	charge	time	space
charge	charge	mass	space	time
time	time	space	mass	charge
space	space	time	charge	mass

There is a suggestion here that, if these parameters are truly and absolutely symmetrical in the way suggested, then each property held by two of the parameters is completely negated by the exactly opposite property held by two others. The motivation for the original view that they were symmetric was the belief that Nature could not be characterized. It was a totality zero in conceptual as well as numerical terms. This motif repeats itself regularly as the mathematical structure unfolds. But this representation is not unique. We can easily rearrange the algebraic symbols to make any of space, time or charge the identity element. For example, we could have made space the identity element by assigning the symbols as follows:

mass	-x	y	-z
time	x	-y	-z
charge	-x	-y	z
space	x	y	z

And, if we switch one of the properties around, we create a dual D_2 group. The easiest to change mathematically is the real / imaginary distinction. In this case, we have:

mass*	x	$-y$	z
time*	$-x$	y	z
charge*	x	y	$-z$
space*	$-x$	$-y$	$-z$

or

mass*	conserved	imaginary	commutative
time*	nonconserved	real	commutative
charge*	conserved	real	anticommutative
space*	nonconserved	imaginary	anticommutative

There is a C_2 symmetry between the dual D_2 structures, and the $D_2 \times D_2$ of order 8 creates a larger structure of the form:

*	M	C	S	T	$M*$	$C*$	$S*$	$T*$
M	M	C	S	T	$M*$	$C*$	$S*$	$T*$
C	C	$M*$	T	$S*$	$C*$	M	$T*$	T
S	S	$T*$	$M*$	C	$S*$	T	M	$C*$
T	T	S	$C*$	$S*$	$T*$	$S*$	C	M
$M*$	$M*$	$C*$	$S*$	$T*$	M	C	S	T
$C*$	$C*$	M	$T*$	S	C	$M*$	T	$T*$
$S*$	$S*$	T	M	$C*$	S	$T*$	$M*$	C
$T*$	$T*$	$S*$	C	S	T	S	$C*$	$M*$

Remarkably, this structure is identical to that of the quaternion group (Q):

*	1	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$	-1	-$\boldsymbol{i}$	-$\boldsymbol{j}$	-$\boldsymbol{k}$
1	1	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$	-1	-$\boldsymbol{i}$	-$\boldsymbol{j}$	-$\boldsymbol{k}$
$\boldsymbol{i}$	$\boldsymbol{i}$	-1	$\boldsymbol{k}$	-$\boldsymbol{j}$	-$\boldsymbol{i}$	1	-$\boldsymbol{k}$	$\boldsymbol{j}$
$\boldsymbol{j}$	$\boldsymbol{j}$	-$\boldsymbol{k}$	-1	$\boldsymbol{i}$	-$\boldsymbol{j}$	$\boldsymbol{k}$	1	-$\boldsymbol{i}$
$\boldsymbol{k}$	$\boldsymbol{k}$	$\boldsymbol{j}$	-$\boldsymbol{i}$	-1	-$\boldsymbol{k}$	-$\boldsymbol{j}$	$\boldsymbol{i}$	1
-1	-1	-$\boldsymbol{i}$	-$\boldsymbol{j}$	-$\boldsymbol{k}$	1	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$
-$\boldsymbol{i}$	-$\boldsymbol{i}$	1	-$\boldsymbol{k}$	$\boldsymbol{j}$	$\boldsymbol{i}$	-1	$\boldsymbol{k}$	-$\boldsymbol{j}$
-$\boldsymbol{j}$	-$\boldsymbol{j}$	$\boldsymbol{k}$	1	-$\boldsymbol{i}$	$\boldsymbol{j}$	-$\boldsymbol{k}$	-1	$\boldsymbol{i}$
-$\boldsymbol{k}$	-$\boldsymbol{k}$	-$\boldsymbol{j}$	$\boldsymbol{i}$	1	$\boldsymbol{k}$	$\boldsymbol{j}$	-$\boldsymbol{i}$	-1

Another fundamental aspect of the symmetry is that it suggests that physics can be reduced to algebra. The parameters are defined by the successive algebras, leading up to a full Clifford algebra, and all the physical properties can be explained entirely by the algebraic ones. The symmetry requires that charge forms a quaternion structure, with units, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, following the well-known multiplication rules:

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1 \tag{1.4}$$

$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k} \tag{1.5}$$

$$\boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i} \tag{1.6}$$

$$\boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}. \tag{1.7}$$

To be fully symmetric with charge, space has to be a multivariate vector, that is, described by 3-D Clifford algebra. The multivariate vector units, **i**, **j**, **k**, i, are effectively complexified quaternions $(i\boldsymbol{i}) = \mathbf{i}$, $(i\boldsymbol{j}) = \mathbf{j}$, $(i\boldsymbol{k}) = \mathbf{k}$, $(i1) = i$, and follow the multiplication rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{1} \tag{1.8}$$

$$\mathbf{ij} = -\mathbf{ji} = i\mathbf{k} \tag{1.9}$$

$$\mathbf{jk} = -\mathbf{kk} = i\mathbf{i} \tag{1.10}$$

$$\mathbf{ki} = -\mathbf{ik} = i\mathbf{j}. \tag{1.11}$$

They are isomorphic to Pauli matrices. If we complexify this algebra, we revert to quaternions, so $i\mathbf{i} = \boldsymbol{i}$, $i\mathbf{j} = \boldsymbol{j}$, $i\mathbf{k} = \boldsymbol{k}$, etc. Multivariate vectors differ from ordinary vectors in having a full (algebraic) product:

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + i\mathbf{a} \times \mathbf{b} \tag{1.12}$$

from which all the rules concerning unit vector multiplication may be derived.[5] They automatically include spin within the structure of 3-D space. Terms $i\mathbf{i}$, $i\mathbf{j}$, $i\mathbf{k}$ are pseudovectors (e.g. area, angular momentum) and i is a pseudoscalar (e.g. volume). The successive algebras now become:

	Units			Parameter
Real numbers	1			Mass
Imaginary numbers	i			Time
Quaternions	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$	Charge
Vectors	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	Space

The first 3 algebras are the 3 subalgebras of the fourth

2 A dual vector space

The units $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ define a complete Clifford algebra of 3D space:

$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	*vector*		
$i\mathbf{i}$	$i\mathbf{j}$	$i\mathbf{k}$	*bivector*	*pseudovector*	*quaternion*
i			*trivector*	*pseudoscalar*	
1			*scalar*		

Pseudovectors and pseudoscalars give us areas and volumes, etc. The intrinsic complexification produces a kind of doubling of the elements. Combining the 3 subalgebras of charge (unit $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$), time (unit i) and mass (unit 1) produces the equivalent of a second complete Clifford algebra of 3D space:

	$\mathbf{I}$	$\mathbf{J}$	$\mathbf{K}$	*vector*		
C	$i\mathbf{I}$	$i\mathbf{J}$	$i\mathbf{K}$	*bivector*	*pseudovector*	*quaternion*
T	i			*trivector*	*pseudoscalar*	
M	1			*scalar*		

There is no physical quantity relating to this space, with units **I**, **J**, **K**, but the charge, time and mass combine algebraically to produce its equivalent. If we wanted to find the group governing the combination of the four symmetric parameters, we would take the algebraic product of the four sets of units, **i**, **j**, **k**, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, i, 1, and obtain + and versions of:

$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	$i\mathbf{i}$	$i\mathbf{j}$	$i\mathbf{k}$	i	1
$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$	$i\boldsymbol{i}$	$i\boldsymbol{j}$	$i\boldsymbol{k}$		
$\mathbf{i}\boldsymbol{i}$	$\mathbf{j}\boldsymbol{i}$	$\mathbf{k}\boldsymbol{i}$	$i\mathbf{i}\boldsymbol{i}$	$i\mathbf{j}\boldsymbol{i}$	$i\mathbf{k}\boldsymbol{i}$		
$\mathbf{i}\boldsymbol{j}$	$\mathbf{j}\boldsymbol{j}$	$\mathbf{k}\boldsymbol{j}$	$i\mathbf{i}\boldsymbol{j}$	$i\mathbf{j}\boldsymbol{j}$	$i\mathbf{k}\boldsymbol{j}$		
$\mathbf{i}\boldsymbol{k}$	$\mathbf{j}\boldsymbol{k}$	$\mathbf{k}\boldsymbol{k}$	$i\mathbf{i}\boldsymbol{k}$	$i\mathbf{j}\boldsymbol{k}$	$i\mathbf{k}\boldsymbol{k}$		

This is a finite group of order 64. Alternatively, we could take the algebraic product of the units of space, **i**, **j**, **k**, and those of the space combining the units of mass, time and charge, **I**, **J**, **K**, to obtain + and versions of:

$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	$i\mathbf{i}$	$i\mathbf{j}$	$i\mathbf{k}$	i	1
$\mathbf{I}$	$\mathbf{J}$	$\mathbf{K}$	$i\mathbf{I}$	$i\mathbf{J}$	$i\mathbf{K}$		
$\mathbf{iI}$	$\mathbf{jI}$	$\mathbf{kI}$	$i\mathbf{iI}$	$i\mathbf{jI}$	$i\mathbf{kI}$		
$\mathbf{iJ}$	$\mathbf{jJ}$	$\mathbf{kJ}$	$i\mathbf{iJ}$	$i\mathbf{jJ}$	$i\mathbf{kJ}$		
$\mathbf{iK}$	$\mathbf{jK}$	$\mathbf{kK}$	$i\mathbf{iK}$	$i\mathbf{jK}$	$i\mathbf{kK}$		

These are the units of a double vector algebra or a double Clifford algebra of 3D space. The group is completely isomorphic to the previous one and each is derivable immediately from the other. In fact, both are also completely isomorphic to a complexified double quaternion algebra, with units:

$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$	$i\boldsymbol{i}$	$i\boldsymbol{j}$	$i\boldsymbol{k}$	i	1
$\boldsymbol{I}$	$\boldsymbol{J}$	$\boldsymbol{K}$	$i\boldsymbol{I}$	$i\boldsymbol{J}$	$i\boldsymbol{K}$		
$\boldsymbol{iI}$	$\boldsymbol{jI}$	$\boldsymbol{kI}$	$i\boldsymbol{iI}$	$i\boldsymbol{jI}$	$i\boldsymbol{kI}$		
$\boldsymbol{iJ}$	$\boldsymbol{jJ}$	$\boldsymbol{kJ}$	$i\boldsymbol{iJ}$	$i\boldsymbol{jJ}$	$i\boldsymbol{kJ}$		
$\boldsymbol{iK}$	$\boldsymbol{jK}$	$\boldsymbol{kK}$	$i\boldsymbol{iK}$	$i\boldsymbol{jK}$	$i\boldsymbol{kK}$		

It is easy to show that the group requires a minimum of 5 generators. One way of setting out the elements of the vector-quaternion version would be:

1	i				-1	$-i$			
$\boldsymbol{i}\mathbf{i}$	$\boldsymbol{i}\mathbf{j}$	$\boldsymbol{i}\mathbf{k}$	$i\boldsymbol{k}$	$\boldsymbol{j}$	$-\boldsymbol{i}\mathbf{i}$	$-\boldsymbol{i}\mathbf{j}$	$-\boldsymbol{i}\mathbf{k}$	$-i\boldsymbol{k}$	$-\boldsymbol{j}$
$\boldsymbol{j}\mathbf{i}$	$\boldsymbol{j}\mathbf{j}$	$\boldsymbol{j}\mathbf{k}$	$i\boldsymbol{i}$	$\boldsymbol{k}$	$-\boldsymbol{j}\mathbf{i}$	$-\boldsymbol{j}\mathbf{j}$	$-\boldsymbol{j}\mathbf{k}$	$-i\boldsymbol{i}$	$-\boldsymbol{k}$
$\boldsymbol{k}\mathbf{i}$	$\boldsymbol{k}\mathbf{j}$	$\boldsymbol{k}\mathbf{k}$	$i\boldsymbol{j}$	$\boldsymbol{i}$	$-\boldsymbol{k}\mathbf{i}$	$-\boldsymbol{k}\mathbf{j}$	$-\boldsymbol{k}\mathbf{k}$	$-i\boldsymbol{j}$	$-\boldsymbol{i}$
$i\boldsymbol{i}\mathbf{i}$	$i\boldsymbol{i}\mathbf{j}$	$i\boldsymbol{i}\mathbf{k}$	$i\mathbf{k}$	$\mathbf{j}$	$-i\boldsymbol{i}\mathbf{i}$	$-i\boldsymbol{i}\mathbf{j}$	-$i\boldsymbol{i}\mathbf{k}$	$-i\mathbf{k}$	$-\mathbf{j}$
$i\boldsymbol{j}\mathbf{i}$	$i\boldsymbol{j}\mathbf{j}$	$i\boldsymbol{j}\mathbf{k}$	$i\mathbf{i}$	$\mathbf{k}$	-$i\boldsymbol{j}\mathbf{i}$	$-i\boldsymbol{j}\mathbf{j}$	-$i\boldsymbol{j}\mathbf{k}$	$-i\mathbf{i}$	$-\mathbf{k}$
$i\boldsymbol{k}\mathbf{i}$	$i\boldsymbol{k}\mathbf{j}$	$i\boldsymbol{k}\mathbf{k}$	$i\mathbf{j}$	$\mathbf{i}$	$-i\boldsymbol{k}\mathbf{i}$	$-i\boldsymbol{k}\mathbf{j}$	$-i\boldsymbol{k}\mathbf{k}$	$-i\mathbf{j}$	$-\mathbf{i}$

Here, we immediately see 12 sets of 5 generators. What we have is the complete algebra of two spaces. Putting charge, time, and mass together gives us a mathematical equivalent to another space $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$ alongside real space, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. Physically, this combination of of i with $i\mathbf{I}$, $i\mathbf{J}$, $i\mathbf{K}$ and 1 is not a space, because it is not a single quantity, and so it will never be measurable or observable in the same way as space. However, mathematically it is the same. In addition, the structure of the group suggests that it is a dual to space, a kind of antispace, creating a zero totality when combined with space.

Now starting with the 8 units needed for the 4 parameters:

$$\begin{array}{cccccccc} i & \mathbf{i} & \mathbf{j} & \mathbf{k} & 1 & \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \text{time} & & \text{space} & & \text{mass} & & \text{charge} & \end{array}$$

we have effectively compactified to the 5 generators by removing the three charge units and attaching one to each of the other three parameters:

$$\begin{array}{ccccc} i & \mathbf{i} & \mathbf{j} & \mathbf{k} & 1 \\ \boldsymbol{k} & & \boldsymbol{i} & & \boldsymbol{j} \end{array}$$

and finally:

$$\begin{array}{ccccc} i\boldsymbol{k} & \boldsymbol{i}\mathbf{i} & \boldsymbol{i}\mathbf{j} & \boldsymbol{i}\mathbf{k} & 1\boldsymbol{j} \end{array}$$

As a result, we have created 3 new composite parameters, each of which has aspects of time, space or mass, but also some characteristics of charge.

$$\begin{array}{ccccc} i\boldsymbol{k} & \boldsymbol{i}\mathbf{i} & \boldsymbol{i}\mathbf{j} & \boldsymbol{i}\mathbf{k} & 1\boldsymbol{j} \\ E & p_x & p_y & p_z & 1m \end{array}$$

The significant thing here is that the physical quantities energy, momentum and rest mass are defined by the algebraic units which arise out of the combination of the four fundamental parameters, not by their scalar values. The scalar quantities E, p_x, p_y, p_z and m have become respective coefficients of the algebraic operators $i\boldsymbol{k}$; $\boldsymbol{i}\mathbf{i}$; $\boldsymbol{i}\mathbf{j}$; $\boldsymbol{i}\mathbf{k}$; $1\boldsymbol{j}$.

The packaging process has transformed time-space-mass into their energy-momentum-rest mass conjugate. But it must also affect charge, for it simultaneously creates three new charge units, which take on the respective characteristics of the parameters with which they are associated.

$$\begin{array}{ccccc} i\boldsymbol{k} & \boldsymbol{i}\mathbf{i} & \boldsymbol{i}\mathbf{j} & \boldsymbol{i}\mathbf{k} & 1\boldsymbol{j} \\ \text{weak charge} & & \text{strong charge} & & \text{electric charge} \\ \text{pseudoscalar} & & \text{vector} & & \text{scalar} \\ SU(2) & & SU(3) & & U(1) \end{array}$$

Effectively, if we preserve the symmetry of space in the packaging process, we cannot also preserve the symmetry of charge. The packaging, as we will see, becomes equivalent to the creation of the fermionic state, and it is in the fermionic state that the symmetry of charge is broken.

The double space gives us another way of creating a D_2 or Klein-4 group representation, This is through the double algebra known as H_4, which can be expressed using 4 units, constructed from two commutative sets of quaternions, 1, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, and 1, $\boldsymbol{I}$, $\boldsymbol{J}$, $\boldsymbol{K}$. The H_4 algebra units can be constructed using coupled quaternions, with units 1, $\boldsymbol{iI}$, $\boldsymbol{jJ}$, $\boldsymbol{kK}$, but the units $\boldsymbol{iI}$, $\boldsymbol{jJ}$, $\boldsymbol{kK}$ commute with each other, unlike the units of their parent systems. So $\boldsymbol{iIjJ} = \boldsymbol{jJiI}$, etc. The algebra is cyclic but commutative, with multiplication rules:

$$\boldsymbol{iIiI} = \boldsymbol{jJjJ} = \boldsymbol{kKkK} = 1 \tag{2.1}$$

$$\boldsymbol{iIjJ} = \boldsymbol{jJiI} = \boldsymbol{kK} \tag{2.2}$$

$$\boldsymbol{jJkK} = \boldsymbol{kKjJ} = \boldsymbol{iI} \tag{2.3}$$

$$\boldsymbol{kKiI} = \boldsymbol{iIkK} = \boldsymbol{jJ} \tag{2.4}$$

The H_4 algebra units become a group with the multiplication table:

*		1	$\boldsymbol{iI}$	$\boldsymbol{jJ}$	$\boldsymbol{kK}$
1		1	$\boldsymbol{iI}$	$\boldsymbol{jJ}$	$\boldsymbol{kK}$
$\boldsymbol{iI}$		$\boldsymbol{iI}$	1	$\boldsymbol{kK}$	$\boldsymbol{jJ}$
$\boldsymbol{jJ}$		$\boldsymbol{jJ}$	$\boldsymbol{kK}$	1	$\boldsymbol{iI}$
$\boldsymbol{kK}$		$\boldsymbol{kK}$	$\boldsymbol{iI}$	$\boldsymbol{jJ}$	1

They are like quaternions with no negative signs. The group here is a subgroup of the 64-part complexified double quaternion representation.An alternative representation is via negative double vector units 1, $-\mathbf{iI}$, $-\mathbf{jJ}$, $-\mathbf{kK}$:

*		1	$-\mathbf{iI}$	$-\mathbf{jJ}$	$-\mathbf{kK}$
1		1	$-\mathbf{iI}$	$-\mathbf{jJ}$	$-\mathbf{kK}$
$-\mathbf{iI}$		$-\mathbf{iI}$	1	$-\mathbf{kK}$	$\mathbf{jJ}$
$-\mathbf{jJ}$		$-\mathbf{jJ}$	$-\mathbf{kK}$	1	$\mathbf{iI}$
$-\mathbf{kK}$		$-\mathbf{kK}$	$-\mathbf{iI}$	$-\mathbf{jJ}$	1

In this sense, it can be seen to emerge out of a double space. The unit 1 can even be seen as equivalent to $-ii$.

Our system has involved 3 of the 4 division algebras: real, complex and quaternion. Is there any place for the fourth division algebra: octonions? This comes when we consider the algebraic base units of mass, charge, time and space. It is easiest to use the complexified double quaternion form, expressing the units of space as complexified quaternions. So 1, $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, i, $i\boldsymbol{i}$, $i\boldsymbol{j}$, $i\boldsymbol{k}$ represent the units of m, $\boldsymbol{s}$, $\boldsymbol{e}$, $\boldsymbol{w}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$. Interestingly, the antiassociative parts of this algebra represent terms with no known physical meaning. The physical meanings are isolated within their individual substructures, with the octonion structure already representing a broken symmetry because it brings into itself the properties associated with its components.

The parameters arranged in algebraic units:

$*$	m	s	e	w	t	x	y	z
m	m	s	e	w	t	x	y	z
s	s	$-m$	w	$-e$	x	t	$-z$	y
e	e	$-w$	$-m$	s	y	z	$-t$	$-x$
w	w	e	$-s$	$-m$	z	$-y$	x	$-t$
t	t	$-x$	$-y$	$-z$	$-m$	s	e	w
x	x	t	$-z$	y	$-s$	$-m$	$-w$	e
y	y	z	t	$-x$	$-e$	w	$-m$	$-s$
z	z	$-y$	x	t	$-w$	$-e$	s	$-m$

The octonion mapping:

$*$	1	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$	$\boldsymbol{e}$	$\boldsymbol{f}$	$\boldsymbol{g}$	$\boldsymbol{h}$
$*$	1	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$	$\boldsymbol{e}$	$\boldsymbol{f}$	$\boldsymbol{g}$	$\boldsymbol{h}$
$\boldsymbol{i}$	$\boldsymbol{i}$	-1	$\boldsymbol{k}$	-$\boldsymbol{j}$	$\boldsymbol{f}$	-$\boldsymbol{e}$	-$\boldsymbol{h}$	$\boldsymbol{g}$
$\boldsymbol{j}$	$\boldsymbol{j}$	-$\boldsymbol{k}$	-1	$\boldsymbol{i}$	$\boldsymbol{g}$	$\boldsymbol{h}$	-$\boldsymbol{e}$	-$\boldsymbol{f}$
$\boldsymbol{k}$	$\boldsymbol{k}$	$\boldsymbol{j}$	-$\boldsymbol{i}$	-1	$\boldsymbol{h}$	-$\boldsymbol{g}$	$\boldsymbol{f}$	-$\boldsymbol{e}$
$\boldsymbol{e}$	$\boldsymbol{e}$	-$\boldsymbol{f}$	-$\boldsymbol{g}$	-$\boldsymbol{h}$	-1	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$
$\boldsymbol{f}$	$\boldsymbol{f}$	$\boldsymbol{e}$	-$\boldsymbol{h}$	$\boldsymbol{g}$	-$\boldsymbol{i}$	-1	-$\boldsymbol{k}$	$\boldsymbol{j}$
$\boldsymbol{g}$	$\boldsymbol{g}$	$\boldsymbol{h}$	$\boldsymbol{e}$	-$\boldsymbol{f}$	-$\boldsymbol{j}$	$\boldsymbol{k}$	-1	-$\boldsymbol{i}$
$\boldsymbol{h}$	$\boldsymbol{h}$	-$\boldsymbol{g}$	$\boldsymbol{f}$	$\boldsymbol{e}$	-$\boldsymbol{k}$	-$\boldsymbol{j}$	$\boldsymbol{i}$	1

If mass, time, charge and space are the truly the fundamental parameters of physics, then the double vector group of order 64 should have some major physical significance at the fundamental level. Here, we ask the question: what is physics about at the fundamental level? The answer so far seems to be: fermions and their interactions (the interactions also generating bosons). If we then ask whether the group we have derived has significance for fermions and their interactions, we immediately see that it does, for it is essentially that of the gamma algebra of the Dirac equation, governing fermions and their interaction. The gamma algebra in full is a set of 4×4 matrices. However, all possible gamma matrices can be derived from the products of two commuting sets of Pauli matrices, say σ_1, σ_2, σ_3 and Σ_1, Σ_2, Σ_3. Pauli matrices are isomorphic to multivariate vectors, so the group formed by multiplying out **i**, **j**, **k** with **I**, **J**, **K** is identical in all respects to the group formed by multiplying out σ_1, σ_2, σ_3 with Σ_1, Σ_2, Σ_3. We can set out the whole algebra of relativistic quantum mechanics by using a dual vector space or its equivalent.

Of course, the gamma algebra also has 5 generators and they can be matched to the vector-quaternion group in a number of ways. One example would be:

$$\gamma_0 = i\boldsymbol{k}; \quad \gamma_1 = \boldsymbol{i}\mathbf{i}; \quad \gamma_2 = \boldsymbol{i}\mathbf{j}; \quad \gamma_3 = \boldsymbol{i}\mathbf{k}; \quad \gamma_5 = i\boldsymbol{j}.$$

We could, of course, exchange vectors and quaternions, + and , complexified and non-complexified, etc., but the overall structure will remain the same. Significantly, 5 is not a truly symmetrical number in nature. Essentially, one set of 3-D operators (here, the quaternions) will have its symmetry broken. The other set (here, the vectors) will have its symmetry preserved. This purely mathematical requirement has major physical consequences. In fact, from now on, we see mathematics developing out of the physics and physics developing out of the mathematics. This symbiotic relationship will be one of the key features of what follows.

3 Nilpotent quantum mechanics

The algebra, and specifically the 5 generators of the group we have chosen, allows us to factorize

$$E^2 - p^2 - m^2 = 0 \tag{3.5}$$

in the form

$$(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{i}\mathbf{j}p_y + \boldsymbol{i}\mathbf{k}p_z + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{i}\mathbf{j}p_y + \boldsymbol{i}\mathbf{k}p_z + \boldsymbol{j}m) = \mathbf{0} \tag{3.6}$$

or

$$(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) = \mathbf{0}. \tag{3.7}$$

These apparently classical expressions can be immediately restructured as relativistic quantum mechanics using a canonical quantization of the first bracket (E becoming $i\partial\ /\ \partial t$, $\boldsymbol{p}$ becoming $ki\nabla$) and its application to a phase factor, which, for a free particle, would be exp $(-i(Et - \mathbf{p.r}))$. So that, including both sign options for E and $\mathbf{p}$, this becomes

$$\left(\mp \boldsymbol{k}\frac{\partial}{\partial t} \mp i\boldsymbol{i}\nabla + \boldsymbol{j}m\right)(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\mathbf{e}^{-i(Et-\mathbf{p.r})} = \mathbf{0} \tag{3.8}$$

for the nilpotent Dirac free particle equation. Here, the first bracket can be considered a row vector and the second a column vector. Defining E and $\boldsymbol{p}$ in the first bracket as operators, we can also write the second equation in the form

$$(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\pm i\boldsymbol{k}\mathbf{E} \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\mathbf{e}^{-i(Et-\mathbf{p.r})} = \mathbf{0} \tag{3.9}$$

If we quantize both brackets of the classical form of the equation, we get the Klein-Gordon equation:

$$\left(\mp \boldsymbol{k}\frac{\partial}{\partial t} \mp i\boldsymbol{i}\nabla + \boldsymbol{j}m\right)\left(\mp \boldsymbol{k}\frac{\partial}{\partial t} \mp i\boldsymbol{i}\nabla + \boldsymbol{j}m\right)e^{-i(Et-\mathbf{p.r})} = 0 \tag{3.10}$$

again with the first bracket a row vector and the second a column vector. Defining E and $\boldsymbol{p}$ in the first bracket as operators, we can write it in the same form as the Dirac equation

$$(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\pm i\boldsymbol{k}\mathbf{E} \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\mathbf{e}^{-i(Et-\mathbf{p.r})} = \mathbf{0} \tag{3.11}$$

We can, of course, obtain the nilpotent Dirac equation by left multiplying the conventional Dirac equation by $-i\gamma_5$:

$$-i\gamma_5\left(\gamma_0\frac{\partial}{\partial t} + \gamma_1\frac{\partial}{\partial x} + \gamma_2\frac{\partial}{\partial y} + \gamma_3\frac{\partial}{\partial z} + im\right) = 0 \tag{3.12}$$

and substituting algebraic operators for the gamma terms, as given above, and transferring the variation of signs in E and $\mathbf{p}$ in the phase term into the differential operator, now reconstructed as a column vector. The four separate sign variations can be immediately identified as representing, typically:

$(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	fermion	spin up
$(i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	fermion	spin down
$(-i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	antifermion	spin down
$(-i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	antifermion	spin up

The brackets, which can be imagined as arranged in a column vector, can be cycled around to represent different real particle states. A column headed by $(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ would indicate a real fermion with spin up. One headed by $(-i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ would be an antifermion with spin down. The appropriate sign variations would then follow automatically.

$$(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{i}p_x + \boldsymbol{i}\mathbf{j}p_y + \boldsymbol{i}\mathbf{k}p_z + \boldsymbol{j}m) = (i\boldsymbol{k}\mathbf{E} + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) \tag{3.13}$$

always squares to zero, we can regard it as a nilpotent, either as a classical object or as a wavefunction $(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\mathbf{e}^{-i(Et-\mathbf{p.r})}$.

As always, mathematical operations have a physical meaning. In the case of the wavefunction, it is Pauli exclusion. Two identical fermion wavefunctions will produce a zero combination state. In addition, nilpotent wavefunctions are always explicit expressions in energy and momentum, never black boxes. We can see an operator of the form $\left(\mp\boldsymbol{k}\frac{\partial}{\partial t} \mp i\boldsymbol{i}\nabla + \boldsymbol{j}m\right)$ as a generic object, which could, for a non-free fermion, incorporate field terms or covariant derivatives, with, for example, E and $\boldsymbol{p}$ becoming, respectively, and $i\partial \;/\; \partial t + ie\phi$ and $ki\nabla + ie\mathbf{A}$. The phase factor to which this is applied would no longer be exp $(-i(Et - \mathbf{p.r}))$, but whatever is needed to create an amplitude of the generic form $\left(\mp\boldsymbol{k}\frac{\partial}{\partial t} \mp i\boldsymbol{i}\nabla + \boldsymbol{j}m\right)$, still squaring to zero, but with eigenvalues E and $\mathbf{p}$ representing more complicated expressions resulting from the presence of the field terms.

Nilpotency, the fact that the wavefunction always squares to zero means that we can introduce an extra constraint such that the relativistic quantum mechanics to be applied to any fermion state is totally defined with the operator and doesn't need an equation at all. An operator of the generic form $\left(\mp\boldsymbol{k}\frac{\partial}{\partial t} \mp i\boldsymbol{i}\nabla + \boldsymbol{j}m\right)$ will always uniquely define the phase factor which makes the amplitude nilpotent, or squaring to zero. So that

$$\text{operator acting on phase factor}^2 = \text{amplitude}^2 = 0. \tag{3.14}$$

A second constraint is that the four terms in the wavefunction are merely distinguished by sign variations. A third is that there is only one phase factor.

4 Vacuum, Pauli exclusion and the dual group

If the object $(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{i}p_x+\boldsymbol{i}\mathbf{j}p_y+\boldsymbol{i}\mathbf{k}p_z+\boldsymbol{j}m) = (i\boldsymbol{k}\mathbf{E}+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)$ is a nilpotent, squaring to 0, it is easy to show that

$$\begin{aligned}
\boldsymbol{k}(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{i}p_x+\boldsymbol{i}\mathbf{j}p_y+\boldsymbol{i}\mathbf{k}p_z+\boldsymbol{j}m) &= \boldsymbol{k}(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)\\
\boldsymbol{i}(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{i}p_x+\boldsymbol{i}\mathbf{j}p_y+\boldsymbol{i}\mathbf{k}p_z+\boldsymbol{j}m) &= \boldsymbol{i}(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)\\
\boldsymbol{k}(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{i}p_x+\boldsymbol{i}\mathbf{j}p_y+\boldsymbol{i}\mathbf{k}p_z+\boldsymbol{j}m) &= \boldsymbol{j}(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)
\end{aligned}$$

are idempotents, squaring to themselves up to a scale factor that can be normalized away. Idempotents have a particular significance in nilpotent quantum mechanics relating to vacuum.

Relativistic quantum mechanics was always assumed to require idempotent, rather than nilpotent wavefunctions, essentially because spinors are built up from primitive idempotents. We can, in fact, make exactly the same equation look either idempotent or nilpotent simply by redistributing a single algebraic unit between the sections of the equation defined as operator and wavefunction. So, we can write the basic equation using either a nilpotent wavefunction:

$$\left(\left(\mp\boldsymbol{k}\frac{\partial}{\partial t}\mp i\boldsymbol{i}\nabla+\boldsymbol{j}m\right)\boldsymbol{j}\boldsymbol{j}\right)(\pm i\boldsymbol{k}E\pm\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)\mathbf{e}^{-i(Et-\mathbf{p}.\mathbf{r})}=\mathbf{0} \tag{4.1}$$

or an idempotent wavefunction:

$$\left(\left(\mp\boldsymbol{k}\frac{\partial}{\partial t}\mp i\boldsymbol{i}\nabla+\boldsymbol{j}m\right)\boldsymbol{j}\right)(\boldsymbol{j}(\pm i\boldsymbol{k}E\pm\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m))\,e^{-i(Et-\mathbf{p}.\mathbf{r})}=0 \tag{4.2}$$

Where velocity operators are not in evidence, we can define a nilpotent amplitude

$$\psi = i\boldsymbol{k}E+\boldsymbol{i}\mathbf{i}P_{\mathbf{1}}+\boldsymbol{i}\mathbf{j}P_{\mathbf{2}}+\boldsymbol{i}\mathbf{k}P_{\mathbf{3}}+\boldsymbol{j}m) \tag{4.3}$$

and an operator

$$D = i\boldsymbol{k}\frac{\partial}{\partial t}-\boldsymbol{i}\mathbf{i}\frac{\partial}{\partial X_{\mathbf{1}}}-\boldsymbol{i}\mathbf{j}\frac{\partial}{\partial X_{\mathbf{2}}}-\boldsymbol{i}\mathbf{k}\frac{\partial}{\partial X_{\mathbf{3}}} \tag{4.4}$$

with

$$\frac{dF}{dt}=[F,H]=[F,E] \tag{4.5}$$

and

$$\frac{\partial F}{\partial X_i}=[F,P_i] \tag{4.6}$$

As we will see, the discrete operators here do not need the i or $i\hbar$ coefficient of conventional quantum mechanics, which means that the equations that result are equally classical and quantum, suggesting that, under appropriate conditions, it can be applied to discrete classical, as well as quantum systems, and, as suggested by Marcer and Rowlands,[7] make nilpotency the key principle governing systems which, like the fermion, become self-ordered by interacting with their environment at many scales. This is because the mass term in the operator disappears because of the use of commutators. After some basic algebraic manipulation, we obtain

$$D = i\psi(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{i}P_{\mathbf{1}} + \boldsymbol{i}\mathbf{j}P_{\mathbf{2}} + \boldsymbol{i}\mathbf{k}P_{\mathbf{3}} + \boldsymbol{j}m) + i(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{i}P_{\mathbf{1}} + \boldsymbol{i}\mathbf{j}P_{\mathbf{2}} + \boldsymbol{i}\mathbf{k}P_{\mathbf{3}} + \boldsymbol{j}m)\psi \\ -2i(E - P_1^2 - P_2^2 - P_3^2 - m^2) \quad (4.7)$$

When is ψ nilpotent, then

$$D\psi = \left(\boldsymbol{k}\frac{\partial}{\partial t} + i\boldsymbol{i}\nabla\right)\psi = 0. \quad (4.8)$$

Generalising this to four states, with D and ψ represented as 4-spinors, then

$$D\psi = \left(\boldsymbol{k}\frac{\partial}{\partial t} \pm i\boldsymbol{i}\nabla\right)(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{i}P_{\mathbf{1}} \pm \boldsymbol{i}\mathbf{j}P_{\mathbf{2}} \pm \boldsymbol{i}\mathbf{k}P_{\mathbf{3}} + \boldsymbol{j}m) = \mathbf{0} \quad (4.9)$$

With the nilpotent structure, we have reached the minimal mathematical structure that is possible for relativistic quantum mechanics. All redundancy (and there is a great deal in the more conventional formulations) is removed. If the minimal mathematical structure is reached, it is also likely to follow that it will also be the most transparent in revealing physical information, and in fact every mathematical development which follows seems to be loaded with physical meaning.

A significant aspect of nilpotent theory is that its vacuum invokes the fundamental idea of totality zero. In nilpotent quantum mechanics the total structure of the universe is exactly zero. We create a fermion in some particular state (determined by added potentials, interaction terms, etc) ab initio, that is, from absolutely nothing, a complete void or totality zero. Vacuum then becomes what is left in nothing that is, everything other than the fermion. It is the rest of the universe which allows that fermion to be created. In this context, of course, $(i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ of course defines conservation of energy, but it is only truly valid over the entire universe, indicating that we need the second law of thermodynamics as well as the first.

Pauli exclusion is an immediate consequence of the nilpotent definition of vacuum, and it effectively says that no two fermions share the same vacuum. Suppose we create a fermion wavefunction of the form $\psi_f = (i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ from absolutely nothing; then we must simultaneously create the dual term, vacuum, $\psi_f = -(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m))$, which negates it both in superposition and combination:

$$\psi_f + \psi_v = (i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) - (i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) = \mathbf{0} \quad (4.10)$$

$$\psi_f\psi_v = -(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) = \mathbf{0} \quad (4.11)$$

This definition of vacuum also gives a new understanding to the meaning of the terms 'local' and 'nonlocal'. The 'local' is defined what happens inside the nilpotent structure $(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, which is, of course, Lorentz-invariant. The nonlocal is defined by what happens outside it. The addition and multiplication of wavefunctions: superposition and combination. Pauli exclusion is a nonlocal correlation which results from the creation of each point-like localized fermion being simultaneous with the creation of its nonlocal vacuum. Local and nonlocal are always linked. Local changes cause nonlocal ones and vice versa. For example, when we block off one slit in a Young's slit experiment we create a local change. This however causes a nonlocal one, removing the superposition and quantum coherence.

Nilpotency isn't the usual way of expressing Pauli exclusion mathematically. In the standard interpretation, wavefunctions or amplitudes are also Pauli exclusive because they are antisymmetric, with nonzero

$$(\psi_1\psi_2 - \psi_2\psi_1) = -(\psi_2\psi_1 - \psi_1\psi_2) \tag{4.12}$$

This, however, is automatic in the nilpotent formalism, where the expression becomes

$$(\pm\boldsymbol{ik}E_1 \pm \boldsymbol{i}\mathbf{p_1} + \boldsymbol{j}m_\mathbf{1})(\pm\boldsymbol{ik}E_\mathbf{2} \pm \boldsymbol{i}\mathbf{p_2} + \boldsymbol{j}m_\mathbf{2}) = \mathbf{4p_1p_2} - \mathbf{4p_2p_1} = \mathbf{8}i\mathbf{p_1}\times\mathbf{p_2} = -\mathbf{8}i\times\mathbf{p_1p_2} \tag{4.13}$$

This result is clearly antisymmetric, but it also has a quite astonishing consequence, for it requires any nilpotent wavefunction to have a p vector, in real space, the one defined by the axes **i**, **j**, **k**, at a different orientation to any other. The wavefunctions of all nilpotent fermions then instantaneously correlate because the planes of their **p** vector directions must all intersect. This is the only source of the entire physical information relating to the fermion, for, at the same time, the nilpotent condition requires the iE, $\mathbf{p}$ and m combinations to be unique, and we can visualize this as constituting a unique direction in vacuum space along a set of axes defined by $\boldsymbol{k}$, $\boldsymbol{i}$, $\boldsymbol{j}$, or **k**, **i**, **j**, with coordinates defined by the values of iE, $\mathbf{p}$ and m.

A number of significant results emerge automatically from the $\boldsymbol{k}$, $\boldsymbol{i}$, $\boldsymbol{j}$ or **k**, **i**, **j** representation. For example, half of the possibilities on one axis (those with $-m$) would be eliminated automatically (as being in the same direction as those with m), providing clear evidence that invariant mass cannot have two signs. Also eliminated would be fermions with zero m (since the directions would all be along the line $E = p$). In addition, such hypothetical massless particles would be impossible for fermions and antifermions with the same helicity, as E, $\mathbf{p}$ has the same direction as $-E$, $\mathbf{p}$.

We now have sufficient information to identify the nature of the elements in the dual group to space, time, mass and charge. The extra quaternion units in the expression $(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ clearly change the norm of the timelike term $(\boldsymbol{ik}E)$ from 1 to 1, and those of the spacelike and masslike terms ($\boldsymbol{i}\mathbf{p}$ and $\boldsymbol{j}m$) from 1 to -1, so making the quantized energy and momentum and rest mass terms equivalent to time*, space* and mass*. The same would be true if we used the nilpotent structure $(\boldsymbol{ik}t + \boldsymbol{i}\mathbf{r} + \boldsymbol{j}\tau)$ for the relativistic space-time invariance, where τ is the proper time.

The quantized angular momentum would then be equivalent to the charge* term. Its 3 dimensions (on axes, $\boldsymbol{k}$, $\boldsymbol{i}$, $\boldsymbol{j}$) become sources for the handedness, direction and magnitude which are separately conserved characteristics of angular momentum, and relate to the fact that weak, strong and electric charges manifest separate characteristics. (Ultimately these aspects of angular momentum carry separate information about the charges.) The group of order 8 incorporating the D_2 parameter group and its mathematical dual, which is isomorphic to the quaternions, would then be the quantized phase space for the fermion.

We now have at least five different meanings for the expression

$$\boldsymbol{j}(i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{j}(i\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\phi = \mathbf{0}$$

with ϕ an (optional) arbitrary scalar factor (phase, etc.):

classical			special relativity
operator	$\times$	operator	Klein-Gordon equation
operator	$\times$	wavefunction	Dirac equation
wavefunction	$\times$	wavefunction	Pauli exclusion
fermion	$\times$	vacuum	thermodynamics

It is characteristic of a theory in which duality is so deeply embedded to create such multiple meanings.

The nilpotent operator can be used to do ordinary relativistic quantum mechanics. We define a probability density for a nilpotent wavefunction, $(\pm i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$. Here, we multiply by the complex quaternion conjugate $(\mp i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ (the extra quaternion resulting from the premultiplication of ψ by a quaternion operator), and normalize to 1. The complex quaternion conjugate then becomes the reciprocal of the original wavefunction. More significantly, the nilpotent formalism not only creates quantum mechanics, but also implies a full quantum field theory in which the operators act on the entire quantum field, without requiring any formal process of second quantization. A nilpotent operator, defined from absolutely nothing, becomes a creation operator acting on vacuum to create the fermion, together with all the interactions in which it is involved. Many standard results follow: spin , helicity, chirality of massless fermions, *zitterbewegung*. In these cases, the nilpotent method is not significantly different from ordinary relativistic quantum mechanics. There is also a nilpotent version of the Dirac prescription for converting the operator to polar coordinates:

$$\left(\pm i\boldsymbol{k}\left(E + \frac{A}{r}\right) \pm \boldsymbol{i}\left(\frac{\partial}{\partial r} + \frac{1}{r} \pm i\left(\frac{j+\frac{1}{2}}{r}\right)\right) + \boldsymbol{j}m\right). \tag{4.14}$$

$$\left(\pm i\boldsymbol{k}\left(E + \frac{A}{r}\right) \pm \boldsymbol{i}\left(\frac{\partial}{\partial r} + \frac{1}{r} \pm i\left(\frac{j+\frac{1}{2}}{r}\right)\right) + \boldsymbol{j}m\right). \tag{4.15}$$

The last prescription produces a new result. It ensures that no nilpotent solution is possible for a point-fermion with spherical symmetry unless a term proportional to r^{-1} (Coulomb potential) is added to the $\boldsymbol{ik}E$ term to counter the similar term in the $\partial\ /\ \partial r$ part of the operator. In fundamental terms it arises because defining a point in any meaningful way in 3-dimensional space requires a dual space which is structured on the basis of point charges. In addition, the only way of fixing a point in a nonconserved space with no identifiable units is to fix it in the space of a conserved quantity which is made to coincide, through nilpotency, with this one. By making this Coulomb component a consequence of nilpotency, we can also see it as a consequence of Pauli exclusion.

5 CPT symmetry

Since the lead term in the fermionic column vector defines the fermion state, then we can show that the remaining terms are equivalent to the lead term, subjected to the respective symmetry transformations, P, T and C, by pre- and post-multiplication by the quaternion units $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, defining the vacuum space:

Parity	P	$\boldsymbol{i}(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)\boldsymbol{i}$	$=$	$(i\boldsymbol{k}E-\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)$
Time reversal	T	$\boldsymbol{k}(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)\boldsymbol{k}$	$=$	$(-i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)$
Charge conjugation	C	$-\boldsymbol{j}(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)\boldsymbol{j}$	$=$	$(-i\boldsymbol{k}E-\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)$

We can easily show that PTC, TCP, andCPT also apply,

$$\begin{array}{lllll} PT & \boldsymbol{i}(-i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)\boldsymbol{i} & = & (-i\boldsymbol{k}E-\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m) \\ TC & \boldsymbol{k}(-i\boldsymbol{k}E-\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)\boldsymbol{k} & = & (i\boldsymbol{k}E-\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m) \\ CP & -\boldsymbol{j}(i\boldsymbol{k}E-\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)\boldsymbol{j} & = & (-i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m) \end{array}$$

and that $TCP = CPT = identity.$

$$\boldsymbol{k}\,(-\boldsymbol{j}\,(\boldsymbol{i}(\pm i\boldsymbol{k}E\pm\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)\boldsymbol{i})\,\boldsymbol{j})\,\boldsymbol{k} = -\boldsymbol{kji}(\pm i\boldsymbol{k}E\pm\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)\boldsymbol{ijk} = (\pm i\boldsymbol{k}\mathbf{E}\pm\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m) \tag{5.1}$$

The relation between the P, T and C transformations and vacuum can be shown in a relatively simple way. If we take (ikE ip + jm) and post-multiply it by the idempotent $\boldsymbol{k}(\pm i\boldsymbol{k}E\pm\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)$ any number of times, the only effect is to introduce a scalar multiple, which can be normalized away.

$$(\pm i\boldsymbol{k}E\pm\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)\boldsymbol{k}(\pm i\boldsymbol{k}E\pm\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)\boldsymbol{k}(\pm i\boldsymbol{k}E\pm\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)\ \ldots\ \to (\pm i\boldsymbol{k}E\pm\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m) \tag{5.2}$$

Similarly with $\boldsymbol{i}(\pm i\boldsymbol{k}E\pm\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)$ and $\boldsymbol{j}(\pm i\boldsymbol{k}E\pm\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)$. All these idempotent quantities can be regarded as vacuum operators, and $\boldsymbol{k}$, $\boldsymbol{i}$ and $\boldsymbol{j}$, or, equivalently, $\mathbf{k}$, $\mathbf{i}$ and $\mathbf{j}$, as coefficients of a vacuum space. The observed particle state relating to any fermion is the first in the column vector, while the others are the accompanying vacuum states, or states into which the observed particle could transform by respective P, T and C transformations. Replacing the observed fermion state spin up with any of the others would simultaneously transform all four states by P, T or C.

The nilpotent formalism defines a continuous vacuum $-(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ to each fermion state $(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, and this vacuum expresses the nonlocal aspect of the state. However, the use of the operators $\boldsymbol{k}$, $\boldsymbol{i}$, $\boldsymbol{j}$ suggests that we can partition this state into discrete components with a dimensional structure. We can now interpret the three terms other than the lead term in the spinor as the vacuum reflections that are created with the particle. We can regard the existence of three vacuum operators as a result of a partitioning of the vacuum as a result of quantization and as a consequence of the 3-part structure observed in the nilpotent fermionic state, while the zitterbewegung can be taken as an indication that the vacuum is active in defining the fermionic state.

Now, we know that the three vacuum coefficients $\boldsymbol{k}$, $\boldsymbol{i}$, $\boldsymbol{j}$ originate in (or are responsible for) the concept of discrete (point-like) charge. However, the operators, $\boldsymbol{k}$, $\boldsymbol{i}$ and $\boldsymbol{j}$, as we are using them here, perform another function of weak, strong and electric charges or sources, in acting to partition the continuous vacuum represented by $-(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ to each fermion state $(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, and responsible for zero-point energy, into discrete components, whose special characteristics are determined by the respective pseudoscalar, vector and scalar natures of their associated terms iE, $\mathbf{p}$ and m.

6 Bosons and baryons

Among the most important of nilpotent quantum mechanics' many new results are the descriptions of three different boson-type states, which are combinations of the fermion state with any of the P, T or C transformed ones, the result being a scalar wavefunction. Many new results also emerge from the nilpotent formalism.

$(\pm\boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\mp\boldsymbol{ik}\mathbf{E} \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	spin 1 boson
$(\pm\boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\mp\boldsymbol{ik}\mathbf{E} \mp \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	spin 0 boson
$(\pm\boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\pm\boldsymbol{ik}\mathbf{E} \mp \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$	fermion-fermion combination

One of the most significant aspects of this formalization is that a spin 1 boson can be massless, but a spin 0 boson cannot, as then $(\pm\boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p})(\mp\boldsymbol{ik}\mathbf{E} \mp \boldsymbol{i}\mathbf{p})$ would immediately zero: hence Goldstone bosons must become Higgs bosons in the Higgs mechanism. We can thus represent the four components of the nilpotent spinor as creation operators for

fermion	spin up	$(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$
fermion	spin down	$(\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$
antifermion	spin down	$(-\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$
antifermion	spin up	$(-\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$

or annihilation operators for

antifermion	spin down	$(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$
antifermion	spin up	$(\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$
fermion	spin up	$(-\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$
fermion	spin down	$(-\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$

Using just the lead terms of the nilpotents, and assuming that we can complete the spinor structures using the 3 conventional sign variations, we could represent any given fermion as:

$$\begin{array}{l}(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)\boldsymbol{k}(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)\boldsymbol{k}(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)...\\(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)\boldsymbol{j}(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)\boldsymbol{j}(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)...\\(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)\boldsymbol{i}(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)\boldsymbol{i}(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)...\end{array}$$

Here, we see that every alternate bracket reverses sign(s) in such a way that it pairs with the fermion on its left to become a boson. In general, a fermion converts to a boson by multiplication by an antifermionic operator $Q^\dagger$; a boson converts to a fermion by multiplication by a fermionic operator Q, and we can represent a sequence such as $(i\boldsymbol{k}E-\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)\boldsymbol{k}(i\boldsymbol{k}E-\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)\boldsymbol{k}(i\boldsymbol{k}E-\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)...$ by the supersymmetric $QQ^\dagger QQ^\dagger QQ^\dagger QQ^\dagger Q$... We can interpret this as the series of boson and fermion loops, of the same energy and momentum, required by the exact supersymmetry which would eliminate the need for self-energy renormalization. The fermions and bosons are their own supersymmetric partners through the creation of vacuum states.

Baryons can be represented as three-component structures in which the vector nature of $\mathbf{p}$ plays an explicit role:

$$(i\boldsymbol{k}E\pm\boldsymbol{i}\mathbf{i}p_x+\boldsymbol{j}m)(i\boldsymbol{k}\mathbf{E}\pm\boldsymbol{i}\mathbf{j}p_y+\boldsymbol{j}m)(i\boldsymbol{k}\mathbf{E}\pm\boldsymbol{i}\mathbf{j}p_z+\boldsymbol{k}m)$$

This has nilpotent solutions when $\mathbf{p}=\pm\mathbf{i}p_x$, $\mathbf{p}=\pm\mathbf{j}p_y$, or $\mathbf{p}=\pm\mathbf{k}p_z$, or when the momentum is directed entirely along the $\mathbf{x}$, $\mathbf{y}$, or $\mathbf{z}$ axes, in either direction, though these, of course, are arbitrarily defined. Using the appropriate normalization, these reduce to

$$\begin{array}{ll}(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{i}p_x+\boldsymbol{j}m) & +\mathrm{RGB}\\(i\boldsymbol{k}E-\boldsymbol{i}\mathbf{i}p_x+\boldsymbol{j}m) & -\mathrm{RBG}\\(i\boldsymbol{k}E-\boldsymbol{i}\mathbf{j}p_y+\boldsymbol{j}m) & +\mathrm{BRG}\\(i\boldsymbol{k}E+\boldsymbol{i}\mathbf{j}p_y+\boldsymbol{j}m) & -\mathrm{GRB}\\(i\boldsymbol{k}E\pm\boldsymbol{i}\mathbf{k}p_z+\boldsymbol{j}m) & +\mathrm{GBR}\\(i\boldsymbol{k}E\pm\boldsymbol{i}\mathbf{k}p_z-\boldsymbol{j}m) & -\mathrm{BGR}\end{array}$$

with the third and fourth changing, very significantly, the sign of the $\mathbf{p}$ component. Because of this, there has to be a maximal superposition of left- and right-handed components, thus explaining the zero observed chirality in the interaction. Gluons would be represented as $(i\boldsymbol{k}E\pm\boldsymbol{i}\mathbf{i}p_x)(-i\boldsymbol{k}\mathbf{E}+\boldsymbol{i}\boldsymbol{i}\mathbf{j}p_y+\boldsymbol{j}m)$, etc.

7 The three gauge interactions

The three gauge interactions can all be seen to arise from nonlocal origins, which then have corresponding local manifestations. All definition of locality at a point with spherical symmetry requires, as we have seen, a Coulomb potential to maintain nilpotency or nonlocal Pauli exclusion. So all gauge interactions require a Coulomb component. Now the strong interaction, reflecting the vector nature of the strong charge, requires a combination state with three components of momentum switching nonlocally between three fermionic brackets. As this is local, there is a force or rate of change of momentum which is not dependent on distance. Locally, this will manifest itself as a linear potential. The weak interaction, reflecting the pseudoscalar nature of the strong charge, emerges from a superposition between fermionic and antifermionic (or + and $\boldsymbol{ik}E$) states. It is built into the spinor structure of the fermionic state itself. The dual structure of the pseudoscalar term means that the weak charge is, at the least, always part of a dipole with its vacuum reflection. To reproduce the nonlocal connection in a local form, this would require at least one extra potential, proportional to r^n, where is a positive or negative integer $\neq 1$, in addition to the Coulomb term.

The nonlocal manifestations of the three gauge interactions can thus be realised first nonlocally through structures that reflect the status of the charges in the fermionic nilpotent as scalar, vector or pseudoscalar, and then through the local interactions which produce the equivalent result.

electric	Coulomb
strong	Coulomb + linear
weak	Coulomb + dipole or higher order terms

In fact, these and only these potentials all provide analytic nilpotent solutions. The Coulomb solution is well known (the hydrogen atom) but the nilpotent method reproduces it in only six lines of calculation. The others are new results. The Coulomb + linear potential produces a phase term which leads to a solution with the required characteristics of infrared slavery and asymptotic freedom, exactly as required in the strong interaction. Coulomb + any other potential or combination of potentials (excluding linear) yields a harmonic oscillator solution with the ($\frac{1}{2}$ unit energy term correlated to fermion spin. This is exactly in line with the weak interaction as a creator and annihilator of boson states, with the spin creating the dipolar connection of the fermion with vacuum through *zitterbewegung*.

The particle-vacuum weak dipole mechanism, as a fundamental ordering mechanism involving annihilation and creation, also connects with similar dipole-generated phenomena at a wide variety of scales, giving support the conjecture by Marcer and Rowlands that nilpotency of some kind is a very general phenomenon which extends well beyond quantum mechanics, and appears as a key factor in creating systems, like the fermion, which self-order with respect to their environment.[7] Since the Coulomb + dipole or higher order terms solution is so general, and the Coulomb term is required for spherical symmetry, then no other nilpotent solution is available for a point particle. Since the last solution is so general, and the Coulomb term is required for spherical symmetry, then no other nilpotent solution is available for a point particle.

Quantum field theory is very amenable to the nilpotent method. Second quantization is redundant as the vacuum is a fundamental part of the structure. Perturbation theory calculations for QED can be done with relative efficiency, along with renormalization of the interactive parts, but the self-energy term cancels at first order.[4] Similarly, cancellation of boson and fermion loops can be carried out at all orders for the self-energy[6] using

$$(\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) = (\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{k}(\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\boldsymbol{k}(\boldsymbol{ik}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)...$$

Also, boson propagators being inverses of terms like $(\pm\boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(\mp\boldsymbol{ik}\mathbf{E} \pm \boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)$, $(\pm\boldsymbol{ik}E\pm\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)(\mp\boldsymbol{ik}\mathbf{E}\mp\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)$, and $(\pm\boldsymbol{ik}E\pm\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)(\pm\boldsymbol{ik}\mathbf{E}\mp\boldsymbol{i}\mathbf{p}+\boldsymbol{j}m)$, have denumerators which are never zero, completely removing the infrared divergence.[4] Propagators can also be written for weak and strong interactions. Weak interactions can be calculated directly from the bosonic states without the use of trace theorems, a method which is capable of much greater generalisation, and probable application in QCD as well. Other significant nilpotent calculations and interpretations involve BRST quantization and the Higgs mechanism. The method is clearly capable of a great deal more generalisation than it has so far received.

8 Dimensions and spinors

With our mapping of the component units of the four parameters onto an octonion structure, it is interesting that Baez and Herta have proposed that an octonion space is the true basis of physics, and that it can be used as a basis to support 2-D strings within the 10-D of string theory.[8] Now, the nilpotent structure is an 8-D object in at least two senses, and it has been apparent from the beginning that $(\pm\boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ can be regarded as 10-D in containing 5-D for E, $\mathbf{p}$), m and 5-D for the charges, and that 6 of the dimensions (all except E and $\mathbf{p}$) are fixed or compactified. However, in $(\pm\boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m$, two of the dimensions are redundant due to nilpotency, and we reduce to 8 using a point source rather than a 2-D string.

The combination of two spaces or a space and antispace in a point source with zero norm is responsible for the symmetry-breaking between them. But there is one aspect of the fermion where they are equal: the angular momentum or spin. A set of primitive idempotents constructing a spinor can be defined in terms of the H_4 algebra, constructed from the dual vector spaces:

$$\begin{gathered}(1-\mathbf{iI}-\mathbf{jJ}-\mathbf{kK})/4\\(1-\mathbf{iI}+\mathbf{jJ}+\mathbf{kK})/4\\(1+\mathbf{iI}-\mathbf{jJ}+\mathbf{kK})/4\\(1+\mathbf{iI}+\mathbf{jJ}-\mathbf{kK})/4\end{gathered}$$

or from coupled quaternions:

$$\begin{gathered}(1+\boldsymbol{iI}+\boldsymbol{jJ}+\boldsymbol{iI})/4\\(1+\boldsymbol{iI}-\boldsymbol{jJ}-\boldsymbol{iI})/4\\(1-\boldsymbol{iI}+\boldsymbol{jJ}-\boldsymbol{iI})/4\\(1-\boldsymbol{iI}-\boldsymbol{jJ}+\boldsymbol{iI})/4\end{gathered}$$

As required the 4 terms add up to 1, and are orthogonal as well as idempotent, all products between them being 0. One of the remarkable things about the structures is that they have the exact form of the components of the two forms of the quartic Berwald-Moor metric of Finsler geometry:

$$(t - x - y - z)(t - x + y + z)(t + x - y + z)(t + x + y - z) \tag{8.1}$$

$$(t + x + y + z)(t + x - y - z)(t - x + y - z)(t - x - y + z) \tag{8.2}$$

If we multiply the 4 components in any order, we will always get zero. In a sense this is like defining a singularity in spinor space. The zero product can thus be interpreted as a fermionic singularity arising from the distortion introduced into the vacuum (or spinor) space by the application of a nilpotent condition. The space becomes quartic because it is created out of two quadratic spaces. The quartic Berwald-Moor metric becomes an expression of the fundamentally rotationally quartic nature of the underlying algebra. While multiplication of the units of the algebra produces rotations in the spaces and identity after a complete cycle, multiplication of the spin metric shows that it describes a singularity. It is fitting that the only place where the perfect symmetry between the two spaces is preserved is in the description of the quantity, angular momentum or spin, which we have shown carries the entire information about the fermion state. It also only occurs where we reduce the information about the real spatial state to a zero size. Ultimately, nilpotent quantum mechanics is an expression of the creation of the perfect self-organizing singularity state.

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