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AN ADAPTED FRAME ON THE INDICATRIX BUNDLE OF A COMPLEX FINSLER SPACE

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Abstract

Following the study of the indicatrix of a complex Finsler space (M, L) initiated in [10], in this paper an adapted frame is introduced on the complexified of the real tangent bundle of the complex Finsler manifold in a manner that makes it easier to study the properties of the indicatrix bundle. The indicatrix IM is studied as a hypersurface of the holomorphic tangent bundle $T'M$ and the adapted frame obtained on it gives simplified expressions of the equations of the subspace. Using them, a link between the curvature and torsion coefficients of the induced tangent connection and the ones existing on the ambient manifold is obtained.

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1 Introduction

The study of the indicatrix of a real Finsler space is one of interest ([3, 5, 7], etc.), mainly because it is a compact and strictly convex set surrounding the origin. For example, the indicatrix plays a special role in the definition of the volume of a Finsler space.

In this paper, based on some ideas from the real case and continuing the existing ones in the complex spaces, the indicatrix bundle of a complex Finsler manifold (M, F) is introduced and several of its properties are obtained. Firstly, we recall some basic notions about complex Finsler geometry (in Section 1). Then, in Section 2 a new local frame of vector fields in $T_CT'M$ is fixed and in this frame,

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the holomorphic tangent bundle of $T'M$ can be written as direct sum of the vertical Liouville vector field and its orthogonal distributions with respect to the Sasaki lifted metric G. With the help of these bases, a local frame is introduced on the indicatrix bundle. In Section 3, by using the new frame introduced in Section 2, the Gauss-Weingarten formulae are expressed and the relation between the local coefficients of the second fundamental form and Weingarten operator will be given. Using the submanifold equations ([9, 10]), the local expressions for the Gauss, H- and A-Codazzi, Ricci equations, corresponding to the indicatrix bundle, are obtained in Section 4.

Now, we will make a short overview of the concepts and terminology used in complex Finsler geometry, for more see [1, 8]. Let M be an $n+1$ dimensional complex manifold and $z := (z^k),\ k = 1,..,n+1$, the complex coordinates on a local chart (U, φ). The complexified of the real tangent bundle $T_{\mathbb{C}}M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$, i.e. $T_{\mathbb{C}}M = T'M \oplus T''M$. The holomorphic tangent bundle $T'M$ is in its turn a $(2n+2)$-dimensional complex manifold and the local coordinates in a local chart in $u \in T'M$ are $u := (z^k, \eta^k),\ k = 1,..,n+1$, where η^k are the components of a (1,0) vector of T_zM, $X_z = \eta^k \frac{\partial}{\partial z^k}$.

Definition 1. *A* complex Finsler space *is a pair* (M, F)*, where* $\mathrm{F} : T'M \to \mathbb{R}^+,\ \mathrm{F} = \mathrm{F}(z, \eta)$ *is a continuous function that satisfies the following conditions:*

i. $L := \mathrm{F}^2$ *is a smooth function on* $\widetilde{T'M} := T'M \setminus \{0\}$*;*

ii. $\mathrm{F}(z, \eta) \geq 0$*, the equality holds if and only if* $\eta = 0$*;*

iii. $\mathrm{F}(z, \lambda\eta) = |\lambda|\mathrm{F}(z, \eta),\ \forall\lambda \in \mathbb{C}$*;*

iv. the Hermitian matrix $\big(g_{i\bar{j}}(z, \eta)\big)$*, with* $g_{i\bar{j}} = \frac{\partial^2 L}{\partial\eta^i\partial\bar{\eta}^j}$ *the fundamental metric tensor is positive definite on* $T'M$*.*

The last condition means that the indicatrix $\mathrm{I}_z = \{\eta \mid g_{i\bar{j}}(z, \eta)\eta^i\bar{\eta}^j = 1\}$ considered in a fixed point is strongly pseudoconvex, for any $z \in M$. Moreover, the positivity of $(g_{i\bar{j}})$ ensures the existence of the inverse $(g^{\bar{j}i})$, with $g^{\bar{j}i}g_{i\bar{k}} = \delta^{\bar{j}}_{\bar{k}}$.

Condition iii. represents the homogeneity of L with respect to the complex norm, $L(z, \lambda\eta) = \lambda\bar{\lambda}L(z, \eta),\ \forall\lambda \in \mathbb{C}$, and by applying Euler's formula we get that:

$$\frac{\partial L}{\partial\eta^k}\eta^k = \frac{\partial L}{\partial\bar{\eta}^k}\bar{\eta}^k = L; \quad \frac{\partial g_{i\bar{j}}}{\partial\eta^k}\eta^k = \frac{\partial g_{i\bar{j}}}{\partial\bar{\eta}^k}\bar{\eta}^k = 0 \quad \text{and} \quad L = g_{i\bar{j}}\eta^i\bar{\eta}^j. \tag{1}$$

Roughly speaking, the geometry of a complex Finsler space consists of the study of the geometric objects of the complex manifold $T'M$ endowed with a Hermitian metric structure defined by $g_{i\bar{j}}$. Regarding this, the first step is the study of the sections of the complexified tangent bundle of $T'M$ which splits into the direct sum $T_{\mathbb{C}}(T'M) = T'(T'M) \oplus T''(T'M)$, where $T''_u(T'M) = \overline{T'_u(T'M)}$. Let $V(T'M) \subset T'(T'M)$ be the vertical bundle, locally spanned by $\left\{\frac{\partial}{\partial\eta^k}\right\}$ and let $V(T''M)$ be its conjugate that contains $(0,1)-$ vector fields.

The idea of complex nonlinear connection, briefly (c.n.c.), is fundamental in "linearization" of this geometry ([8]). A (c.n.c.) is a supplementary complex subbundle to $V(T'M)$ in $T'(T'M)$, i.e. $T'(T'M) = H(T'M) \oplus V(T'M)$. The horizontal distribution $H_u(T'M)$ is locally spanned by $\left\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\right\}$, where $N_k^j(z,\eta)$ are the coefficients of the (c.n.c.). Then, we will call the adapted frame of the (c.n.c.) the pair $\left\{\delta_k := \frac{\delta}{\delta z^k},\, \dot{\partial}_k := \frac{\partial}{\partial \eta^k}\right\}$, which obeys the change rules $\delta_k = \frac{\partial z'^j}{\partial z^k}\delta'_j$ and $\dot{\partial}_k = \frac{\partial z'^j}{\partial z^k}\dot{\partial}'_j$. By conjugation everywhere we get an adapted frame $\{\delta_{\bar{k}},\, \dot{\partial}_{\bar{k}}\}$ on $T''_u(T'M)$. The dual adapted bases are $\{\mathrm{d}z^k,\, \delta\eta^k := \mathrm{d}\eta^k + N_j^k \mathrm{d}z^j\}$, respectively $\{\mathrm{d}\bar{z}^k,\, \delta\bar{\eta}^k\}$, where $\delta\bar{\eta}^k = \mathrm{d}\bar{\eta}^k + N_{\bar{j}}^{\bar{k}}\mathrm{d}\bar{z}^j$.

Let us consider on $T'M$ the Hermitian metric structure G, named the Sasaki type lift of the metric tensor $g_{i\bar{j}}$, as

$$G = g_{i\bar{j}}\mathrm{d}z^i \otimes \mathrm{d}\bar{z}^k + g_{i\bar{j}}\delta\eta^i \otimes \delta\bar{\eta}^j. \tag{2}$$

On the sections of $T_{\mathrm{C}}(T'M)$ bundle, two almost complex structures act. One is the natural complex structure J on the complex manifold $T'M$, given by $J(\partial_k) = \mathrm{i}\partial_k,\ J(\partial_{\bar{k}}) = -\mathrm{i}\partial_{\bar{k}},\ J(\dot{\partial}_k) = \mathrm{i}\dot{\partial}_k,\ J(\dot{\partial}_{\bar{k}}) = -\mathrm{i}\dot{\partial}_{\bar{k}}$,where $\partial_k = \frac{\partial}{\partial z^k}$. With respect to the adapted frames of a (c.n.c.), J is given by

$$J(\delta_k) = \mathrm{i}\delta_k, \quad J(\dot{\partial}_k) = \mathrm{i}\dot{\partial}_k, \quad J(\delta_{\bar{k}}) = -\mathrm{i}\delta_{\bar{k}}, \quad J(\dot{\partial}_{\bar{k}}) = -\mathrm{i}\dot{\partial}_{\bar{k}}. \tag{3}$$

The second almost complex structure is

$$F(\delta_k) = -\dot{\partial}_k, \quad F(\dot{\partial}_k) = \delta_k, \quad F(\delta_{\bar{k}}) = -\dot{\partial}_{\bar{k}}, \quad F(\dot{\partial}_{\bar{k}}) = \delta_{\bar{k}}. \tag{4}$$

Further on, to avoid a possible confusion with the fundamental function F, we will use the notation (M,L) for the complex Finsler space. Thus, (M,F,G) is an almost Hermitian structure on $T'M$ and its integrability involves the integrability of the horizontal distribution.

One main problem of this geometry is to determine a (c.n.c) related only by the fundamental function of a complex Finsler space (M,L); one almost classical now is the Chern-Finsler (c.n.c) ([1, 8]), in brief C-F (c.n.c.):

$$N_j^k = g^{\bar{m}k}\frac{\partial g_{l\bar{m}}}{\partial z^j}\eta^l. \tag{5}$$

The next step is to specify the derivation law D on sections of $T_{\mathrm{C}}(T'M)$. A Hermitian connection D, of $(1,0)$-type, which satisfies $D_{JX}Y = JD_XY$, for all horizontal vectors X and J the natural complex structure of the manifold, will be the *Chern-Finsler linear connection*, locally given by the next set of coefficients (notations from [8]):

$$L^i_{jk} = g^{\bar{l}i}\left(\delta_k g_{j\bar{l}}\right), \quad C^i_{jk} = g^{\bar{l}i}(\dot{\partial}_k g_{j\bar{l}}), \quad L^{\bar{\imath}}_{\bar{j}k} = 0, \quad C^{\bar{\imath}}_{\bar{j}k} = 0, \tag{6}$$

where $D_{\delta_k}\delta_j = L^i_{jk}\delta_i,\ D_{\delta_k}\dot{\partial}_j = L^i_{jk}\dot{\partial}_i,\ D_{\dot{\partial}_k}\dot{\partial}_j = C^i_{jk}\dot{\partial}_i,\ D_{\dot{\partial}_k}\delta_j = C^i_{jk}\delta_i$. Of course, there is also $\overline{D_XY} = D_{\bar{X}}\bar{Y}$. From the homogeneity conditions (1) it takes: $C^i_{jk}\eta^j = C^i_{jk}\eta^k = 0$. Moreover, considering that N^i_k is $(1,0)$-homogeneous, i.e. $(\dot{\partial}_k N^i_j)\eta^k = N^i_j$ and $(\dot{\partial}_{\bar{k}} N^i_j)\eta^{\bar{k}} = 0$ ([2]), it takes place $\eta^j L^i_{jk} = N^i_k$ and $L^i_{jk} = \dot{\partial}_j N^i_k$.

Further we will use the following notation $\bar{\eta}^j =: \eta^{\bar{j}}$ to note a conjugate object.

2 A frame on the indicatrix bundle of a complex Finsler manifold

In the following, we consider $(\widetilde{T'M}, G)$ the slit holomorphic tangent bundle of the Finsler manifold M endowed with the Sasaki lift (2), which is a Hermitian metric structure on $\widetilde{T'M} = T'M\backslash\{0\}$. Considering that $\dim_{\mathbb{C}} T'M = 2n+2$, where $\dim_C M = n+1$, we take on $T'M$ the local coordinates (z^k, η^k), with $k = 1,..,n+1$.

We denote by IM the hypersurface of $\widetilde{T'M}$ given by

$$IM = \underset{z\in M}{\cup} I_z M, \quad I_z M = \left\{\eta \in T'_z M \mid \mathrm{F}(z,\eta) = 1\right\},$$

which will be called the *indicatrix bundle* of the complex Finsler space (M, F). The above condition can be written, for any $z \in M$, as

$$L(z,\eta) = 1 \quad \text{or} \quad g_{i\bar{j}}(z,\eta)\eta^i\bar{\eta}^j = 1.$$

In the following, considering the results form the real case [3], we will determine the normal vector of the indicatrix bundle. First, it can be noticed that the inclusion $IM \overset{i}{\hookrightarrow} T'M$ takes place. Locally, we can consider a parametrization of this submanifold as:

$$z^i = z^i(v^a), \quad \eta^i = \eta^i(v^a), \quad a \in \{1,2,..,2n+1\}.$$

Differentiating $\mathrm{F}^2(z,\eta) = 1$ with respect to v^a we obtain: $\frac{\partial \mathrm{F}^2}{\partial z^i}\frac{\partial z^i}{\partial v^a} + \frac{\partial \mathrm{F}^2}{\partial \eta^i}\frac{\partial \eta^i}{\partial v^a} = 0$. Using $\mathrm{F}^2 = L$, we can rewrite:

$$\frac{\partial L}{\partial z^i}\frac{\partial z^i}{\partial v^a} + \frac{\partial L}{\partial \eta^i}\frac{\partial \eta^i}{\partial v^a} = 0.$$

From the homogeneity relations we define: $\eta_i := g_{i\bar{j}}\bar{\eta}^j = \frac{\partial L}{\partial \eta^i}$. Furthermore, on $T'M$ we consider the Chern-Finsler (c.n.c.) such that $\frac{\delta}{\delta z^i} = \frac{\partial}{\partial z^i} - N^k_i \frac{\partial}{\partial \eta^k}$ and $\frac{\delta L}{\delta z^i} = 0$. Then the above relations can be written as $\left(\frac{\delta L}{\delta z^i} + N^k_i \frac{\partial L}{\partial \eta^k}\right)\frac{\partial z^i}{\partial v^a} + \frac{\partial L}{\partial \eta^i}\frac{\partial \eta^i}{\partial v^a} = 0$, that is equivalent to

$$\left(N^k_i \frac{\partial z^i}{\partial v^a} + \frac{\partial \eta^k}{\partial v^a}\right)\eta_k = 0. \tag{7}$$

The natural frame field on IM is represented by

$$\frac{\partial}{\partial v^a} = \frac{\partial z^i}{\partial v^a}\frac{\partial}{\partial z^i} + \frac{\partial \eta^i}{\partial v^a}\frac{\partial}{\partial \eta^i} = \frac{\partial z^i}{\partial v^a}\frac{\delta}{\delta z^i} + \left(N^k_i \frac{\partial z^i}{\partial v^a} + \frac{\partial \eta^k}{\partial v^a}\right)\frac{\partial}{\partial \eta^k}.$$

Then, by (7), we have

$$G\left(\frac{\partial}{\partial v^a}, \bar{\eta}^l \frac{\partial}{\partial \bar{\eta}^l}\right) = \left(N_i^k \frac{\partial z^i}{\partial v^a} + \frac{\partial \eta^k}{\partial v^a}\right) \bar{\eta}^l g_{k\bar{l}} = 0,$$

where G is the Sasaki lift. Then it follows that the vertical Liouville vector field $C = \eta^l \frac{\partial}{\partial \eta^l}$ is orthogonal to $T'(IM)$, i.e. it is normal to the indicatrix. Thus, we can state:

Lemma 1. *With respect to the Sasaki lift G given by (2), the vertical Liouville vector field is everywhere orthogonal to the indicatrix bundle, i.e. $G(X, \bar{C}) = 0$, for any vector fields $X \in T'(IM)$. The vector field $N = \frac{1}{\mathrm{F}}C$ is a unit normal vector field orthogonal of the indicatrix bundle.*

The unit horizontal Liouville vector field $\xi = \frac{1}{\mathrm{F}}\eta^i \frac{\delta}{\delta z^i}$ is tangent to IM since $G(\xi, \bar{N}) = 0$. On $T'M$ we have the natural complex structure J and the almost complex structure F, given in (3) and (4), respectively. Thus, we notice that $F(N) = \xi$.

Consider the vertical Liouville distribution on $T'M$ defined by

$$V'_N = \left\{Z \in \Gamma\left(V(T'M)\right) \mid G(Z, \bar{N}) = 0\right\}. \tag{8}$$

Since $\dim_C V'_N = n$, we can assume that the orthogonal vertical distribution to N in $V'(TM)$ with respect to the Sasaki lift G has a local frame taken as follows ([4]):

$$V'_N = span\left\{\frac{\partial}{\partial \theta^\alpha}\right\}, \alpha = 1, .., n.$$

Considering the fact that V'_N is a hypersurface of $V(T'M)$, the inclusion $V'_N \overset{i}{\hookrightarrow} V(T'M)$ takes place and as $V(T'M) = span\{\dot{\partial}_i\},\ i = 1, .., n+1$, there are the projection factors defined as $B^i_\alpha(\theta) = \frac{\partial \eta^i}{\partial \theta^\alpha}$ such that $\dot{\partial}_\alpha = B^i_\alpha \dot{\partial}_i$, where $\dot{\partial}_\alpha = \frac{\partial}{\partial \theta^\alpha}$. Note that $rank(B^i_\alpha) = n$.

We denote $\mu(Z) = G(Z, \bar{N})$, for any $Z \in \Gamma\left(V(T'M)\right)$, which is a vertical 1-form. We notice that $\mu\left(N\right) = 1$, and, considering that $V(T'M) = span\{\dot{\partial}_i\}$, we have $\mu(\dot{\partial}_i) = G\left(\dot{\partial}_i, \frac{1}{\mathrm{F}}\bar{\eta}^k \dot{\partial}_{\bar{k}}\right) = \frac{1}{\mathrm{F}}\eta_i$.

According to Vattamánny [11] or Bejancu [5], we can consider that any vertical vector field $Z = Z^i \dot{\partial}_i \in \Gamma\left(V(T'M)\right)$ admits the decomposition

$$Z = PZ + \mu(Z)N, \tag{9}$$

where the map $P : V(T'M) \to V'_N$, given by $P := Id - \mu \otimes N$ is the projector on the indicatrix bundle, i.e. $P^2 = P$. Thus, considering this projection with $PZ \in V'_N,\ \forall Z \in V(T'M)$ and that $V'_N = span\{\dot{\partial}_\alpha\}$, there are the factors P^α_i with $rank(P^\alpha_i) = n$, such that $P(\dot{\partial}_i) = P^\alpha_i \dot{\partial}_\alpha$ and hence $Z = Z^i \dot{\partial}_i \overset{P}{\longmapsto} P(Z^i \dot{\partial}_i) = Z^i P(\dot{\partial}_i) = Z^i P^\alpha_i \dot{\partial}_\alpha$. In the particular case of $Z = \dot{\partial}_i$, by applying the decomposition (9) we get

$$\dot{\partial}_i = P(\dot{\partial}_i) + \mu(\dot{\partial}_i)N \;\text{ i.e. } \dot{\partial}_i = P^\alpha_i \dot{\partial}_\alpha + \frac{1}{L}\eta_i \eta^j \dot{\partial}_j.$$

Using that $\dot{\partial}_\alpha = B^j_\alpha \dot{\partial}_j$, we obtain

$$P^\alpha_i B^j_\alpha = \delta^j_i - \frac{1}{L}\eta_i \eta^j. \tag{10}$$

In order to build the horizontal distribution we use the complex structure F, given in (4). Therefore, we denote $H'_\xi = F(V'_N)$ and $\{\xi\} = \{F(N)\}$. So $H'_\xi = span\{F(\dot{\partial}_\alpha) =: \tilde{\delta}_\alpha\}$, more precisely $\tilde{\delta}_\alpha = B^i_\alpha \delta_i$. We easily have $\{\xi\} \perp H'_\xi$, by $\frac{1}{\mathrm{F}} B^i_\alpha \eta_i = 0$ which holds because $\{N\} \perp V'_N$.

Therefore, we obtain

$$T'\left(T'M\right) = \{N\} \oplus V'_N \oplus \{\xi\} \oplus H'_\xi. \tag{11}$$

By conjugation, we get that $T''\left(T'M\right) = \{\bar{N}\} \oplus V''_N \oplus \{\bar{\xi}\} \oplus H''_\xi$, where $V''_N = span\{\dot{\partial}_{\bar{\alpha}} = B^{\bar{\imath}}_{\bar{\alpha}} \dot{\partial}_{\bar{\imath}}\}$ and $H''_\xi = span\{\tilde{\delta}_{\bar{\alpha}} = B^{\bar{\imath}}_{\bar{\alpha}} \delta_{\bar{\imath}}\}$.

Due to the fact that

$$T'\left(T'M\right) = \{N\} \oplus T'\left(IM\right), \tag{12}$$

from (11) we have:

$$T'(IM) = \{\xi\} \oplus H'_\xi \oplus V'_N. \tag{13}$$

and, by conjugation, $T''(IM) = \{\bar{\xi}\} \oplus H''_\xi \oplus V''_N$. Thereby,

$$T_C\left(IM\right) = \{\xi\}_C \oplus H_\xi \oplus V_N, \tag{14}$$

where $\{\xi\}_C = \{\xi\} \oplus \{\bar{\xi}\}$, $H_\xi = H'_\xi \oplus H''_\xi$, $V_N = V'_N \oplus V''_N$. So, we can state that:

$$T_C\left(IM\right) = span\left\{\xi,\ \bar{\xi},\ \tilde{\delta}_\alpha = B^i_\alpha \delta_i,\ \tilde{\delta}_{\bar{\alpha}} = B^{\bar{\imath}}_{\bar{\alpha}} \delta_{\bar{\imath}},\ \dot{\partial}_\alpha = B^i_\alpha \dot{\partial}_i,\ \dot{\partial}_{\bar{\alpha}} = B^{\bar{\imath}}_{\bar{\alpha}} \dot{\partial}_{\bar{\imath}}\right\}. \tag{15}$$

The Sasaki lift G on $T'M$ can be written in the new adapted frame as

$$G = g_{\alpha\bar{\beta}} d\tilde{z}^\alpha \otimes d\tilde{z}^{\bar{\beta}} + g_{\alpha\bar{\beta}} \delta\theta^\alpha \otimes \delta\theta^{\bar{\beta}} + \rho \otimes \bar{\rho} + \mu \otimes \bar{\mu},$$

where $\alpha, \beta \in \{1, 2, ..., n\}$, $g_{\alpha\bar{\beta}} = G(\tilde{\delta}_\alpha, \tilde{\delta}_{\bar{\beta}}) = G(\dot{\partial}_\alpha, \dot{\partial}_{\bar{\beta}}) = B^i_\alpha B^{\bar{j}}_{\bar{\beta}} g_{i\bar{j}}$, $B^{\bar{k}}_{\bar{\beta}} = \overline{B^k_\beta}$, ρ and μ represent the dual 1-forms of the unit horizontal and vertical Liouville vector fields, which can be computed by $\rho(X) = G(X, \bar{\xi})$ and $\mu(X) = G(X, \bar{N})$ and locally are given by

$$\rho := \frac{1}{\mathrm{F}} \bar{\eta}^k g_{i\bar{k}} dz^i \quad \text{and} \quad \mu := \frac{1}{\mathrm{F}} \bar{\eta}^k g_{i\bar{k}} \delta\eta^i.$$

Since $F(z, \eta) = 1$ on the indicatrix bundle IM, the Sasaki lift induced on indicatrix bundle may be considered:

$$\tilde{G} = g_{\alpha\bar{\beta}} d\tilde{z}^\alpha \otimes d\tilde{z}^{\bar{\beta}} + g_{\alpha\bar{\beta}} \delta\theta^\alpha \otimes \delta\theta^{\bar{\beta}} + \rho^* \otimes \bar{\rho}^*,$$

where $\rho^* = \bar{\eta}^k g_{i\bar{k}} dz^i$ is the restriction of the 1-form ρ to the indicatrix bundle, the dual of $\xi = \eta^i \delta_i$ on the indicatrix.

Let us consider the frame $\mathcal{R} = \left\{ \dot{\partial}_\alpha = B^k_\alpha \frac{\partial}{\partial \eta^k},\ N = \frac{1}{\mathrm{F}} \eta^k \frac{\partial}{\partial \eta^k} \right\}$ along $VT'M$ and then $\mathcal{R}^{-1} = \{P^\alpha_k,\ \frac{1}{\mathrm{F}}\eta_k\}^t$ are the inverse matrices of this frame, that is:

$$B^k_\beta P^\alpha_k = \delta^\alpha_\beta, \quad \frac{1}{\mathrm{F}} P^\alpha_k \eta^k = 0, \quad \frac{1}{\mathrm{F}} B^k_\alpha \eta_k = 0, \quad B^k_\alpha P^\alpha_j + \frac{1}{L}\eta^k \eta_j = \delta^k_j, \quad \frac{1}{L}\eta_k \eta^k = 1. \tag{16}$$

We can easily obtain that $g^{\bar{\beta}\alpha} = g^{\bar{j}i} P^\alpha_i P^{\bar{\beta}}_{\bar{j}}$ is the inverse of $g_{\alpha\bar{\beta}}$, where $P^{\bar{\beta}}_{\bar{j}} = \overline{P^\beta_j}$. Moreover, along the indicatrix bundle we have $g^{\bar{j}i} = B^i_\alpha B^{\bar{j}}_{\bar{\beta}} g^{\bar{\beta}\alpha} + \frac{1}{L}\eta^i \eta^{\bar{j}}$ and $g_{k\bar{h}} = \tilde{g}_{k\bar{h}} + \frac{1}{L}\eta_k \eta_{\bar{h}}$, where $\tilde{g}_{k\bar{h}} = P^\alpha_k P^{\bar{\beta}}_{\bar{h}} g_{\alpha\bar{\beta}}$.

Considering that restricted on the indicatrix bundle $\mathrm{F} = 1$ and that for the study of properties it does not affect whether the vertical or horizontal Liouville vector fields are unit or not, to simplify the calculus we further consider $N = \eta^i \dot{\partial}_i$ and $\xi = \eta^i \delta_i$. Taking into account the local components of the Lie brackets, in adapted frame fields of a (c.n.c.), from [8], pp. 43, we can compute forward the Lie brackets of the vector fields tangent to the indicatrix bundle, from which we can obtain the local expressions for torsion and curvature.

Thus, considering that according to [8] for the complex Chern-Finsler connection $R^i_{jk} := \delta_k N^i_j - \delta_j N^i_k = 0$, we obtain

$$\begin{aligned}
[\tilde{\delta}_\alpha, \tilde{\delta}_\beta] &= (\tilde{\delta}_\alpha B^i_\beta - \tilde{\delta}_\beta B^i_\alpha)\delta_i;\\
[\tilde{\delta}_\alpha, \tilde{\delta}_{\bar{\beta}}] &= (\tilde{\delta}_\alpha B^{\bar{\imath}}_{\bar{\beta}})\delta_{\bar{\imath}} - (\tilde{\delta}_{\bar{\beta}} B^i_\alpha)\delta_i + B^i_\alpha B^{\bar{j}}_{\bar{\beta}}[\delta_i, \delta_{\bar{j}}];\\
[\dot{\partial}_\alpha, \dot{\partial}_\beta] &= (\dot{\partial}_\alpha B^i_\beta - \dot{\partial}_\beta B^i_\alpha)\dot{\partial}_i; \qquad [\dot{\partial}_\alpha, \dot{\partial}_{\bar{\beta}}] = (\dot{\partial}_\alpha B^{\bar{\imath}}_{\bar{\beta}})\dot{\partial}_{\bar{\imath}} - (\dot{\partial}_{\bar{\beta}} B^i_\alpha)\dot{\partial}_i;\\
[\tilde{\delta}_\alpha, \dot{\partial}_\beta] &= \left(\tilde{\delta}_\alpha B^k_\beta + B^i_\alpha B^j_\beta (\dot{\partial}_j N^k_i)\right)\dot{\partial}_k - (\dot{\partial}_\beta B^i_\alpha)\delta_i;\\
[\tilde{\delta}_\alpha, \dot{\partial}_{\bar{\beta}}] &= (\tilde{\delta}_\alpha B^{\bar{k}}_{\bar{\beta}})\dot{\partial}_{\bar{k}} + B^i_\alpha B^{\bar{j}}_{\bar{\beta}}(\dot{\partial}_{\bar{j}} N^k_i)\dot{\partial}_k - (\dot{\partial}_{\bar{\beta}} B^i_\alpha)\delta_i;\\
[\tilde{\delta}_\alpha, \xi] &= -\left(B^i_\alpha N^j_i + \xi(B^j_\alpha)\right)\delta_j; \qquad [\xi, \xi] = 0;\\
[\tilde{\delta}_\alpha, \bar{\xi}] &= -\bar{\xi}(B^i_\alpha)\delta_i + B^i_\alpha \eta^{\bar{j}}[\delta_i, \delta_{\bar{j}}]; \qquad [\xi, \bar{\xi}] = \eta^i \bar{\eta}^j [\delta_i, \delta_{\bar{j}}].\\
[\dot{\partial}_\alpha, \xi] &= \tilde{\delta}_\alpha - \left(\xi(B^j_\alpha) + B^i_\alpha \eta^k (\dot{\partial}_i N^j_k)\right)\dot{\partial}_j;\\
[\dot{\partial}_\alpha, \bar{\xi}] &= -\bar{\xi}(B^i_\alpha)\dot{\partial}_i - B^i_\alpha \bar{\eta}^j (\dot{\partial}_i N^{\bar{k}}_{\bar{j}})\dot{\partial}_{\bar{k}},
\end{aligned}$$

where $[\delta_j, \delta_{\bar{k}}] = (\delta_{\bar{k}} N^i_j)\dot{\partial}_i - (\delta_j N^{\bar{\imath}}_{\bar{k}})\dot{\partial}_{\bar{\imath}}$. In addition, we consider

$$\begin{aligned}
&[\tilde{\delta}_\alpha, N] = -N(B^i_\alpha)\delta_i; && [\tilde{\delta}_\alpha, \bar{N}] = -\bar{N}(B^i_\alpha)\delta_i;\\
&[\dot{\partial}_\alpha, N] = \dot{\partial}_\alpha - N(B^i_\alpha)\dot{\partial}_i; && [\dot{\partial}_\alpha, \bar{N}] = -\bar{N}(B^i_\alpha)\dot{\partial}_i;\\
&[\xi, N] = -\xi; && [\xi, \bar{N}] = [N, N] = [N, \bar{N}] = 0.
\end{aligned}$$

3 The Gauss-Weingarten formulae of the indicatrix bundle

In this section the Gauss-Weingarten formulae of the indicatrix relative to the adapted frame introduced on the indicatrix bundle in the previous section, will be deduced first, followed in the next Section by the Gauss, H- and A-Codazzi, and Ricci equations.

Taking into account that the indicatrix IM can be regarded as a hypersurface of the holomorphic tangent bundle $T'M$ and considering the general framework of the geometry of subspaces [9], by restriction to the complexified vector fields and with respect to the Chern-Finsler complex linear connection of coefficients (6), for any $X, Y \in \Gamma(T_C IM)$ we have *the Gauss formula* of the immersed subspace IM:

$$D_X Y = \tilde{D}_X Y + h(X, Y), \qquad \forall X, Y \in \Gamma(T_{\mathrm{C}} IM), \tag{17}$$

where $\tilde{D}_X Y \in \Gamma(T_C IM)$ is the tangential component, also called *the induced tangent connection* of the indicatrix bundle, and $h(X, Y) \in \Gamma(T_C^{\perp} IM)$ is the normal part of the vector field $D_X Y$. The map $h : \Gamma(T_C IM) \times \Gamma(T_C IM) \to \Gamma(T_C^{\perp} IM)$ is $\mathcal{F}(I)$-bilinear and it represents *the second fundamental form* of the indicatrix subspace.

With respect to the adapted local basis frame of IM from (15)

$$\{\tilde{\delta}_i, \dot{\partial}_\alpha, \tilde{\delta}_{\bar{\imath}}, \dot{\partial}_{\bar{\alpha}}\}, \quad \text{where} \quad \tilde{\delta}_i = \begin{cases} \tilde{\delta}_\alpha, & \text{for } i \in \{1, .., n\} \\ \xi, & \text{for } i = n+1. \end{cases}, \tag{18}$$

and the normal frame given by $span\{N, \bar{N}\}$, the second fundamental form h is well-determined by the following set of coefficients:

$$\begin{array}{llll} h(\tilde{\delta}_j, \tilde{\delta}_i) = h_{ij} N, & h(\tilde{\delta}_{\bar{j}}, \tilde{\delta}_i) = h_{i\bar{j}} N, & h(\dot{\partial}_\beta, \dot{\partial}_\alpha) = h_{\alpha\beta} N, & h(\dot{\partial}_{\bar{\beta}}, \dot{\partial}_\alpha) = h_{\alpha\bar{\beta}} N, \\ h(\tilde{\delta}_j, \dot{\partial}_\alpha) = h_{\alpha j} N, & h(\tilde{\delta}_{\bar{j}}, \dot{\partial}_\alpha) = h_{\alpha\bar{j}} N, & h(\dot{\partial}_\beta, \tilde{\delta}_i) = h_{i\beta} N, & h(\dot{\partial}_{\bar{\beta}}, \tilde{\delta}_i) = h_{i\bar{\beta}} N, \end{array} \tag{19}$$

such that $\overline{h_{ij}} = h_{\bar{\imath}\bar{j}}$, $\overline{h_{i\bar{j}}} = h_{\bar{\imath}j}$, *etc.* Moreover, it takes place: $G(D_X Y, \bar{N}) = G(h(X, Y), \bar{N})$ and using

$$\begin{array}{ll} (\dot{\partial}_k B^j_\alpha)\eta_j + B^j_\alpha(\dot{\partial}_k g_{j\bar{m}})\eta^{\bar{m}} = 0, & (\dot{\partial}_{\bar{k}} B^j_\alpha)\eta_j = -B^j_\alpha g_{j\bar{k}}, \\ (\delta_k B^j_\alpha)\eta_j + B^j_\alpha(\delta_k g_{j\bar{m}})\eta^{\bar{m}} = 0, & (\delta_{\bar{k}} B^j_\alpha)\eta_j = 0, \end{array} \tag{20}$$

and (15), we compute the coefficients of the second fundamental form and obtain

Proposition 1. *The coefficients of the second fundamental form are*

$$h_{ij} = h_{i\bar{j}} = h_{\alpha\beta} = h_{\alpha j} = h_{\alpha\bar{j}} = h_{i\beta} = h_{i\bar{\beta}} = 0,$$
$$h_{\alpha\bar{\beta}} = -\frac{1}{L} g_{\alpha\bar{\beta}}.$$

Then, from these relations and the Gauss formula we have

$$\begin{array}{ll} \tilde{D}_{\tilde{\delta}_j} \tilde{\delta}_i = D_{\tilde{\delta}_j} \tilde{\delta}_i; & \tilde{D}_{\tilde{\delta}_{\bar{j}}} \tilde{\delta}_i = D_{\tilde{\delta}_{\bar{j}}} \tilde{\delta}_i \\ \tilde{D}_{\dot{\partial}_\beta} \dot{\partial}_\alpha = D_{\dot{\partial}_\beta} \dot{\partial}_\alpha; & \tilde{D}_{\dot{\partial}_{\bar{\beta}}} \dot{\partial}_\alpha = D_{\dot{\partial}_{\bar{\beta}}} \dot{\partial}_\alpha + \frac{1}{L} g_{\alpha\bar{\beta}}; \\ \tilde{D}_{\tilde{\delta}_j} \dot{\partial}_\alpha = D_{\tilde{\delta}_j} \dot{\partial}_\alpha; & \tilde{D}_{\tilde{\delta}_{\bar{j}}} \dot{\partial}_\alpha = D_{\tilde{\delta}_{\bar{j}}} \dot{\partial}_\alpha; \\ \tilde{D}_{\dot{\partial}_\beta} \tilde{\delta}_i = D_{\dot{\partial}_\beta} \tilde{\delta}_i; & \tilde{D}_{\dot{\partial}_{\bar{\beta}}} \tilde{\delta}_i = D_{\dot{\partial}_{\bar{\beta}}} \tilde{\delta}_i. \end{array}$$

Considering this, we observe that the induced tangent connection $\tilde{D} : \Gamma(T_C IM) \to \Gamma(T_C IM \otimes T_C IM^*)$ preserves the distributions given in (15), thus it is a d-(c.l.c) and in the adapted local basis (18), $\tilde{D}$ is well defined by the next set of coefficients:

$$\begin{array}{llll}
\tilde{D}_{\tilde{\delta}_k}\tilde{\delta}_j = \overset{1}{\tilde{L}}{}^i_{jk}\tilde{\delta}_i, & \tilde{D}_{\dot{\partial}_\gamma}\tilde{\delta}_j = \overset{1}{\tilde{C}}{}^i_{j\gamma}\tilde{\delta}_i, & \tilde{D}_{\tilde{\delta}_{\bar{k}}}\tilde{\delta}_j = \overset{3}{\tilde{L}}{}^i_{j\bar{k}}\tilde{\delta}_i, & \tilde{D}_{\dot{\partial}_{\bar{\gamma}}}\tilde{\delta}_j = \overset{3}{\tilde{C}}{}^i_{j\bar{\gamma}}\tilde{\delta}_i, \\
\tilde{D}_{\tilde{\delta}_k}\dot{\partial}_\beta = \overset{2}{\tilde{L}}{}^\alpha_{\beta k}\dot{\partial}_\alpha, & \tilde{D}_{\dot{\partial}_\gamma}\dot{\partial}_\beta = \overset{2}{\tilde{C}}{}^\alpha_{\beta\gamma}\dot{\partial}_\alpha, & \tilde{D}_{\tilde{\delta}_{\bar{k}}}\dot{\partial}_\beta = \overset{4}{\tilde{L}}{}^\alpha_{\beta\bar{k}}\dot{\partial}_\alpha, & \tilde{D}_{\dot{\partial}_{\bar{\gamma}}}\dot{\partial}_\beta = \overset{4}{\tilde{C}}{}^\alpha_{\beta\bar{\gamma}}\dot{\partial}_\alpha, \\
\tilde{D}_{\tilde{\delta}_k}\tilde{\delta}_{\bar{j}} = \overset{3}{\tilde{L}}{}^{\bar{\imath}}_{\bar{j}k}\tilde{\delta}_{\bar{\imath}}, & \tilde{D}_{\dot{\partial}_\gamma}\tilde{\delta}_{\bar{j}} = \overset{3}{\tilde{C}}{}^{\bar{\imath}}_{\bar{j}\gamma}\tilde{\delta}_{\bar{\imath}}, & \tilde{D}_{\tilde{\delta}_{\bar{k}}}\tilde{\delta}_{\bar{j}} = \overset{1}{\tilde{L}}{}^{\bar{\imath}}_{\bar{j}\bar{k}}\tilde{\delta}_{\bar{\imath}}, & \tilde{D}_{\dot{\partial}_{\bar{\gamma}}}\tilde{\delta}_{\bar{j}} = \overset{1}{\tilde{C}}{}^{\bar{\imath}}_{\bar{j}\bar{\gamma}}\tilde{\delta}_{\bar{\imath}}, \\
\tilde{D}_{\tilde{\delta}_k}\dot{\partial}_{\bar{\beta}} = \overset{4}{\tilde{L}}{}^{\bar{\alpha}}_{\bar{\beta}k}\dot{\partial}_{\bar{\alpha}}, & \tilde{D}_{\dot{\partial}_\gamma}\dot{\partial}_{\bar{\beta}} = \overset{4}{\tilde{C}}{}^{\bar{\alpha}}_{\bar{\beta}\gamma}\dot{\partial}_{\bar{\alpha}}, & \tilde{D}_{\tilde{\delta}_{\bar{k}}}\dot{\partial}_{\bar{\beta}} = \overset{2}{\tilde{L}}{}^{\bar{\alpha}}_{\bar{\beta}\bar{k}}\dot{\partial}_{\bar{\alpha}}, & \tilde{D}_{\dot{\partial}_{\bar{\gamma}}}\dot{\partial}_{\bar{\beta}} = \overset{2}{\tilde{C}}{}^{\bar{\alpha}}_{\bar{\beta}\bar{\gamma}}\dot{\partial}_{\bar{\alpha}}.
\end{array}$$

By applying the Gauss formula, we can compute the above coefficients, and we obtain:

$$\begin{array}{ll}
\overset{1}{\tilde{L}}{}^\alpha_{\beta\gamma} = \overset{2}{\tilde{L}}{}^\alpha_{\beta\gamma} = P^\alpha_i B^k_\gamma(\delta_k B^i_\beta + B^j_\beta L^i_{jk}); & \overset{1}{\tilde{L}}{}^{n+1}_{\beta\gamma} = \overset{1}{\tilde{L}}{}^\alpha_{n+1\gamma} = \overset{1}{\tilde{L}}{}^{n+1}_{n+1\gamma} = 0; \\
\overset{1}{\tilde{L}}{}^\alpha_{\beta\ n+1} = \overset{2}{\tilde{L}}{}^\alpha_{\beta\ n+1} = P^\alpha_i \eta^k(\delta_k B^i_\beta + B^j_\beta L^i_{jk}); & \overset{1}{\tilde{L}}{}^{n+1}_{\beta\ n+1} = \overset{1}{\tilde{L}}{}^\alpha_{n+1\ n+1} = \overset{1}{\tilde{L}}{}^{n+1}_{n+1\ n+1} = 0; \\
\overset{3}{\tilde{L}}{}^\alpha_{\beta\bar{\gamma}} = P^\alpha_i B^{\bar{k}}_{\bar{\gamma}}(\delta_{\bar{k}} B^i_\beta) & \overset{3}{\tilde{L}}{}^{n+1}_{\beta\bar{\gamma}} = -\frac{1}{L}g_{\beta\bar{\gamma}}; \quad \overset{3}{\tilde{L}}{}^\alpha_{n+1\bar{\gamma}} = \overset{3}{\tilde{L}}{}^{n+1}_{n+1\bar{\gamma}} = 0; \\
\overset{3}{\tilde{L}}{}^\alpha_{\beta\ \overline{n+1}} = P^\alpha_i \eta^{\bar{k}}(\delta_{\bar{k}} B^i_\beta); & \overset{3}{\tilde{L}}{}^{n+1}_{\beta\ \overline{n+1}} = \overset{3}{\tilde{L}}{}^\alpha_{n+1\ \overline{n+1}} = \overset{3}{\tilde{L}}{}^{n+1}_{n+1\ \overline{n+1}} = 0; \\
\overset{4}{\tilde{L}}{}^\alpha_{\beta\bar{\gamma}} = P^\alpha_i B^{\bar{k}}_{\bar{\gamma}}(\delta_{\bar{k}} B^i_\beta); & \overset{4}{\tilde{L}}{}^\alpha_{\beta\ \overline{n+1}} = P^\alpha_i \eta^{\bar{k}}(\delta_{\bar{k}} B^i_\beta); \\
\overset{1}{\tilde{C}}{}^\alpha_{\beta\gamma} = \overset{2}{\tilde{C}}{}^\alpha_{\beta\gamma} = P^\alpha_i B^k_\gamma(\dot{\partial}_k B^i_\beta + B^j_\beta C^i_{jk}); & \overset{1}{\tilde{C}}{}^{n+1}_{\beta\gamma} = \overset{1}{\tilde{C}}{}^{n+1}_{n+1\gamma} = 0; \qquad \overset{1}{\tilde{C}}{}^\alpha_{n+1\gamma} = \delta^\alpha_\gamma; \\
\overset{3}{\tilde{C}}{}^\alpha_{\beta\bar{\gamma}} = \overset{4}{\tilde{C}}{}^\alpha_{\beta\bar{\gamma}} = P^\alpha_i B^{\bar{k}}_{\bar{\gamma}}(\dot{\partial}_{\bar{k}} B^i_\beta); & \overset{3}{\tilde{C}}{}^{n+1}_{\beta\bar{\gamma}} = -\frac{1}{L}g_{\beta\bar{\gamma}}; \quad \overset{3}{\tilde{C}}{}^\alpha_{n+1\bar{\gamma}} = \overset{3}{\tilde{C}}{}^{n+1}_{n+1\bar{\gamma}} = 0;
\end{array} \tag{21}$$

In order to obtain the coefficients of the induced (c.n.c.), we consider the adapted local frame $\{\tilde{\delta}_i, \dot{\partial}_\alpha, \tilde{\delta}_{\bar{\imath}}, \dot{\partial}_{\bar{\alpha}}\}$, as in (18). Then, its dual frame is $\{\mathrm{d}\tilde{z}^i,\ \delta\theta^\alpha = \mathrm{d}\theta^\alpha + \tilde{N}^\alpha_i \mathrm{d}\tilde{z}^i\}$, with $\mathrm{d}\tilde{z}^i = \left\{\begin{array}{ll} \mathrm{d}\tilde{z}^\alpha = P^\alpha_j dz^j, & \text{for } i \in \{1,..,n\} \\ \rho = \bar{\eta}^k g_{i\bar{k}} dz^i, & \text{for } i = n+1. \end{array}\right.$ and $d\theta^\alpha = P^\alpha_j d\eta^j$. $\tilde{N}^\alpha_i$ are called the *coefficients of the induced (c.n.c.)* iff $\delta\theta^\alpha = P^\alpha_k \delta\eta^k$, namely $\mathrm{d}\theta^\alpha + \tilde{N}^\alpha_i \mathrm{d}\tilde{z}^i = P^\alpha_k(\mathrm{d}\eta^k + N^k_i \mathrm{d}z^i)$ and using (16) we get

$$\tilde{N}^\alpha_i = \left\{\begin{array}{ll} \tilde{N}^\alpha_\beta = \frac{1}{L}\eta^j P^\alpha_k N^k_j, & \text{for } i \in \{1,..,n\} \\ \tilde{N}^\alpha_{n+1} = P^\alpha_k B^j_\beta N^k_j, & \text{for } i = n+1. \end{array}\right. \tag{22}$$

Next, let us consider *the Weingarten formula* of the immersed subspace IM:

$$D_Z W = -A_W Z + D^\perp_Z W,\ \forall Z \in T_C\,(IM)\,,\ \forall W \in span\{N\}_C, \tag{23}$$

where $A_W Z \in \Gamma(T_C IM)$ is the tangential component and $D^\perp_X Z \in \Gamma(T^\perp_C IM)$ is the normal part, with $D^\perp$ the induced normal connection from the Chern-Finsler complex linear connection D. The map $A : \Gamma(T^\perp_C IM) \times \Gamma(T_C IM) \to \Gamma(T_C IM)$ is $\mathcal{F}(IM)$-bilinear, $A_W X = A(W, X)$ and A_W is called *the shape operator* (or Weingarten operator). It can be noticed that the space $T^\perp_C IM$ is spanned by N, $\bar{N}$, i.e. has only vertical component and then we can conclude $D^{\perp}_X W \in \Gamma(V^{\perp}_C IM)$ and $A : \Gamma(V^{\perp}_C IM) \times \Gamma(T_C IM) \to \Gamma(V_C IM)$. Thus, as above, we express the action of the shape operator $A_N(X) := A(X) \in VIM$ on $\tilde{\delta}_k$ and $\dot{\partial}_\alpha$ as follows:

$$\begin{aligned} A_N(\tilde{\delta}_k) &= A^{\alpha}_k \dot{\partial}_\alpha; \qquad A_N(\dot{\partial}_\beta) = A^{\alpha}_\beta \dot{\partial}_\alpha; \\ A_N(\tilde{\delta}_{\bar{k}}) &= A^{\alpha}_{\bar{k}} \dot{\partial}_\alpha; \qquad A_N(\dot{\partial}_{\bar{\beta}}) = A^{\alpha}_{\bar{\beta}} \dot{\partial}_\alpha, \end{aligned}$$

such that $\overline{A^{\alpha}_k} = A^{\bar{\alpha}}_{\bar{k}}$, $\overline{A^{\alpha}_{\bar{k}}} = A^{\bar{\alpha}}_k$, *etc.* By direct computation, using $G(D_X N, \dot{\partial}_{\bar{\beta}}) = -G(A(X), \dot{\partial}_{\bar{\beta}})$, we obtain

Proposition 2. *The coefficients of the shape operator are:*

$$A^{\alpha}_k = A^{\alpha}_{\bar{k}} = A^{\alpha}_{\bar{\beta}} = 0 \quad \textit{and} \quad A^{\alpha}_\beta = -\delta^{\alpha}_\beta. \tag{24}$$

Moreover, it can be noticed that

$$D^{\perp}_Z W = 0, \quad \forall Z \in T_C(IM), \ \forall W \in span\{N\}_C, \tag{25}$$

and thus, the Weingarten formula becomes $D_Z W = -A_W Z$, $\forall Z \in T_C(IM)$, $\forall W \in span\{N\}_C$.

Considering that the Chern-Finsler connection D is metrical with respect to the Sasaki lift G (2), i.e. $(D_X G)(Y, \bar{N}) = 0$, $\forall X, Y \in \Gamma(T'IM)$, and by applying the Gauss and Weingarten formulae for the immersed subspace IM in $(\widetilde{T'M}, G)$, between Weingarten operator and the second fundamental tensor the following relation exists:

$$G\left(A_N X, \bar{Y}\right) = G(N, h(X, \bar{Y})) \quad \text{and} \quad G(Y, A_{\bar{N}} X) = G(h(X, Y), \bar{N}),$$

and their conjugates, for all $X, Y \in \Gamma(T'IM)$. Thereby, their nonzero components satisfy

$$h_{\alpha\bar{\beta}} = \frac{1}{L} A^{\bar{\gamma}}_{\bar{\beta}} g_{\alpha\bar{\gamma}}, \text{ that is equivalent to } A^{\alpha}_\beta = L h_{\bar{\gamma}\beta} g^{\bar{\gamma}\alpha}.$$

4 Gauss, Codazzi and Ricci equations

In order to introduce Gauss, Codazzi and Ricci equations on the indicatrix hypersurface we consider $\tilde{D}$ and $D^{\perp}$ the induced tangent and normal connection on IM of the Chern-Finsler (c.l.c), as above. To get a link between curvatures $R(X,Y)Z$ of D connection and $\tilde{R}(X,Y)Z$ of $\tilde{D}$ connection, for $X, Y, Z \in \Gamma(T_C IM)$ we act similar steps as in [9, 10]. First, the covariant derivative of the second fundamental form is being defined as $(D_X h)(Y, Z) = D^{\perp}_X(h(Y,Z)) - h(\tilde{D}_X Y, Z) - h(Y, \tilde{D}_X Z)$ and using (25), we get :

$$(D_X h)(Y, Z) = -h(\tilde{D}_X Y, Z) - h(Y, \tilde{D}_X Z).$$

Using curvature and torsion definitions $R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}$ and $T(X,Y) = D_X Y - D_Y X - [X,Y]$, respectively, for $X, Y, Z \in \Gamma(T_C IM)$ and applying Gauss-Weingarten formulae (17) and (23), we get:

$$\begin{aligned} R(X,Y)Z &= \tilde{R}(X,Y)Z + A(h(X,Z),Y) - A(h(Y,Z),X) + (D_X h)(Y,Z) - \\ &- (D_Y h)(X,Z) + h(\tilde{T}(X,Y),Z) \end{aligned}$$

Equating the components from $T_C IM$ and $T_C^{\perp} IM$ with the help of the metric structures G and $\tilde{G}$ introduced in previous sections, it is obtained

$$G\left(R(X,Y)Z,U\right) = \tilde{G}(\tilde{R}(X,Y)Z,U) + \tilde{G}\left(A_{h(X,Z)}Y - A_{h(Y,Z)}X,U\right) \tag{26}$$

where $X,Y \in \Gamma\left(T_C IM\right)$, $Z \in \Gamma(T^{'} IM)$, $U \in \Gamma(T^{''} IM)$, and respectively, using that $T_C^{\perp} IM = span\{N,\ \bar{N}\}$ and taking now $Z \in \Gamma\left(V' IM\right)$

$$G\left(R(X,Y)Z,\bar{N}\right) = G\left(\left(D_X h\right)(Y,Z) - \left(D_Y h\right)(X,Z),\bar{N}\right) + G(h(\tilde{T}(X,Y),Z),\bar{N}) \tag{27}$$

called *the Gauss equations*, respectively *H-Codazzi* equations of the indicatrix bundle.

Analogously, for normal curvatures $R\left(X,Y\right)N$ and $\tilde{R}\left(X,Y\right)N$, defining the covariant derivative of the shape operator as $\left(D_X A\right)(N,Y) = \tilde{D}_X\left(A_N Y\right) - A(D_X^{\perp} N,Y) - A(N,\tilde{D}_X Y)$ and considering the curvature form $R^{\perp}$ of the normal Finsler connection, we apply (25) and we obtain for each $X,Y \in \Gamma\left(T_C IM\right)$

$$\left(D_X A\right)(N,Y) = \tilde{D}_X\left(A_N Y\right) - A(N,\tilde{D}_X Y) \text{ and } R^{\perp}(X,Y)N = 0.$$

Thus, using the Gauss-Weingarten equations it is obtained that:

$$\begin{aligned} R(X,Y)N = \ & h\left(Y,A_N X\right) - h\left(X,A_N Y\right) + \\ & + \left(D_Y A\right)(N,X) - \left(D_X A\right)(N,Y) - A_N(\tilde{T}\left(X,Y\right)). \end{aligned}$$

Equating their components from $T_C IM$ and $T_C^{\perp} IM$, we have

$$G\left(R(X,Y)N,Z\right) = \tilde{G}\left(\left(D_Y A\right)(N,X) - \left(D_X A\right)(N,Y),Z\right) - \tilde{G}(A_N(\tilde{T}\left(X,Y\right)),Z), \tag{28}$$

where $X,Y \in \Gamma\left(T_C IM\right),\ Z \in \Gamma\left(V'' IM\right)$, and,

$$G\left(R(X,Y)N,\bar{N}\right) = G\left(h\left(Y,A_N X\right) - h\left(X,A_N Y\right),\bar{N}\right) \tag{29}$$

called the *A-Codazzi equations*, respectively *Ricci equations*.

Therefore, we suggest obtaining propose to obtain local expressions of these equations in the adapted frames orthogonal to *span*$\{N,\bar{N}\}$, given by (18) with respect to the Chern-Finsler (c.l.c.). First, for a simplified writing of the equations, we recall, that according to [8], the Chern-Finsler connection has the following nonzero curvature coefficients

$$\begin{aligned} R^i_{j\bar{k}h} &= -\delta_{\bar{k}} L^i_{jh} - \delta_{\bar{k}}(N^l_h) C^i_{j\gamma}, \quad P^i_{j\bar{k}h} = -\delta_{\bar{k}} C^i_{jh}, \\ Q^i_{j\bar{k}h} &= -\dot{\partial}_k L^i_{jh} - \dot{\partial}_{\bar{k}}(N^l_h) C^i_{jl}, \qquad S^i_{j\bar{k}h} = -\dot{\partial}_{\bar{k}} C^i_{jh}. \end{aligned}$$

With this setting and considering the notations of the curvature and torsion coefficients introduced in [8], pp. 44, by direct calculation with respect to the vector fields of the adapted frame (18) of IM and taking into account the second fundamental form and the shape operator coefficients from (19) and (24), respectively, the local expressions for the equations (26)-(29) are:

Theorem 1. *With respect to the induced tangent Chern-Finsler connection $\tilde{D}$ defined in (17) on the indicatrix bundle IM by the Chern-Finsler connection (6) on (M, L), we have the local expressions for:*

i. *the Gauss equations*

$$
\begin{array}{ll}
P_i^{\alpha} B_{\beta}^{j} B_{\bar{\gamma}}^{\bar{k}} B_{\sigma}^{h} R_{j\bar{k}h}^{i} = \tilde{R}_{\beta\bar{\gamma}\sigma}^{\alpha}; & \eta_i B_{\beta}^{j} B_{\bar{\gamma}}^{\bar{k}} B_{\sigma}^{h} R_{j\bar{k}h}^{i} = \tilde{R}_{\beta\bar{\gamma}\sigma}^{n+1}; \\
P_i^{\alpha} \eta^{j} B_{\bar{\gamma}}^{\bar{k}} B_{\sigma}^{h} R_{j\bar{k}h}^{i} = \tilde{R}_{n+1\bar{\gamma}\sigma}^{\alpha}; & \eta_i \eta^{j} B_{\bar{\gamma}}^{\bar{k}} B_{\sigma}^{h} R_{j\bar{k}h}^{i} = \tilde{R}_{n+1\bar{\gamma}\sigma}^{n+1}; \\
P_i^{\alpha} B_{\beta}^{j} \eta^{\bar{k}} B_{\sigma}^{h} R_{j\bar{k}h}^{i} = \tilde{R}_{\beta\overline{n+1}\sigma}^{\alpha}; & \eta_i B_{\beta}^{j} \eta^{\bar{k}} B_{\sigma}^{h} R_{j\bar{k}h}^{i} = \tilde{R}_{\beta\overline{n+1}\sigma}^{n+1}; \\
P_i^{\alpha} B_{\beta}^{j} B_{\bar{\gamma}}^{\bar{k}} \eta^{h} R_{j\bar{k}h}^{i} = \tilde{R}_{\beta\bar{\gamma}n+1}^{\alpha}; & \eta_i B_{\beta}^{j} B_{\bar{\gamma}}^{\bar{k}} \eta^{h} R_{j\bar{k}h}^{i} = \tilde{R}_{\beta\bar{\gamma}n+1}^{n+1}; \\
P_i^{\alpha} \eta^{j} \eta^{\bar{k}} B_{\sigma}^{h} R_{j\bar{k}h}^{i} = \tilde{R}_{n+1\overline{n+1}\sigma}^{\alpha}; & \eta_i \eta^{j} \eta^{\bar{k}} B_{\sigma}^{h} R_{j\bar{k}h}^{i} = \tilde{R}_{n+1\overline{n+1}\sigma}^{n+1}; \\
P_i^{\alpha} B_{\beta}^{j} \eta^{\bar{k}} \eta^{h} R_{j\bar{k}h}^{i} = \tilde{R}_{\beta\overline{n+1}n+1}^{\alpha}; & \eta_i B_{\beta}^{j} \eta^{\bar{k}} \eta^{h} R_{j\bar{k}h}^{i} = \tilde{R}_{\beta\overline{n+1}n+1}^{n+1}; \\
P_i^{\alpha} \eta^{j} B_{\bar{\gamma}}^{\bar{k}} \eta^{h} R_{j\bar{k}h}^{i} = \tilde{R}_{n+1\bar{\gamma}n+1}^{\alpha}; & \eta_i \eta^{j} B_{\bar{\gamma}}^{\bar{k}} \eta^{h} R_{j\bar{k}h}^{i} = \tilde{R}_{n+1\bar{\gamma}n+1}^{n+1}; \\
P_i^{\alpha} \eta^{j} \eta^{\bar{k}} \eta^{h} R_{j\bar{k}h}^{i} = \tilde{R}_{n+1\overline{n+1}n+1}^{\alpha}; & \eta_i \eta^{j} \eta^{\bar{k}} \eta^{h} R_{j\bar{k}h}^{i} = \tilde{R}_{n+1\overline{n+1}n+1}^{n+1};
\end{array}
$$

$$
\begin{array}{ll}
P_i^{\alpha} B_{\beta}^{j} B_{\bar{\gamma}}^{\bar{k}} B_{\sigma}^{h} P_{j\bar{k}h}^{i} = \tilde{P}_{\beta\bar{\gamma}\sigma}^{\alpha}; & \eta_i B_{\beta}^{j} B_{\bar{\gamma}}^{\bar{k}} B_{\sigma}^{h} P_{j\bar{k}h}^{i} = \tilde{P}_{\beta\bar{\gamma}\sigma}^{n+1}; \\
P_i^{\alpha} \eta^{j} B_{\bar{\gamma}}^{\bar{k}} B_{\sigma}^{h} P_{j\bar{k}h}^{i} = \tilde{P}_{n+1\bar{\gamma}\sigma}^{\alpha}; & \eta_i \eta^{j} B_{\bar{\gamma}}^{\bar{k}} B_{\sigma}^{h} P_{j\bar{k}h}^{i} = \tilde{P}_{n+1\bar{\gamma}\sigma}^{n+1}; \\
P_i^{\alpha} B_{\beta}^{j} \eta^{\bar{k}} B_{\sigma}^{h} P_{j\bar{k}h}^{i} = \tilde{P}_{\beta\overline{n+1}\sigma}^{\alpha}; & \eta_i B_{\beta}^{j} \eta^{\bar{k}} B_{\sigma}^{h} P_{j\bar{k}h}^{i} = \tilde{P}_{\beta\overline{n+1}\sigma}^{n+1}; \\
P_i^{\alpha} \eta^{j} \eta^{\bar{k}} B_{\sigma}^{h} P_{j\bar{k}h}^{i} = \tilde{P}_{n+1\overline{n+1}\sigma}^{\alpha}; & \eta_i \eta^{j} \eta^{\bar{k}} B_{\sigma}^{h} P_{j\bar{k}h}^{i} = \tilde{P}_{n+1\overline{n+1}\sigma}^{n+1};
\end{array}
$$

$$
\begin{array}{ll}
P_i^{\alpha} B_{\beta}^{j} B_{\bar{\gamma}}^{\bar{k}} B_{\sigma}^{h} Q_{j\bar{k}h}^{i} = \tilde{Q}_{\beta\bar{\gamma}\sigma}^{\alpha}; & \eta_i B_{\beta}^{j} B_{\bar{\gamma}}^{\bar{k}} B_{\sigma}^{h} Q_{j\bar{k}h}^{i} = \tilde{Q}_{\beta\bar{\gamma}\sigma}^{n+1}; \\
P_i^{\alpha} \eta^{j} B_{\bar{\gamma}}^{\bar{k}} B_{\sigma}^{h} Q_{j\bar{k}h}^{i} = \tilde{Q}_{n+1\bar{\gamma}\sigma}^{\alpha}; & \eta_i \eta^{j} B_{\bar{\gamma}}^{\bar{k}} B_{\sigma}^{h} Q_{j\bar{k}h}^{i} = \tilde{Q}_{n+1\bar{\gamma}\sigma}^{n+1}; \\
P_i^{\alpha} B_{\beta}^{j} B_{\bar{\gamma}}^{\bar{k}} \eta^{h} Q_{j\bar{k}h}^{i} = \tilde{Q}_{\beta\bar{\gamma}n+1}^{\alpha}; & \eta_i B_{\beta}^{j} B_{\bar{\gamma}}^{\bar{k}} \eta^{h} Q_{j\bar{k}h}^{i} = \tilde{Q}_{\beta\bar{\gamma}n+1}^{n+1}; \\
P_i^{\alpha} \eta^{j} B_{\bar{\gamma}}^{\bar{k}} \eta^{h} Q_{j\bar{k}h}^{i} = \tilde{Q}_{n+1\bar{\gamma}n+1}^{\alpha}; & \eta_i \eta^{j} B_{\bar{\gamma}}^{\bar{k}} \eta^{h} Q_{j\bar{k}h}^{i} = \tilde{Q}_{n+1\bar{\gamma}n+1}^{n+1};
\end{array}
$$

$$
\begin{array}{ll}
P_i^{\alpha} B_{\beta}^{j} B_{\bar{\gamma}}^{\bar{k}} B_{\sigma}^{h} S_{j\bar{k}h}^{i} = \tilde{S}_{\beta\bar{\gamma}\sigma}^{\alpha}; & \eta_i B_{\beta}^{j} B_{\bar{\gamma}}^{\bar{k}} B_{\sigma}^{h} S_{j\bar{k}h}^{i} = \tilde{S}_{\beta\bar{\gamma}\sigma}^{n+1}; \\
P_i^{\alpha} \eta^{j} B_{\bar{\gamma}}^{\bar{k}} B_{\sigma}^{h} S_{j\bar{k}h}^{i} = \tilde{S}_{n+1\bar{\gamma}\sigma}^{\alpha}; & \eta_i \eta^{j} B_{\bar{\gamma}}^{\bar{k}} B_{\sigma}^{h} S_{j\bar{k}h}^{i} = \tilde{S}_{n+1\bar{\gamma}\sigma}^{n+1}; \\
B_{\beta}^{j} B_{\bar{\gamma}}^{\bar{k}} B_{\sigma}^{h} S_{j\bar{k}h}^{i} = \mathbb{S}_{\beta\bar{\gamma}\sigma}^{\alpha} B_{\alpha}^{i} - \frac{1}{L} g_{\beta\bar{\gamma}} B_{\sigma}^{i}, & \textit{where } R(\dot{\partial}_{\sigma}, \dot{\partial}_{\bar{\gamma}})\dot{\partial}_{\beta} = \mathbb{S}_{\beta\bar{\gamma}\sigma}^{\alpha} \dot{\partial}_{\alpha};
\end{array}
$$

ii. *the H-Codazzi equations*

$$
\begin{array}{ll}
\eta_i B_{\beta}^{j} B_{\bar{\gamma}}^{\bar{k}} B_{\sigma}^{h} R_{j\bar{k}h}^{i} = -\tilde{\Theta}_{\bar{\gamma}\sigma}^{\bar{\mu}} g_{\beta\bar{\mu}}; & \eta_i B_{\beta}^{j} \eta^{\bar{k}} B_{\sigma}^{h} R_{j\bar{k}h}^{i} = -\tilde{\Theta}_{\overline{n+1}\sigma}^{\bar{\mu}} g_{\beta\bar{\mu}}; \\
\eta_i B_{\beta}^{j} B_{\bar{\gamma}}^{\bar{k}} \eta^{h} R_{j\bar{k}h}^{i} = -\tilde{\Theta}_{\bar{\gamma}n+1}^{\bar{\mu}} g_{\beta\bar{\mu}}; & \eta_i B_{\beta}^{j} \eta^{\bar{k}} \eta^{h} R_{j\bar{k}h}^{i} = -\tilde{\Theta}_{\overline{n+1}n+1}^{\bar{\mu}} g_{\beta\bar{\mu}}; \\
\eta_i B_{\beta}^{j} B_{\bar{\gamma}}^{\bar{k}} B_{\sigma}^{h} P_{j\bar{k}h}^{i} = -\tilde{\rho}_{\bar{\gamma}\sigma}^{\bar{\mu}} g_{\beta\bar{\mu}}; & \eta_i B_{\beta}^{j} \eta^{\bar{k}} B_{\sigma}^{h} P_{j\bar{k}h}^{i} = -\tilde{\rho}_{\overline{n+1}\sigma}^{\bar{\mu}} g_{\beta\bar{\mu}};
\end{array}
$$

$$
\begin{array}{rcl}
\eta_i B_{\beta}^{j} B_{\bar{\gamma}}^{\bar{k}} B_{\sigma}^{h} Q_{j\bar{k}h}^{i} & = & \overset{4}{\tilde{L}}{}_{\bar{\gamma}\sigma}^{\bar{\mu}} g_{\beta\bar{\mu}} + \overset{2}{\tilde{L}}{}_{\beta\sigma}^{\mu} g_{\mu\bar{\gamma}} - \tilde{\Sigma}_{\bar{\gamma}\sigma}^{\bar{\mu}} g_{\beta\bar{\mu}}; \\
\eta_i B_{\beta}^{j} B_{\bar{\gamma}}^{\bar{k}} \eta^{h} Q_{j\bar{k}h}^{i} & = & \overset{4}{\tilde{L}}{}_{\bar{\gamma}n+1}^{\bar{\mu}} g_{\beta\bar{\mu}} + \overset{2}{\tilde{L}}{}_{\beta n+1}^{\mu} g_{\mu\bar{\gamma}} - \tilde{\Sigma}_{\bar{\gamma}n+1}^{\bar{\mu}} g_{\beta\bar{\mu}}; \\
\eta_i B_{\beta}^{j} B_{\bar{\gamma}}^{\bar{k}} B_{\sigma}^{h} S_{j\bar{k}h}^{i} & = & \overset{4}{\tilde{C}}{}_{\bar{\gamma}\sigma}^{\bar{\mu}} g_{\beta\bar{\mu}} + \overset{2}{\tilde{C}}{}_{\beta\sigma}^{\mu} g_{\mu\bar{\gamma}} - \tilde{\chi}_{\bar{\gamma}\sigma}^{\bar{\mu}} g_{\beta\bar{\mu}};
\end{array}
$$

$$
0 = P_{\bar{k}}^{\bar{\mu}} (\dot{\partial}_{\bar{\sigma}} B_{\bar{\gamma}}^{\bar{k}} - \dot{\partial}_{\bar{\gamma}} B_{\bar{\sigma}}^{\bar{k}}) g_{\beta\bar{\mu}} + \overset{4}{\tilde{C}}{}_{\beta\bar{\sigma}}^{\mu} g_{\mu\bar{\gamma}} - \overset{4}{\tilde{C}}{}_{\beta\bar{\gamma}}^{\mu} g_{\mu\bar{\sigma}} - \tilde{S}_{\bar{\gamma}\bar{\sigma}}^{\bar{\mu}} g_{\beta\bar{\mu}};
$$

iii. *the A-Codazzi equations*

$$P_i^\alpha \eta^j B_{\bar\gamma}^{\bar k} B_\beta^h R_{j\bar kh}^i = -\tilde\Theta_{\beta\bar\gamma}^\alpha; \quad P_i^\alpha \eta^j \eta^{\bar k} B_\beta^h R_{j\bar kh}^i = -\tilde\Theta_{\beta\overline{n+1}}^\alpha;$$
$$P_i^\alpha \eta^j \eta^h B_{\bar\gamma}^{\bar k} R_{j\bar kh}^i = -\tilde\Theta_{n+1\bar\gamma}^\alpha; \quad P_i^\alpha \eta^j \eta^{\bar k} \eta^h R_{j\bar kh}^i = -\tilde\Theta_{n+1\overline{n+1}}^\alpha;$$
$$P_i^\alpha \eta^j B_{\bar\gamma}^{\bar k} B_\beta^h P_{j\bar kh}^i = -\tilde\Sigma_{\beta\bar\gamma}^\alpha; \quad P_i^\alpha \eta^j \eta^{\bar k} B_\beta^h P_{j\bar kh}^i = -\tilde\Sigma_{\beta\overline{n+1}}^\alpha;$$
$$P_i^\alpha \eta^j B_{\bar\gamma}^{\bar k} B_\beta^h Q_{j\bar kh}^i = -\tilde P_{\beta\bar\gamma}^\alpha; \quad P_i^\alpha \eta^j \eta^h B_{\bar\gamma}^{\bar k} Q_{j\bar kh}^i = -\tilde P_{n+1\bar\gamma}^\alpha;$$
$$P_i^\alpha \eta^j B_{\bar\gamma}^{\bar k} B_\beta^h S_{j\bar kh}^i = -\tilde\chi_{\beta\bar\gamma}^\alpha;$$

iv. *the Ricci equations*

$$\eta_i \eta^j B_{\bar\gamma}^{\bar k} B_\sigma^h R_{j\bar kh}^i = 0; \quad \eta_i \eta^j \eta^{\bar k} B_\sigma^h R_{j\bar kh}^i = 0; \quad \eta_i \eta^j B_{\bar\gamma}^{\bar k} \eta^h R_{j\bar kh}^i = 0;$$
$$\eta_i \eta^j \eta^{\bar k} \eta^h R_{j\bar kh}^i = 0; \quad \eta_i \eta^j B_{\bar\gamma}^{\bar k} B_\sigma^h P_{j\bar kh}^i = 0; \quad \eta_i \eta^j \eta^{\bar k} B_\sigma^h P_{j\bar kh}^i = 0;$$
$$\eta_i \eta^j B_{\bar\gamma}^{\bar k} B_\sigma^h Q_{j\bar kh}^i = 0; \quad \eta_i \eta^j B_{\bar\gamma}^{\bar k} \eta^h Q_{j\bar kh}^i = 0; \quad \eta_i \eta^j B_{\bar\gamma}^{\bar k} B_\sigma^h S_{j\bar kh}^i = g_{\sigma\bar\gamma}.$$

By comparing the local expressions of the Ricci equations with the Gauss, H- and A-Codazzi equations, using the curvature coefficients expressions and the homogeneity conditions (1), we notice that some of the torsion and curvature components of the induced tangent connection become zero, and we get:

Theorem 2. *With respect to the adapted frame introduced on IM, the above equations give the following nonzero relations*

$$P_i^\alpha B_\beta^j B_{\bar\gamma}^{\bar k} B_\sigma^h R_{j\bar kh}^i = \tilde R_{\beta\bar\gamma\sigma}^\alpha; \quad \eta_i B_\beta^j B_{\bar\gamma}^{\bar k} B_\sigma^h R_{j\bar kh}^i = \tilde R_{\beta\bar\gamma\sigma}^{n+1} = -\tilde\Theta_{\bar\gamma\sigma}^{\bar\mu} g_{\beta\bar\mu};$$
$$P_i^\alpha B_\beta^j \eta^{\bar k} B_\sigma^h R_{j\bar kh}^i = \tilde R_{\beta\overline{n+1}\sigma}^\alpha; \quad \eta_i B_\beta^j \eta^{\bar k} B_\sigma^h R_{j\bar kh}^i = \tilde R_{\beta\overline{n+1}\sigma}^{n+1} = -\tilde\Theta_{\overline{n+1}\sigma}^{\bar\mu} g_{\beta\bar\mu};$$
$$P_i^\alpha B_\beta^j B_{\bar\gamma}^{\bar k} \eta^h R_{j\bar kh}^i = \tilde R_{\beta\bar\gamma n+1}^\alpha; \quad \eta_i B_\beta^j B_{\bar\gamma}^{\bar k} \eta^h R_{j\bar kh}^i = \tilde R_{\beta\bar\gamma n+1}^{n+1} = -\tilde\Theta_{\bar\gamma n+1}^{\bar\mu} g_{\beta\bar\mu};$$
$$P_i^\alpha B_\beta^j \eta^{\bar k} \eta^h R_{j\bar kh}^i = \tilde R_{\beta\overline{n+1}n+1}^\alpha; \quad \eta_i B_\beta^j \eta^{\bar k} \eta^h R_{j\bar kh}^i = \tilde R_{\beta\overline{n+1}n+1}^{n+1} = -\tilde\Theta_{\overline{n+1}n+1}^{\bar\mu} g_{\beta\bar\mu};$$
$$P_i^\alpha \eta^j B_{\bar\gamma}^{\bar k} B_\beta^h R_{j\bar kh}^i = \tilde R_{n+1\bar\gamma\beta}^\alpha = -\tilde\Theta_{\beta\bar\gamma}^\alpha;$$
$$P_i^\alpha \eta^j \eta^{\bar k} B_\beta^h R_{j\bar kh}^i = \tilde R_{n+1\overline{n+1}\beta}^\alpha = -\tilde\Theta_{\beta\overline{n+1}}^\alpha;$$
$$P_i^\alpha \eta^j B_{\bar\gamma}^{\bar k} \eta^h R_{j\bar kh}^i = \tilde R_{n+1\bar\gamma n+1}^\alpha = -\tilde\Theta_{n+1\bar\gamma}^\alpha;$$
$$P_i^\alpha \eta^j \eta^{\bar k} \eta^h R_{j\bar kh}^i = \tilde R_{n+1\overline{n+1}n+1}^\alpha = -\tilde\Theta_{n+1\overline{n+1}}^\alpha;$$

$$P_i^\alpha B_\beta^j B_{\bar\gamma}^{\bar k} B_\sigma^h P_{j\bar kh}^i = \tilde P_{\beta\bar\gamma\sigma}^\alpha; \quad \eta_i B_\beta^j B_{\bar\gamma}^{\bar k} B_\sigma^h P_{j\bar kh}^i = \tilde P_{\beta\bar\gamma\sigma}^{n+1} = -\tilde\rho_{\bar\gamma\sigma}^{\bar\mu} g_{\beta\bar\mu};$$
$$P_i^\alpha B_\beta^j \eta^{\bar k} B_\sigma^h P_{j\bar kh}^i = \tilde P_{\beta\overline{n+1}\sigma}^\alpha; \quad \eta_i B_\beta^j \eta^{\bar k} B_\sigma^h P_{j\bar kh}^i = \tilde P_{\beta\overline{n+1}\sigma}^{n+1} = -\tilde\rho_{\overline{n+1}\sigma}^{\bar\mu} g_{\beta\bar\mu};$$
$$P_i^\alpha \eta^j B_{\bar\gamma}^{\bar k} B_\sigma^h P_{j\bar kh}^i = \tilde P_{n+1\bar\gamma\sigma}^\alpha = -\tilde\Sigma_{\beta\bar\gamma}^\alpha;$$
$$P_i^\alpha \eta^j \eta^{\bar k} B_\beta^h P_{j\bar kh}^i = \tilde P_{n+1\overline{n+1}\beta}^\alpha = -\tilde\Sigma_{\beta\overline{n+1}}^\alpha;$$

$$P_i^\alpha B_\beta^j B_{\bar\gamma}^{\bar k} B_\sigma^h Q_{j\bar kh}^i = \tilde Q_{\beta\bar\gamma\sigma}^\alpha; \quad P_i^\alpha \eta^j B_{\bar\gamma}^{\bar k} B_\beta^h Q_{j\bar kh}^i = \tilde Q_{n+1\bar\gamma\beta}^\alpha = -\tilde P_{\beta\bar\gamma}^\alpha;$$
$$P_i^\alpha B_\beta^j B_{\bar\gamma}^{\bar k} \eta^h Q_{j\bar kh}^i = \tilde Q_{\beta\bar\gamma n+1}^\alpha; \quad P_i^\alpha \eta^j B_{\bar\gamma}^{\bar k} \eta^h Q_{j\bar kh}^i = \tilde Q_{n+1\bar\gamma n+1}^\alpha = -\tilde P_{n+1\bar\gamma}^\alpha;$$

$$P_i^\alpha B_\beta^j B_{\bar\gamma}^{\bar k} B_\sigma^h S_{j\bar kh}^i = \tilde S_{\beta\bar\gamma\sigma}^\alpha; \quad P_i^\alpha \eta^j B_{\bar\gamma}^{\bar k} B_\beta^h S_{j\bar kh}^i = \tilde S_{n+1\bar\gamma\beta}^\alpha = -\tilde\chi_{\beta\bar\gamma}^\alpha;$$
$$\eta_i \eta^j B_{\bar\gamma}^{\bar k} B_\sigma^h S_{j\bar kh}^i = \tilde S_{n+1\bar\gamma\sigma}^{n+1} = g_{\sigma\bar\gamma};$$

$$B^j_\beta B^{\bar{k}}_{\bar{\gamma}} B^h_\sigma S^i_{j\bar{k}h} = \tilde{\mathbb{S}}^\alpha_{\beta\bar{\gamma}\sigma} B^i_\alpha - \frac{1}{L} g_{\beta\bar{\gamma}} B^i_\sigma, \quad where\ R(\dot{\partial}_\sigma, \dot{\partial}_{\bar{\gamma}})\dot{\partial}_\beta = \tilde{\mathbb{S}}^\alpha_{\beta\bar{\gamma}\sigma}\dot{\partial}_\alpha;$$

$$\eta_i B^j_\beta B^{\bar{k}}_{\bar{\gamma}} B^h_\sigma Q^i_{j\bar{k}h} = \tilde{Q}^{n+1}_{\beta\bar{\gamma}\sigma} = \overset{4}{\tilde{L}}{}^{\bar{\mu}}_{\bar{\gamma}\sigma} g_{\beta\bar{\mu}} + \overset{2}{\tilde{L}}{}^{\mu}_{\beta\sigma} g_{\mu\bar{\gamma}} - \tilde{\Sigma}^{\bar{\mu}}_{\bar{\gamma}\sigma} g_{\beta\bar{\mu}};$$

$$\eta_i B^j_\beta B^{\bar{k}}_{\bar{\gamma}} \eta^h Q^i_{j\bar{k}h} = \tilde{Q}^{n+1}_{\beta\bar{\gamma}n+1} = \overset{4}{\tilde{L}}{}^{\bar{\mu}}_{\bar{\gamma}n+1} g_{\beta\bar{\mu}} + \overset{2}{\tilde{L}}{}^{\mu}_{\beta n+1} g_{\mu\bar{\gamma}} - \tilde{\Sigma}^{\bar{\mu}}_{\bar{\gamma}n+1} g_{\beta\bar{\mu}};$$

$$\eta_i B^j_\beta B^{\bar{k}}_{\bar{\gamma}} B^h_\sigma S^i_{j\bar{k}h} = \tilde{S}^{n+1}_{\beta\bar{\gamma}\sigma} = \overset{4}{\tilde{C}}{}^{\bar{\mu}}_{\bar{\gamma}\sigma} g_{\beta\bar{\mu}} + \overset{2}{\tilde{C}}{}^{\mu}_{\beta\sigma} g_{\mu\bar{\gamma}} - \tilde{\chi}^{\bar{\mu}}_{\bar{\gamma}\sigma} g_{\beta\bar{\mu}};$$

$$0 = P^{\bar{\mu}}_{\bar{k}}(\dot{\partial}_{\bar{\sigma}} B^{\bar{k}}_{\bar{\gamma}} - \dot{\partial}_{\bar{\gamma}} B^{\bar{k}}_{\bar{\sigma}}) g_{\beta\bar{\mu}} + \overset{4}{\tilde{C}}{}^{\mu}_{\beta\bar{\sigma}} g_{\mu\bar{\gamma}} - \overset{4}{\tilde{C}}{}^{\mu}_{\beta\bar{\gamma}} g_{\mu\bar{\sigma}} - \tilde{S}^{\bar{\mu}}_{\bar{\gamma}\bar{\sigma}} g_{\beta\bar{\mu}}.$$

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