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THE GAUSS-WEINGARTEN FORMULAE FOR THE HOMOGENEOUS LIFT TO THE OSCULATOR BUNDLE OF A FINSLER METRIC

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Abstract

In this article we present a study of the subspaces of the manifold OscM, the total space of the osculator bundle of a real manifold M. We obtain the induced connections of the canonical metrical N-linear connection determined by the homogeneous prolongation of a Finsler metric to the manifold OscM. We present the Gauss-Weingarten equations of the associated osculator submanifold.

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1 Introduction

The Sasaki N-prolongation $\mathbb{G}$ to the osculator bundle without the null section $\widetilde{OscM} = OscM \setminus \{0\}$ of a Finslerian metric g_{ab} on the manifold M given by

$$\mathbb{G} = g_{ab}\,(x, y)\, dx^a \otimes dx^b + g_{ab}\,(x, y)\, \delta y^a \otimes \delta y^b \qquad (*)$$

is a Riemannian structure on $\widetilde{OscM}$, which depends only on the metric g_{ab}.

The tensor $\mathbb{G}$ is not invariant with respect to the homothetis on the fibres of $\widetilde{OscM}$, because $\mathbb{G}$ is not homogeneous with respect to the variable y^a.

In this paper, we use a new kind of prolongation $\mathring{\mathbb{G}}$ to $\widetilde{OscM}$, ([8]), which depends only on the metric g_{ab}. Thus, $\mathring{\mathbb{G}}$ determines on the manifold $\widetilde{OscM}$ a Riemannian structure which is 0-homogeneous on the fibres of $OscM$.

Some geometrical properties of $\mathring{\mathbb{G}}$ are studied: the canonical metrical N-linear connection, the induced linear connections etc.

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2 Preliminaries

As far as we know the general theory of submanifolds (in particular the Finsler submanifolds or the complex Finsler submanifolds) is far from being settled ([1],[10], [3],[11], [12]). In [9] and [10] R.Miron and M. Anastasiei give the theory of subspaces in generalized Lagrange spaces. Also, in [6] and [5] R. Miron presented the theory of subspaces in higher order Finsler and Lagrange spaces respectively.

If $\check{M}$ is an immersed manifold in manifold M, a nonlinear connection on $OscM$ induces a nonlinear connection $\check{N}$ on $Osc\check{M}$.

The d-tensor $\mathbb{G}$ from (*) is not homogeneous with respect to the variable y^a. This is an incovenient from the point of view of mechanics. Moreover, the physical dimensions of the terms of $\mathbb{G}$ are not the same. This disadvantage was corrected by R. Miron. He took a new kind of prolongation $\mathring{\mathbb{G}}$ to $\widetilde{OscM}$ of the fundamental tensor of a Finsler space, ([8]) (5), which depends only on the metric g_{ab}. Thus, $\mathring{\mathbb{G}}$ determines on the manifold $\widetilde{OscM}$ a Riemannian structure which is 0-homogeneous on the fibres of $OscM$ and p is a positive constant required by applications in order that the physical dimensions of the terms of $\mathring{\mathbb{G}}$ be the same. He proved that there exist metrical N-linear connections with respect to the metric tensor $\mathring{\mathbb{G}}$.

We take this canonical N-linear metric connection D on the manifold $OscM$ and obtain the induced tangent and normal connections and the relative covariant derivation in the algebra of d-tensor fields ([13], [16]).

In this paper we get the Gauss-Weingarten formulae of submanifold $Osc\check{M}$.

Let us consider $F^n = (M, F)$ a Finsler space ([10]), and $F : TM = OscM \to \mathbb{R}$ the fundamental function. F is a C^∞ function on the manifold $OscM$ and it is continuous on the null section of the projection $\pi : OscM \to M$. The fundamental tensor on F^n is

$$g_{ab}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^a \partial y^b}, \ \ \forall (x, y) \in OscM.$$

The lagrangian $F^2(x, y)$ determines the canonical spray $S = y^a \dfrac{\partial}{\partial x^a} - 2G^a \dfrac{\partial}{\partial y^a}$ with the coefficients $G^a = \dfrac{1}{2}\gamma^a_{bc}(x, y)\, y^b y^c$, where $\gamma^a_{bc}(x, y)$ are the Christoffels symbols of the metric tensor $g_{ab}(x, y)$. The Cartan nonlinear connection N of the space F^n has the coefficients

$$N^a{}_b = \frac{\partial G^a}{\partial y^b}. \tag{1}$$

N determines a distribution on the manifold $\widetilde{OscM}$, ([10],[9]), which is supplementary to the vertical distribution V. We have the next decomposition

$$T_w\widetilde{OscM} = N_w \oplus V_w, \forall w = (x, y) \in \widetilde{OscM}. \tag{2}$$

The adapted basis of this decomposition is $\left\{ \frac{\delta}{\delta x^a}, \frac{\partial}{\partial y^a} \right\}$, $(a = 1, .., n)$ and its dual basis is $(dx^a, \delta y^a)$, where

$$\begin{cases} \frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - N^b{}_a \frac{\delta}{\delta y^b}, \\ \\ \frac{\partial}{\partial y^a} = \qquad\qquad \frac{\partial}{\partial y^a} \end{cases} \tag{3}$$

and

$$\begin{cases} dx^a = \qquad\quad dx^a, \\ \delta y^a = dy^a + N^a{}_b dx^b. \end{cases} \tag{4}$$

We use the next notations:

$$\delta_a = \frac{\delta}{\delta x^a}, \ \ \dot{\partial}_{1a} = \frac{\partial}{\partial y^a}.$$

The fundamental tensor g_{ab} determines on the manifold $\widetilde{OscM}$ the homogeneous N-lift $\overset{0}{\mathbb{G}}$,[8],

$$\overset{0}{\mathbb{G}} = g_{ab}(x, y)\, dx^a \otimes dx^b + h_{ab}(x, y)\, \delta y^a \otimes \delta y^b, \tag{5}$$

where

$$h_{ab}(x, y) = \frac{p^2}{\|y\|^2} g_{ab}(x, y), \tag{6}$$

$$\|y\|^2 = g_{ab}(x, y)\, y^a y^b.$$

This is homogeneous with respect to y, and p is a positive constant required by applications in order that the physical dimensions of the terms of $\overset{\circ}{\mathbb{G}}$ be the same.

Let $\check{M}$ be a real, m-dimensional manifold, immersed in M through the immersion $i : \check{M} \to M$. Localy, i can be given in the form

$$x^a = x^a\left(u^1, ..., u^m\right), \qquad rank \left\| \frac{\partial x^a}{\partial u^\alpha} \right\| = m.$$

The indices $a, b, c, ...$run over the set $\{1, ..., n\}$ and $\alpha, \beta, \gamma, ...$ run on the set $\{1, ..., m\}$. We assume $1 < m < n$. We take the immersed submanifold $Osc\check{M}$ of the manifold $OscM$, by the immersion Osc$i$: $Osc\check{M} \to OscM$. The parametric equations of the submanifold $Osc\check{M}$ are

$$\begin{cases} x^a = x^a\left(u^1, ..., u^m\right), rang \left\| \frac{\partial x^a}{\partial u^\alpha} \right\| = m \\ \\ y^a = \frac{\partial x^a}{\partial u^\alpha} v^\alpha. \end{cases} \tag{7}$$

The restriction of the fundamental function F to the submanifold $\widetilde{Osc\check{M}}$ is

$$\check{F}(u,v) = F(x(u), y(u,v))$$

and we call $\check{F}^m = (\check{M}, \check{F})$ the **induced Finsler subspaces** of F^n and $\check{F}$ the **induced fundamental function**.

Let $B^a_\alpha(u) = \dfrac{\partial x^a}{\partial u^\alpha}$ and $g_{\alpha\beta}$ the induced fundamental tensor,

$$g_{\alpha\beta}(u,v) = g_{ab}(x(u), y(u,v)) B^a_\alpha B^b_\beta. \tag{8}$$

We obtain a system of d-vectors $\{B^a_\alpha, B^a_{\bar{\alpha}}\}$ which determines a moving frame $\mathcal{R} = \{(u,v); B^a_\alpha(u), B^a_{\bar{\alpha}}(u,v)\}$ in $OscM$ along to the submanifold $Osc\check{M}$.

Its dual frame will be denoted by $\mathcal{R}^* = \{B^\alpha_a(u,v), B^{\bar{\alpha}}_a(u,v)\}$. This is also defined on an open set $\check{\pi}^{-1}(\check{U}) \subset Osc\check{M}$, $\check{U}$ being a domain of a local chart on the submanifold $\check{M}$.

The conditions of duality are given by:

$$B^a_\beta B^\alpha_a = \delta^\alpha_\beta, \quad B^a_\beta B^{\bar{\alpha}}_a = 0, \quad B^\alpha_a B^a_{\bar{\beta}} = 0, \quad B^{\bar{\alpha}}_a B^a_{\bar{\beta}} = \delta^{\bar{\alpha}}_{\bar{\beta}}$$

$$B^a_\alpha B^\alpha_b + B^a_{\bar{\alpha}} B^{\bar{\alpha}}_b = \delta^a_b.$$

The restriction of the nonlinear connection N to $\widetilde{Osc\check{M}}$ uniquely determines an induced nonlinear connection $\check{N}$ on $\widetilde{Osc\check{M}}$

$$\check{N}^\alpha{}_\beta = B^\alpha_a \left(B^a_{0\beta} + N^a{}_b B^b_\beta \right). \tag{9}$$

The cobasis $(dx^i, \delta y^a)$ restricted to $Osc\check{M}$ is uniquely represented in the moving frame $\mathcal{R}$ in the following form:

$$\begin{cases} dx^a = B^a_\beta du^\beta \\ \delta y^a = B^a_\alpha \delta v^\alpha + B^a_{\bar{\alpha}} K^{\bar{\alpha}}_\beta du^\beta \end{cases} \tag{10}$$

where

$$K^{\bar{\alpha}}_\beta = B^{\bar{\alpha}}_a \left(B^a_{0\beta} + M^a_b B^b_\beta \right), \; B^a_{0\beta} = B^a_{\alpha\beta} v^a.$$

A linear connection D on the manifold $OscM$ is called **metrical N-linear connection** with respect to $\overset{\circ}{\mathbb{G}}$, if $D\overset{\circ}{\mathbb{G}} = 0$ and D preserves by parallelism the distributions N and V. The coefficients of the N-linear connections $D\Gamma(N)$ will be denoted with $\left(\underset{(00)}{\overset{H}{L}}{}^a_{bc}, \underset{(10)}{\overset{V}{L}}{}^a_{bc}, \underset{(01)}{\overset{H}{C}}{}^a_{bc}, \underset{(11)}{\overset{V}{C}}{}^a_{bc} \right)$.

Theorem 1.1([8]) *There exist metrical N-linear connections $D\Gamma(N)$ on $\widetilde{OscM}$, with respect to the homogeneous prolongation $\overset{\circ}{\mathbb{G}}$, which depend only on the metric*

$g_{ab}(x,y)$. *One of these connections has the "horizontal" coefficients*

$$\begin{aligned} \underset{(00)}{\overset{H}{L}}{}^{a}_{bc} &= \frac{1}{2} g^{ad} \left(\delta_b g_{dc} + \delta_c g_{bd} - \delta_d g_{bc} \right) \\ \underset{(10)}{\overset{V}{L}}{}^{a}_{bc} &= \frac{1}{2} h^{ad} \left(\delta_b h_{dc} + \delta_c h_{bd} - \delta_d h_{bc} \right) \end{aligned} \tag{11}$$

and the "vertical" coefficients:

$$\begin{aligned} \underset{(01)}{\overset{H}{C}}{}^{a}_{bc} &= \frac{1}{2} g^{ad} \left(\dot{\partial}_b g_{dc} + \dot{\partial}_c g_{bd} - \dot{\partial}_d g_{bc} \right) \\ \underset{(11)}{\overset{V}{C}}{}^{a}_{bc} &= \frac{1}{2} h^{ad} \left(\dot{\partial}_b h_{dc} + \dot{\partial}_c h_{bd} - \dot{\partial}_d h_{bc} \right). \end{aligned} \tag{12}$$

It is called the **Cartan metrical N-linear connection**. This linear connection will be used throughout this paper.

For this N-linear connection, we have the operators $\overset{H}{D}$ and $\overset{V}{D}$ which are given by the following relations

$$\left.\begin{aligned} \overset{H}{D} X^a &= dX^a + \overset{H}{\omega}{}^a_b X^b \\ \overset{V}{D} X^a &= dX^a + \overset{V}{\omega}{}^a_b X^b. \end{aligned}\right. \quad \forall X \in \mathcal{F}\left(\widetilde{OscM}\right) \tag{13}$$

We call these operators the **horizontal** and **vertical covariant differentials**. The 1-forms which define these operators will be called the **horizontal** and **vertical 1-form**, where

$$\begin{aligned} \overset{H}{\omega}{}^a_b &= \underset{(00)}{\overset{H}{L}}{}^{a}_{bc} dx^c + \underset{(01)}{\overset{H}{C}}{}^{a}_{bc} \delta y^c \\ \overset{V}{\omega}{}^a_b &= \underset{(10)}{\overset{V}{L}}{}^{a}_{bc} dx^c + \underset{(11)}{\overset{V}{C}}{}^{a}_{bc} \delta y^c. \end{aligned} \tag{14}$$

We have

Theorem 1.2[16] *The d-tensors of torsion of the Cartan metrical N-linear connection D have the next expresions:*

$$\begin{aligned} &\underset{(00)}{\overset{H}{T}}{}^{a}_{bc} = \underset{(00)}{\overset{H}{L}}{}^{a}_{bc} - \underset{(00)}{\overset{H}{L}}{}^{a}_{cb}, \qquad \underset{(01)}{\overset{V}{T}}{}^{a}_{bc} = R^a_{bc}, \\ &\underset{(10)}{\overset{H}{P}}{}^{a}_{bc} = \underset{(01)}{\overset{H}{C}}{}^{a}_{bc}, \qquad \underset{(11)}{\overset{V}{P}}{}^{a}_{bc} = \underset{(11)}{B}{}^{a}_{bc} - \underset{(10)}{\overset{V}{L}}{}^{a}_{cb} \\ &\underset{(11)}{\overset{V}{S}}{}^{a}_{bc} = \underset{(11)}{\overset{V}{C}}{}^{a}_{bc} - \underset{(11)}{\overset{V}{C}}{}^{a}_{cb}. \end{aligned} \tag{15}$$

Theorem 1.3[16] *The Cartan metrical N-linear connection D has, in the adapted bases* $\left\{\delta_a, \dot{\partial}_{1a}\right\}$, *the following d-tensors of curvature "horizontals"*

$$
\begin{aligned}
\underset{(00)}{\overset{H}{R}}{}_b{}^a{}_{cd} &= \delta_d \underset{(00)}{\overset{H}{L}}{}^a_{bc} - \delta_c \underset{(00)}{\overset{H}{L}}{}^a_{bd} + \underset{(00)}{\overset{H}{L}}{}^f_{bc} \underset{(00)}{\overset{H}{L}}{}^a_{fd} - \underset{(00)}{\overset{H}{L}}{}^f_{bd} \underset{(00)}{\overset{H}{L}}{}^a_{fc} + \\
&\quad + \underset{(01)}{\overset{H}{C}}{}^a_{bf} R^f_{cd}, \\
\underset{(10)}{\overset{H}{P}}{}_b{}^a{}_{cd} &= \dot{\partial}_{1d} \underset{(00)}{\overset{H}{L}}{}^a_{bc} - \underset{(01)}{\overset{H}{C}}{}^a_{bd|0c} + \underset{(01)}{\overset{H}{C}}{}^a_{bf} \underset{(11)}{\overset{H}{P}}{}^f_{cd}, \\
\underset{(10)}{\overset{H}{S}}{}_b{}^a{}_{cd} &= \dot{\partial}_{1d} \underset{(01)}{\overset{H}{C}}{}^a_{bc} - \dot{\partial}_{1c} \underset{(01)}{\overset{H}{C}}{}^a_{bd} + \underset{(01)}{\overset{H}{C}}{}^f_{bc} \underset{(01)}{\overset{H}{C}}{}^a_{fd} - \underset{(01)}{\overset{H}{C}}{}^f_{bd} \underset{(01)}{\overset{H}{C}}{}^a_{fc},
\end{aligned}
\tag{16}
$$

and the "verticals"

$$
\begin{aligned}
\underset{(01)}{\overset{V}{R}}{}_b{}^a{}_{cd} &= \delta_d \underset{(10)}{\overset{V}{L}}{}^a_{bc} - \delta_c \underset{(10)}{\overset{V}{L}}{}^a_{bd} + \underset{(10)}{\overset{V}{L}}{}^f_{bc} \underset{(10)}{\overset{V}{L}}{}^a_{fd} - \underset{(10)}{\overset{V}{L}}{}^f_{bd} \underset{(10)}{\overset{V}{L}}{}^a_{fc} + \\
&\quad + \underset{(11)}{\overset{V}{C}}{}^a_{bf} R^f_{cd}, \\
\underset{(11)}{\overset{V}{P}}{}_b{}^a{}_{cd} &= \dot{\partial}_{1d} \underset{(10)}{\overset{V}{L}}{}^a_{bc} - \underset{(11)}{\overset{V}{C}}{}^a_{bd|1c} + \underset{(11)}{\overset{V}{C}}{}^a_{bf} \underset{(11)}{\overset{V}{P}}{}^f_{cd}, \\
\underset{(11)}{\overset{V}{S}}{}_b{}^a{}_{cd} &= \dot{\partial}_{1d} \underset{(11)}{\overset{V}{C}}{}^a_{bc} - \dot{\partial}_{1c} \underset{(11)}{\overset{V}{C}}{}^a_{bd} + \underset{(11)}{\overset{V}{C}}{}^f_{bc} \underset{(11)}{\overset{V}{C}}{}^a_{fd} - \underset{(11)}{\overset{V}{C}}{}^f_{bd} \underset{(11)}{\overset{V}{C}}{}^a_{fc}.
\end{aligned}
\tag{17}
$$

3 The relative covariant derivatives

Let $D\Gamma(N)$, the Cartan metrical N-linear connection of the manifold $OscM$. A classical method to determine the laws of derivation on a Finsler submanifold is the type of the coupling.

Theorem 2.1 *The coupling of the N-linear connection D to the induced nonlinear connection* $\check{N}$ *along* $\widetilde{Osc\check{M}}$ *is locally given by the set of coefficients* $\check{D}\Gamma\left(\check{N}\right) =$

$\left(\underset{(00)}{\overset{H}{\check{L}}}{}^{a}_{b\delta}, \underset{(10)}{\overset{V}{\check{L}}}{}^{a}_{b\delta}, \underset{(01)}{\overset{H}{\check{C}}}{}^{a}_{b\delta}, \underset{(11)}{\overset{V}{\check{C}}}{}^{a}_{b\delta}\right)$, *where*

$$\begin{cases} \underset{(00)}{\overset{H}{\check{L}}}{}^{a}_{b\delta} = \underset{(00)}{\overset{H}{L}}{}^{a}_{bd} B^{d}_{\delta} + \underset{(01)}{\overset{H}{C}}{}^{a}_{bd} B^{d}_{\bar{\delta}} K^{\bar{\delta}}_{\delta} \\ \underset{(10)}{\overset{V}{\check{L}}}{}^{a}_{b\delta} = \underset{(10)}{\overset{V}{L}}{}^{a}_{bd} B^{d}_{\delta} + \underset{(11)}{\overset{V}{C}}{}^{a}_{bd} B^{d}_{\bar{\delta}} K^{\bar{\delta}}_{\delta} \\ \underset{(01)}{\overset{H}{\check{C}}}{}^{a}_{b\delta} = \underset{(01)}{\overset{H}{C}}{}^{a}_{bd} B^{d}_{\delta} \\ \underset{(11)}{\overset{V}{\check{C}}}{}^{a}_{b\delta} = \underset{(11)}{\overset{V}{C}}{}^{a}_{bd} B^{d}_{\delta}. \end{cases} \tag{18}$$

Definition 2.2 *We call the **induced tangent connection** on $\widetilde{Osc\check{M}}$ by the metrical N-linear connection D, the couple of operators $\overset{H}{D}{}^{\top}$, $\overset{V}{D}{}^{\top}$ which are defined by*

$$\begin{matrix} \overset{H}{D}{}^{\top} X^{\alpha} = B^{\alpha}_{b} \overset{H}{\check{D}} X^{b}, \\ \overset{V}{D}{}^{\top} X^{\alpha} = B^{\alpha}_{b} \overset{V}{\check{D}} X^{b}, \end{matrix} \qquad for X^{a} = B^{a}_{\gamma} X^{\gamma}$$

where

$$\overset{H}{D}{}^{\top} X^{\alpha} = dX^{\alpha} + X^{\beta} \overset{H}{\omega}{}^{\alpha}_{\beta}$$

$$\overset{V}{D}{}^{\top} X^{\alpha} = dX^{\alpha} + X^{\beta} \overset{V}{\omega}{}^{\alpha}_{\beta}$$

*and $\overset{H}{\omega}{}^{\alpha}_{\beta}$, $\overset{V}{\omega}{}^{\alpha}_{\beta}$ are called the **tangent connection 1-forms**.*

We have

Theorem 2.3 *The tangent connections 1-forms are as follows:*

$$\begin{matrix} \overset{H}{\omega}{}^{\alpha}_{\beta} = \underset{(00)}{\overset{H}{L}}{}^{\alpha}_{\beta\delta} du^{\delta} + \underset{(01)}{\overset{H}{C}}{}^{\alpha}_{\beta\delta} \delta v^{\delta} \\ \overset{V}{\omega}{}^{\alpha}_{\beta} = \underset{(10)}{\overset{V}{L}}{}^{\alpha}_{\beta\delta} du^{\delta} + \underset{(11)}{\overset{V}{C}}{}^{\alpha}_{\beta\delta} \delta v^{\delta}, \end{matrix} \tag{19}$$

where

$$\underset{(00)}{\overset{H}{L}}{}^{\alpha}_{\beta\delta} = B^{\alpha}_{d}\left(B^{d}_{\beta\delta} + B^{f}_{\beta}\underset{(00)}{\overset{H}{\check{L}}}{}^{d}_{f\delta}\right),$$

$$\underset{(10)}{\overset{V}{L}}{}^{\alpha}_{\beta\delta} = B^{\alpha}_{d}\left(B^{d}_{\beta\delta} + B^{f}_{\beta}\underset{(10)}{\overset{V}{\check{L}}}{}^{d}_{f\delta}\right),$$

$$\underset{(01)}{\overset{H}{C}}{}^{\alpha}_{\beta\delta} = B^{\alpha}_{d}B^{f}_{\beta}\underset{(01)}{\overset{H}{\check{C}}}{}^{d}_{f\delta},\tag{20}$$

$$\underset{(11)}{\overset{V}{C}}{}^{\alpha}_{\beta\delta} = B^{\alpha}_{d}B^{f}_{\beta}\underset{(11)}{\overset{V}{\check{C}}}{}^{d}_{f\delta}.$$

Definition 2.4 *We call the* ***induced normal connection*** *on $\widetilde{Osc\check{M}}$ by the metrical N-linear connection D, the couple of operators $\overset{H}{D}{}^{\perp}$, $\overset{V}{D}{}^{\perp}$ which are defined by*

$$\overset{H}{D}{}^{\perp}X^{\overline{\alpha}} = B^{\alpha}_{b}\overset{H}{\check{D}}X^{b} \qquad \textit{for } X^{a} = B^{a}_{\bar{\gamma}}X^{\bar{\gamma}}$$
$$\overset{V}{D}{}^{\perp}X^{\overline{\alpha}} = B^{\alpha}_{b}\overset{V}{\check{D}}X^{b},$$

where

$$\overset{H}{D}{}^{\perp}X^{\overline{\alpha}} = dX^{\overline{\alpha}} + X^{\overline{\beta}}\overset{H}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}}$$

$$\overset{V}{D}{}^{\perp}X^{\overline{\alpha}} = dX^{\overline{\alpha}} + X^{\overline{\beta}}\overset{V}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}}$$

and $\overset{H}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}}$, $\overset{V}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}}$ are called the ***normal connection 1-forms****.*

We have

Theorem 2.5 *The normal connections 1-forms are as follows:*

$$\overset{H}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}} = \underset{(00)}{\overset{H}{L}}{}^{\overline{\alpha}}_{\overline{\beta}\delta}du^{\delta} + \underset{(01)}{\overset{H}{C}}{}^{\overline{\alpha}}_{\overline{\beta}\delta}\delta v^{\delta}$$
$$\overset{V}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}} = \underset{(10)}{\overset{V}{L}}{}^{\overline{\alpha}}_{\overline{\beta}\delta}du^{\delta} + \underset{(11)}{\overset{V}{C}}{}^{\overline{\alpha}}_{\overline{\beta}\delta}\delta v^{\delta},\tag{21}$$

where

$$\underset{(00)}{\overset{H}{L}}{}^{\overline{\alpha}}_{\overline{\beta}\delta} = B^{\overline{\alpha}}_{d}\left(\frac{\delta B^{d}_{\overline{\beta}}}{\delta u^{\delta}} + B^{f}_{\overline{\beta}}\underset{(00)}{\overset{H}{\check{L}}}{}^{d}_{f\delta}\right),$$

$$\underset{(10)}{\overset{V}{L}}{}^{\overline{\alpha}}_{\overline{\beta}\delta} = B^{\overline{\alpha}}_{d}\left(\frac{\delta B^{d}_{\overline{\beta}}}{\delta u^{\delta}} + B^{f}_{\overline{\beta}}\underset{(10)}{\overset{V}{\check{L}}}{}^{d}_{f\delta}\right),$$

$$\underset{(01)}{\overset{H}{C}}{}^{\overline{\alpha}}_{\overline{\beta}\delta} = B^{\overline{\alpha}}_{d}\left(\frac{\partial B^{d}_{\overline{\beta}}}{\partial v^{\delta}} + B^{f}_{\overline{\beta}}\underset{(01)}{\overset{H}{\check{C}}}{}^{d}_{f\delta}\right), \tag{22}$$

$$\underset{(11)}{\overset{V}{C}}{}^{\overline{\alpha}}_{\overline{\beta}\delta} = B^{\overline{\alpha}}_{d}\left(\frac{\partial B^{d}_{\overline{\beta}}}{\partial v^{\delta}} + B^{f}_{\overline{\beta}}\underset{(11)}{\overset{V}{\check{C}}}{}^{d}_{f\delta}\right).$$

Now, we can define the relative (or mixed) covariant derivatives $\overset{H}{\nabla}$ and $\overset{V}{\nabla}$.

Theorem 2.6 *The relative covariant (mixed) derivatives in the algebra of mixed d-tensor fields are the operators $\overset{H}{\nabla}$, $\overset{V}{\nabla}$ for which the following properties hold*:

$$\overset{H}{\nabla} f = df, \qquad \overset{V}{\nabla} f = df, \qquad \forall f \in \mathcal{F}\left(\widetilde{Osc\check{M}}\right)$$

$$\overset{H}{\nabla} X^{a} = \overset{H}{\check{D}} X^{a}, \quad \overset{H}{\nabla} X^{\alpha} = \overset{H}{D}{}^{\intercal} X^{\alpha}, \quad \overset{H}{\nabla} X^{\overline{\alpha}} = \overset{H}{D}{}^{\perp} X^{\overline{\alpha}},$$

$$\overset{V}{\nabla} X^{a} = \overset{V}{\check{D}} X^{a}, \quad \overset{V}{\nabla} X^{\alpha} = \overset{V}{D}{}^{\intercal} X^{\alpha}, \quad \overset{V}{\nabla} X^{\overline{\alpha}} = \overset{H}{D}{}^{\perp} X^{\overline{\alpha}}.$$

$\overset{H}{\check{\omega}}{}^{a}_{b}, \overset{V}{\check{\omega}}{}^{a}_{b}, \overset{H}{\omega}{}^{\alpha}_{\beta}, \overset{V}{\omega}{}^{\alpha}_{\beta}, \overset{H}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}}, \overset{V}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}}$ *are called the* ***connection 1-forms*** *of* $\overset{H}{\nabla}, \overset{V}{\nabla}$.

4 The Gauss-Weingarten formulae

As usual in the theory of the submanifolds we are interesed in finding the moving equations of the moving frame $\mathcal{R}$ along $Osc\check{M}$.

These equations, called also Gauss-Weingarten formulae, are obtained when the relative covariant derivatives of the vector fields from $\mathcal{R}$ are expressed again in the frame $\mathcal{R}$.

Thus we have

Theorem 3.1 *The following Gauss-Weingarten formulae hold:*

$$\overset{V_i}{\nabla} B^{a}_{\alpha} = B^{a}_{\bar{\delta}} \overset{V_i}{\Pi}{}^{\bar{\delta}}_{\alpha}, \tag{23}$$

$$\overset{V_i}{\nabla} B^{a}_{\bar{\alpha}} = -B^{a}_{\delta} \overset{V_i}{\Pi}{}^{\delta}_{\bar{\alpha}}, \tag{24}$$

where

$$\begin{aligned}\overset{V_i}{\Pi}{}^{\bar\delta}_{\alpha} &= \underset{(0)}{\overset{V_i}{H}}{}_{\alpha}{}^{\bar\delta}{}_{\beta} du^{\beta} + \underset{(1)}{\overset{V_i}{H}}{}_{\alpha}{}^{\bar\delta}{}_{\beta}\delta v^{\beta} \\ \overset{V_i}{\Pi}{}^{\alpha}_{\bar\delta} &= g^{\alpha\sigma}\delta_{\bar\delta\bar\sigma}\overset{V_i}{\Pi}{}^{\bar\sigma}_{\sigma},\end{aligned} \tag{25}$$

and the d-tensors

$$\begin{aligned}&\underset{(0)}{\overset{H}{H}}{}_{\alpha}{}^{\bar\delta}{}_{\beta} = B^{\bar\delta}_{d}\left(B^{d}_{\alpha\beta} + B^{f}_{\alpha}\underset{(00)}{\overset{H}{\check L}}{}^{d}_{f\beta}\right) \qquad \underset{(0)}{\overset{V}{H}}{}_{\alpha}{}^{\bar\delta}{}_{\beta} = B^{\bar\delta}_{d}\left(B^{d}_{\alpha\beta} + B^{f}_{\alpha}\underset{(10)}{\overset{V}{\check L}}{}^{d}_{f\beta}\right) \\ &\underset{(1)}{\overset{H}{H}}{}_{\alpha}{}^{\bar\delta}{}_{\beta} = B^{\bar\delta}_{d}B^{f}_{\alpha}\underset{(01)}{\overset{H}{\check C}}{}^{d}_{f\beta} \qquad \underset{(1)}{\overset{V}{H}}{}_{\alpha}{}^{\bar\delta}{}_{\beta} = B^{\bar\delta}_{d}B^{f}_{\alpha}\underset{(11)}{\overset{V}{\check C}}{}^{d}_{f\beta},\end{aligned} \tag{26}$$

are ***the fundamental d-tensors of the second order*** of manifold $\widetilde{OscM}$, $(i = 0, 1, V_0 = H, V_1 = V)$.

Proof From (11) and (12) we have

$$\begin{aligned}\overset{H}{\nabla}B^{a}_{\alpha} &= B^{a}_{\alpha|0\beta}du^{\beta} + B^{a}_{\alpha}\,|_{0\beta}\,\delta v^{\delta} \\ &= \left(\frac{\delta B^{a}_{\alpha}}{\delta u^{\beta}} + \underset{(00)}{\overset{H}{\check L}}{}^{a}_{b\beta}B^{b}_{\alpha} - \underset{(00)}{\overset{H}{L}}{}^{\delta}_{\alpha\beta}B^{a}_{\delta}\right)du^{\beta} + \\ &+ \left(\frac{\delta B^{a}_{\alpha}}{\delta v^{\beta}} + \underset{(01)}{\overset{H}{\check C}}{}^{a}_{b\beta}B^{b}_{\alpha} - \underset{(01)}{\overset{H}{C}}{}^{\delta}_{\alpha\beta}B^{a}_{\delta}\right)\delta v^{\beta} \\ &= B^{a}_{\alpha\beta}du^{\beta} + B^{b}_{\alpha}\left(\underset{(00)}{\overset{H}{\check L}}{}^{a}_{b\beta}du^{\beta} + \underset{(01)}{\overset{H}{\check C}}{}^{a}_{b\beta}\delta v^{\beta}\right) - \\ &-B^{a}_{\delta}\left[B^{\delta}_{d}\left(B^{d}_{\alpha\beta} + B^{f}_{\alpha}\underset{(00)}{\overset{H}{\check L}}{}^{d}_{f\beta}\right)du^{\beta} + B^{\delta}_{d}B^{f}_{\alpha}\underset{(01)}{\overset{H}{\check C}}{}^{d}_{f\beta}\delta v^{\beta}\right]\end{aligned}$$

Using (25) we get relation (23) for $V_0 = H$.

Now, by applying $\overset{H}{\nabla}$ to both sides of the equations

$$g_{ab}B^{a}_{\alpha}B^{b}_{\bar\beta} = 0$$

one gets

$$g_{ab}B^{a}_{\bar\delta}\overset{H}{\Pi}{}^{\bar\delta}_{\alpha}B^{b}_{\bar\beta} + g_{ab}B^{a}_{\alpha}\overset{H}{\Pi}B^{b}_{\bar\beta} = 0.$$

Multiplying these relation with B^{α}_{d} we obtain

$$g_{bd}\overset{H}{\nabla}B^{b}_{\bar\beta} - B^{a}_{\bar\delta}B^{\bar\delta}_{d}g_{ab}\overset{H}{\nabla}B^{b}_{\bar\beta} = -B^{\alpha}_{d}\delta_{\bar\beta\bar\gamma}\overset{H}{\Pi}{}^{\bar\gamma}_{\alpha}.$$

But $B^{a}_{\bar{\delta}} B^{\bar{\delta}}_{d} g_{ab} \overset{H}{\nabla} B^{b}_{\bar{\beta}} = 0$. Consequently, we obtain the relations (24) for $V_0 = H$. Analogously, for the operator $\overset{V}{\nabla}$ one gets the other relations.

References

[1] Atanasiu, Gh., *The homogeneous prolongation to the second order tangent bundle T^2M of a Finsler metric*, Proc. of The Modern Trends in Geometry and Topology, 5-11 September, 2005, Deva, 63-78.

[2] Atanasiu, Gh. and Oana, A., *The Gauss Weingarten formulae of second order*, BSG Proceedings 15, Geometry Balkan Press, Bucharest 2007, 19-33.

[3] Bejancu A. and Faran H.R., *The Geometry of Pseudo-Finsler Submanifolds*, Kluwer Acad. Publ., 2000.

[4] Matsumoto, M., *The induced and intrinsic Finsler connections of a hypersurface and Finslerian projective geometry*, Journ.b of Math. of Kyoto Univ., **25** (1985), 107-144.

[5] Miron R., *The geometry of higher order Lagrange spaces. Applications to Mechanics and Physics*, Kluwer Acd. Publ., FTPH **82**, 1997.

[6] Miron, R., *The geometry of higher order Finsler spaces*, Hadronic Press, USA, 1998.

[7] Miron R. and Atanasiu Gh., *Differential geometry of the k-osculator bundle*, Rev. Roumaine, Math Pures et Appl., 41, (1996), no. 3-4, 205-236.

[8] Miron R., Shimada, H., Sabău, V.S., *New lifts of generalized Lagrange metrics*, Geom. At Hakodate, 1999.

[9] Miron R. and Anastasiei M., *The geometry of Lagrange spaces: Theory and applications*, Kluwer Acd. Publ., FTPH, **59**, 1994.

[10] Miron R. and Anastasiei M., *Vector bundles. Lagrange spaces. Applications to relativity*, Ed. Academiei Romane, 1987.

[11] Munteanu, Gh., *The equations of a holomorphic subspace in a complex Finsler space*, Per. Math. Hungarica, 55 (2007), no. 1, 97-112.

[12] Munteanu, Gh., *Totally geodesics holomorphic subspaces*, Nonlinear Analysis: Real World Appl. **8** (2007), 1132-1143.

[13] Oană, A., *Submanifolds in the osculator bundle for the homogeneous prolongation*, Bull. Transilvania Univ., Braşov, Ser. III: Math. Info. Phys. **1(50)**, (2008), 263-268.

[14] Oană, A., *The relative covariant derivate and induced connections in the theory of embeddings in the 2-osculator bundle*, Bull. Transilvania Univ., Brasov, Ser. B1 III: Math. Info. Phys. **14(49)** (2007), 211-226.

[15] Oană, A., *On Ricci identities for submanifolds in the 2-osculator bundle*, Iran. J. of Math. Sci. and Inf., **8**, (2013), no. 2, 1-21.

[16] Oană, A., *About intrinsic Finsler connections for the homogeneous lift to the osculator bundle of a Finsler metric*, Bull. Transilvania Univ., Braşov, Ser. III: Math. Info. Phys. **6(55)** (2013), no. 1, 37-54.