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ON LIFTS OF LEFT-INVARIANT HOLOMORPHIC VECTOR FIELDS IN COMPLEX LIE GROUPS

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Abstract

In this paper the complete, vertical and horizontal lifts of left invariant holomorphic vector fields to the holomorphic tangent bundle $T^{1,0}G$ of a complex Lie group G are studied.

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1 Introduction

The study of complete, vertical and horizontal lifts of left-invariant vector fields on both tangent and tensor bundles of $(2,0)$ type over a real Lie group was initiated and intensively studied in [6, 7, 8]. The aim of this note is to obtain a complex analytic version of these notions on the holomorphic tangent bundle of a complex Lie group.

The paper is organized as follows. In the second section we present the complex Lie group structure of the holomorphic tangent bundle $T^{1,0}G$ of a complex Lie group G and we construct the complete and vertical lifts of left-invariant holomorphic vector fields on $T^{1,0}G$. In the third section we consider a holomorphic horizontal distribution on $T^{1,0}G$ defined by a linear holomorphic connection on G. In the last section we construct horizontal lifts of left-invariant holomorphic vector fields on $T^{1,0}G$ and we give necessary and sufficient conditions for the horizontal lifts of left-invariant holomorphic vector fields to be left-invariant on $T^{1,0}G$.

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2 The complex Lie group structure of the holomorphic tangent bundle $T^{1,0}G$

Let G be a complex Lie group. Let us denote by $(u,v) \to w = uv$ the composition law of the complex Lie group G, by S – the inverse mapping $u \to u^{-1}$ and by R_a and L_a, the left and right transitions of the group, respectively, where $a \in G$. These mappings are holomorphic, see [5]. We can now define a composition law "$\circ$" on $T^{1,0}G$.

Let $U = (u, \eta_u), V = (v, \eta_v)$ be two holomorphic vector fields on $T^{1,0}G$. Then

$$(u, \eta_u) \circ (v, \eta_v) = (uv, L_*(u)\eta_v + R_*(v)\eta_u) \tag{1}$$

defines a holomorphic composition law on $T^{1,0}G$. In local coordinates, we have

$$w^k = \varphi^k(u^i, v^j),\ \ \eta^i_w = L^i_s(u)\eta^s_v + R^k_s(v)\eta^s_u. \tag{2}$$

Theorem 1. *The holomorphic tangent bundle $T^{1,0}G$ is a complex Lie group with respect to the composition law defined in* (1)*.*

Proof. The identity of the group $T^{1,0}G$ is $E = (e, 0)$, where e is the identity of G. Indeed, one has

$$(u, \eta_u) \circ (e, 0) = (ue, L_*(u) \cdot 0 + R_*(e) \cdot \eta_u) = (u, \eta_u).$$

Similarly, $(e, 0) \circ (u, \eta_u) = (u, \eta_u)$.

For the inverse of $U \in T^{1,0}G$, $U \circ V = E$ yields $uv = e$ and $L_*(u)\eta_v + R_*(v)\eta_u = 0$. These imply $v = u^{-1}$ and $\eta_v = -L^{-1}_*(u)R^{-1}_*(u)\eta_u = S_*\eta_u$. Therefore,

$$U^{-1} = (u^{-1}, S_*\eta_u), \tag{3}$$

where $S_* = -L^{-1}_*(u)R^{-1}_*(u)$.

In order to prove the associativity of the composition law (1), one has, on the one hand,

$$(u, \eta_u) \circ (v, \eta_v) = (uv, L_*(u)\eta_v + R_*(v)\eta_u) = (uv, \tau_{uv}),$$

$$((u, \eta_u) \circ (v, \eta_v)) \circ (z, \eta_z) = ((uv)z, L_*(uv)\eta_z + R_*(z)\tau_{uv})$$

and, on the other hand,

$$(v, \eta_v) \circ (z, \eta_z) = (vz, L_*(v)\eta_z + R_*(z)\eta_v) = (vz, \zeta_{vz}),$$

$$(u, \eta_u) \circ ((v, \eta_v) \circ (z, \eta_z)) = (u(vz), L_*(u)\zeta_{vz} + R_*(vz)\eta_u).$$

But G is a complex Lie group and by using $L_*(uv) = L_*(u)L_*(v)$, $R_*(uv) = R_*(u)R_*(v)$ and $L_*(u)R_*(v) = R_*(v)L_*(u)$, the associativity is also proved. Therefore, $T^{1,0}G$ is a complex Lie group with the composition law (1). □

Remark 1. *Let $\omega_u \in (T^{1,0}G)^*$ be a holomorphic 1–form on G. One has*

$$\omega_u(\eta_u) = \omega_u(S_*\eta_{u^{-1}}) = S^*\omega_u(\eta_{u^{-1}}) = \omega_{u^{-1}}(\eta_{u^{-1}}),$$

such that $\omega_u(\eta_u) = \omega_{u^{-1}}(S_*\eta_u)$. Therefore,

$$(u, \omega_u)^{-1} = (u^{-1}, S^{*-1}\omega_u) \tag{4}$$

is the inverse of $(u, \omega_u) \in (T^{1,0}G)^*$.

Let us now extend the notion of left-invariance on Lie groups to holomorphic vector fields on complex Lie groups. Recall that a holomorphic vector field ξ on G is called left-invariant if

$$L_*(a)\xi(u) = \xi(au)$$

for any $u \in G$. For $u = e$, we have

$$\xi(a) = L_*(a)z,$$

where z is a holomorphic vector field on the complex Lie group G. In local coordinates,

$$\xi^i(a) = L^i_j(a)z^j,$$

where

$$L^i_j(a) = (\partial_{u^j}\varphi^j(a, u))_e,$$

and $\partial_{u^j} = \partial_j = \frac{\partial}{\partial u^j}$.

Now we can apply these considerations to the complex Lie group $T^{1,0}G$. If we denote by $L(A)$ the matrix of the holomorphic composition law (1), locally given by (2), then a left-invariant holomorphic vector field ξ satisfies

$$\xi(A) = L_*(A)Z,$$

where $A \in T^{1,0}G$ and Z is a holomorphic vector field. If we put $U = A$ and $V = E$ in (1), its Jacobi matrix is

$$L_*(A) = \begin{pmatrix} L_*(a) & 0 \\ (\partial_u R_*(u))_e\eta_a & L_*(a) \end{pmatrix}. \tag{5}$$

From (2), we obtain the following local representations:

$$L_*(A) = (L^i_j(a)), \quad (\partial_u R_*(u))_e\eta_a = (R^i_{sj}(a)\eta^j_a), \tag{6}$$

where

$$R^i_{sj}(a) = \left(\frac{\partial^2\varphi^i(a, u)}{\partial u^s \partial a^j}\right)_{u=e}. \tag{7}$$

As a consequence, one has

$$\xi(A) = L^i_j(a)z^j\partial_i + [R^k_{si}(a)\eta^i_a z^s + L^k_i(a)\dot{z}^i]\dot{\partial}_k, \tag{8}$$

where $\dot{\partial}_k = \frac{\partial}{\partial \eta^k}$ and $(z^i, \dot{z}^j)$ are the components of $Z \in T^{1,0}G$.

Let us denote by $E_\alpha(A) = (e_i(A), \dot{e}_j(A))$, where

$$e_i(A) = L_i^j(a)\partial_j + R_{ij}^k(a)\eta_a^j\dot{\partial}_k, \quad \dot{e}_j(A) = L_j^s(a)\dot{\partial}_s \tag{9}$$

are called the *complete* and *vertical lifts* of A, respectively.

With these notations, formula (8) suggests the following decomposition:

$$Z(A) = z^i e_i(A) + \dot{z}^j \dot{e}_j(A).$$

A similar calculation as in the real case, see [6], leads to the following expression of Lie brackets of holomorphic vector fields given by (9):

$$[e_i, e_j] = c_{ij}^k e_k, \ [e_i, \dot{e}_j] = c_{ij}^k \dot{e}_k, \ [\dot{e}_i, \dot{e}_j] = 0, \tag{10}$$

where c_{ij}^k are the usual structure constants of the complex Lie group G.

Also, the structure equations of the complex Lie group $T^{1,0}G$ with respect to the dual basis $\{\widetilde{\omega}^i = (\omega^i)^v, \widetilde{\omega}^{n+i} = (\omega^i)^c\}$ of $\{e_i, \dot{e}_j\}$, given by vertical and complete lifts of the 1-forms $\{\omega^i\}$ on G, can be expressed as follows:

$$\partial\widetilde{\omega}^i = -\frac{1}{2}c_{jk}^i\widetilde{\omega}^j \wedge \widetilde{\omega}^k, \ \partial\widetilde{\omega}^{n+i} = -\frac{1}{2}c_{jk}^i\widetilde{\omega}^j \wedge \widetilde{\omega}^{n+k}. \tag{11}$$

3 Holomorphic connections on $T^{1,0}G$

Let us consider the holomorphic projection $\pi : T^{1,0}G \to G$. Its holomorphic tangent map $\pi_* : T^{1,0}(T^{1,0}G) \to T^{1,0}G$ is a morphism of holomorphic tangent bundles, which maps a holomorphic vector U at point $Z \in T^{1,0}G$ to a holomorphic vector $u = \pi_* U$ at point $\pi(Z)$. As a result, we have the vertical subbundle

$$V^{1,0}(T^{1,0}G) = \ker \pi_* \subset T^{1,0}(T^{1,0}G),$$

which is holomorphic, and its sections are called *vertical vector fields* on $T^{1,0}G$. Vertical subspaces make up an involutive distribution on the manifold $T^{1,0}G$.

The holomorphic tangent bundle $T^{1,0}G$ is said to be endowed with a *complex nonlinear connection* if there is a complex distribution $H^{1,0}(T^{1,0}G)$ which is complementary to the vertical distribution, that is

$$T^{1,0}(T^{1,0}G) = H^{1,0}(T^{1,0}G) \oplus V^{1,0}(T^{1,0}G).$$

A *horizontal distribution* $H^{1,0}(T^{1,0}G)$ on the holomorphic tangent bundle $T^{1,0}G$ can be locally specified by the projected vector fields

$$\partial_i^H = \partial_i - N_i^j(Z)\dot{\partial}_j,$$

which are π-connected with the vector fields ∂_i of the natural frame field on the base manifold $T^{1,0}G$.

Generally, we notice that the horizontal distribution $H^{1,0}(T^{1,0}G)$ is not a holomorphic one. If the functions $N_i^j(Z)$ depend linearly and uniformly on the fiber coordinates η^j, that is,

$$N_j^i(z^k, \eta_z^k) = N_{il}^j(z^k)\eta_z^l,$$

the connection is said to be *linear*. Thus, the linear connection is specified by the functions $N_{il}^j(z)$, called the components of the linear connection. If, moreover, the linear connection is holomorphic, then the horizontal distribution defined by it is a holomorphic one.

Since any complex Lie group is a complex parallelizable manifold, see [10], there are canonical linear holomorphic connections on it. Let us consider the left connection $\widehat{\nabla}$ with respect to which the left-invariant vector fields are absolutely parallel:

$$\widehat{\nabla}_{\partial_i} L_j^k \partial_k = (L_j^r \widehat{\Gamma}_{ir}^k + \partial_i L_j^k)\partial_k = 0.$$

Thus, the coefficients of the left holomorphic connection have the form

$$\widehat{\Gamma}_{ij}^k(z) = -\widetilde{L}_j^r(z)\partial_i L_r^k(z) = L_r^k(z)\partial_i \widetilde{L}_j^r(z), \tag{12}$$

where $(\widetilde{L}_j^r(z))$ is the inverse of the matrix $(L_j^r(z))$.

4 Horizontal and vertical lifts

Let $U = U^i\dot{\partial}_i \in V_E^{1,0}(T^{1,0}G)$ be an arbitrary vertical holomorphic vector field, acted upon by the differential of the left translation (5). Then

$$U(Z) = L_*(Z)U = L_j^i(z)U^j\dot{\partial}_i,$$

which shows that $U(Z) \in V_Z^{1,0}(T^{1,0}G)$. Thus, we have

Proposition 1. *The vertical distribution $V^{1,0}(T^{1,0}G) \subset T^{1,0}G$ is left-invariant.*

In the following, we consider a holomorphic horizontal distribution defined by a linear holomorphic connection. Let $E_i(Z) = e_i^H(Z)$ be the horizontal lift of a left-invariant holomorphic vector field $e_i(z)$ on G. The mapping of the horizontal lift, i.e. the linear isomorphism $H : T_z^{1,0}G \to H_Z^{1,0}(T^{1,0}G)$, commutes with the differential of the left translation:

$$E_i(Z) = e_i^H(Z) = (L_*(z)\partial_i)^H = L_*(Z)\partial_i^H.$$

We shall now analize the conditions under which E_i are left-invariant vector fields. The condition of left-invariance of E_i is

$$E_i(AZ) = L_*(A)E_i(Z), \tag{13}$$

where $A \in T^{1,0}G$. In local coordinates with respect to the natural field of frames E_i, the left-invariance condition has the form

$$E_i(Z) = L_i^k(z)(\partial_k - N_{kl}^j(z)\eta_z^l\dot{\partial}_j).$$

Then

$$L_*(A)E_i(Z) = L_j^k(a)L_i^j(z)\partial_k - (R_{sj}^l(a)\eta_a^s L_i^j(z) + L_s^l(a)L_i^k(z)N_{kr}^s\eta_z^r)\dot{\partial}_l. \quad (14)$$

On the other hand,

$$E_i(AZ) = L_i^k(az)(\partial_k - N_{kl}^j(az)\eta_{az}^l\dot{\partial}_j). \quad (15)$$

Note that formula (1) implies that

$$\eta_{az}^l = L_s^l(a)\eta_z^s + R_s^l(z)\eta_a^s.$$

Therefore,

$$E_i(AZ) = L_i^k(az)\partial_k - (L_i^k(az)N_{kl}^j(az)L_s^l(a)\eta_z^s + L_i^k(az)N_{kl}^j(az)R_s^l(z)\eta_a^s)\dot{\partial}_j \quad (16)$$

By setting $X = E = (e, 0)$ in (14) and (15), one obtains

$$L_*(A)E_i(E) = L_i^k(a)\partial_k - R_{si}^l(a)\eta_a^s\dot{\partial}_l$$

and

$$E_i(A) = L_i^k(a)\partial_k - L_i^k(a)N_{kl}^j(a)\eta_a^l\dot{\partial}_j.$$

Combining the last two formula yields

$$R_{si}^l(a)\eta_a^s = L_i^k(a)N_{ks}^l(a)\eta_a^s,$$

which in turn implies

$$N_{is}^j(a) = -\widetilde{L}_i^k(a)R_{sk}^j(a). \quad (17)$$

Thus, we have

Theorem 2. *A necessary and sufficient condition for the horizontal lifts of left-invariant holomorphic vector fields to be left-invariant is that the coefficients of the linear holomorphic connection are given by* (17).

Corollary 1. *The field of holomorphic frames E_i is the left-invariant field of frames of the holomorphic horizontal distribution $H^{1,0}(T^{1,0}G)$.*

Let us now consider the vertical vector fields $\dot{E}_h(Z) = L_h^l(z)\dot{\partial}_l$. According to Proposition 1, we have

Corollary 2. *The field of holomorphic frames $\dot{E}_h$ is the left-invariant field of frames of the holomorphic vertical distribution $V^{1,0}(T^{1,0}G)$.*

Thus, we have constructed the left-invariant and adapted field of holomorphic frames $E_A = (E_k, \dot{E}_h)$, where

$$\begin{cases} E_k(Z) = L_k^i(z)\partial_i^H, \\ \dot{E}_h(Z) = L_h^l(z)\dot{\partial}_l. \end{cases} \quad (18)$$

Finally, by similar calculations as in the real case for the tensor bundle of type $(2,0)$ of a Lie group, we obtain

Proposition 2. *The Lie brackets of the vector fields defined in* (18) *are:*

$$[E_k, E_h] = c^i_{kh} E_i,\ [E_k, \dot{E}_h] = \dot{c}^i_{kh} \dot{E}_i,\ [\dot{E}_k, \dot{E}_h] = 0, \tag{19}$$

where c^i_{jk} *are the usual constants structure of* G *and* $\dot{c}^r_{kh} = (\partial_i L^r_h(z))_e + N^r_{ij}(e)$.

Remark 2. *From the first identity in* (19) *it follows that the holomorphic horizontal distribution defined by a linear holomorphic connection with the coefficients given by* (17) *is integrable.*

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