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ON A CLASS OF LP-SASAKIAN MANIFOLDS

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Abstract

The object of the present paper is to study projective curvature tensor in LP-Sasakian manifolds. LP-Sasakian manifolds satisfying $P.R = 0$, $R.P = 0$ and $P.S = 0$ are also considered. ϕ-Ricci symmetric LP-Sasakian manifolds have been studied. In all the cases the manifold becomes an Einstein manifold. Next we study 3-dimensional LP-Sasakian manifold satisfying $divP = 0$. Finally, we construct an example of a 3-dimensional LP-Sasakian manifold which verifies our result.

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1 Introduction

In 1989 Matsumoto [7] introduced the notion of Lorentzian para-Sasakian manifolds. Then Mihai and Rosca [10] defined the same notion independently and they obtained several results in this manifold. LP-Sasakian manifolds have also been studied by Matsumoto and Mihai [8], Matsumoto, Mihai and Rosca [9], Mihai, Shaikh and De [11], De and Shaikh ([2],[4]), Ozgur [13] and many others. After the conformal curvature tensor the projective curvature tensor is an important tensor from the differential geometric point of view. Let M be an $n-$dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighborhood of M and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 3$, M is locally projectively flat if and only if the well known projective curvature tensor P vanishes. Here P is defined by [12]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\}, \tag{1.1}$$

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for $X, Y, Z\varepsilon T(M)$, where R is the curvature tensor and S is the Ricci tensor. In fact, M is projectively flat (that is, $P = 0$) if and only if the manifold is of constant curvature (pp. 84-85 of [16]). Thus, the projective curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.
A Riemannian or a semi-Riemannian manifold is said to be semi-symmetric ([14],[6]) if $R(X, Y).R = 0$, where R is the Riemannian curvature tensor and $R(X, Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y. If a Riemannian manifold satisfies $R(X, Y).P = 0$, then the manifold is said to be projectively semi-symmetric manifold.

Motivated by the above works we study some properties of projective curvature tensor in LP-Sasakian manifolds.

An LP-Sasakian manifold is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_X R)(Y, Z)W) = 0, \tag{1.2}$$

for all vector fields X, Y, Z, W orthogonal to ξ. This notion was introduced for Sasakian manifolds by Takahashi [15]. Later in [1], Blair, Koufogiorgos and Sharma studied locally ϕ-symmetric contact metric manifolds. In (1.2), if X, Y, Z and W are not horizontal vectors then we call the manifold globally ϕ-symmetric.

An LP-Sasakian manifold is said to be ϕ-Ricci symmetric if the Ricci operator Q satisfies

$$\phi^2(\nabla_X Q)(Y) = 0, \tag{1.3}$$

for all vector fields $X, Y \in T(M)$ and the Ricci operator Q is defined by $S(X, Y) = g(QX, Y)$, where S is the Ricci tensor. If X, Y are orthogonal to ξ, then the manifold is said to be locally ϕ-Ricci symmetric. From the definition it follows that ϕ-symmetric implies ϕ-Ricci symmetric, but the converse, is not, in general true. ϕ-Ricci symmetric Sasakian manifolds have been studied by De and Sarkar [3].

Again an LP-Sasakian manifold is called Einstein if the Ricci tensor S is of the form $S = \lambda g$, where λ is a constant.

The paper is organized as follows:
In section 2, some preliminary results are recalled. After preliminaries, we study LP-Sasakian manifolds satisfying $P.R = 0$ and $R.P = 0$. Section 4 deals with LP-Sasakian manifolds satisfying $P.S = 0$. In the next section, we prove that an n-dimensional LP-Sasakian manifold is ϕ-Ricci symmetric if and only if it is an Einstein manifold. Next we study 3-dimensional LP-Sasakian manifold satisfying $divP = 0$ and prove that in that case the manifold is a space form. Finally, we construct some examples of LP-Sasakian manifold which verifies our result.

2 Preliminaries

Let M^n be an n-dimensional differentiable manifold endowed with a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g of type $(0,2)$ such that for each point $p \,\varepsilon M$, the tensor g_p: $T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature $(-,+,+,.....,+)$, where T_pM denotes the tangent space of M at p and $\mathbb{R}$ is the real number space which satisfies

$$\phi^2(X) = X + \eta(X)\xi, \eta(\xi) = -1, \tag{2.1}$$

$$g(X,\xi) = \eta(X), g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{2.2}$$

for all vector fields X, Y. Then such a structure (ϕ, ξ, η, g) is termed as Lorentzian almost paracontact structure and the manifold M^n with the structure (ϕ, ξ, η, g) is called Lorentzian almost paracontact manifold [7]. In the Lorentzian almost paracontact manifold M^n, the following relations hold [7] :

$$\phi\xi = 0, \eta(\phi X) = 0, \tag{2.3}$$

$$\Omega(X,Y) = \Omega(Y,X), \tag{2.4}$$

where $\Omega(X,Y) = g(X, \phi Y)$.

Let $\{e_i\}$ be an orthonormal basis such that $e_1 = \xi$. Then the Ricci tensor S and the scalar curvature r are defined by

$$S(X,Y) = \sum_{i=1}^{n} \epsilon_i g(R(e_i, X)Y, e_i)$$

and

$$r = \sum_{i=1}^{n} \epsilon_i S(e_i, e_i),$$

where we put $\epsilon_i = g(e_i, e_i)$, that is, $\epsilon_1 = -1, \epsilon_2 = \cdots = \epsilon_n = 1$.

A Lorentzian almost paracontact manifold M^n equipped with the structure (ϕ, ξ, η, g) is called Lorentzian paracontact manifold if

$$\Omega(X,Y) = \frac{1}{2}\{(\nabla_X \eta)Y + (\nabla_Y \eta)X\}.$$

A Lorentzian almost paracontact manifold M^n equipped with the structure (ϕ, ξ, η, g) is called an LP-Sasakian manifold [7] if

$$(\nabla_X \phi)Y = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X.$$

In an LP-Sasakian manifold the 1- form η is closed. Also in [7], it is proved that if an n- dimensional Lorentzian manifold (M^n, g) admits a timelike unit vector field ξ such that the 1- form η associated to ξ is closed and satisfies

$$(\nabla_X \nabla_Y \eta)Z = g(X,Y)\eta(Z) + g(X,Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z),$$

then M^n admits an LP-Sasakian structure. Further, on such an LP-Sasakian manifold M^n (ϕ, ξ, η, g), the following relations hold [7]:

$$\eta(R(X,Y)Z) = [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.5}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{2.6}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{2.7}$$

$$R(X,Y)\xi = [\eta(Y)X - \eta(X)Y], \tag{2.8}$$

$$R(\xi, X)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.9}$$

$$(\nabla_X \phi)(Y) = [g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X], \tag{2.10}$$

for all vector fields X, Y, Z, where R, S denote respectively the curvature tensor and the Ricci tensor of the manifold. Also since the vector field η is closed in an LP-Sasakian manifold, we have ([8],[7])

$$(\nabla_X \eta)Y = \Omega(X,Y), \tag{2.11}$$

$$\Omega(X,\xi) = 0, \tag{2.12}$$

$$\nabla_X \xi = \phi X, \tag{2.13}$$

for any vector field X and Y.
We now give some examples of LP-Sasakian manifolds both in odd and even dimensions.

Example 1: [9] Let $\mathbb{R}^5$ be the 5- dimensional real number space with a coordinate system (x, y, z, t, s). Denoting

$$\eta = ds - ydx - tdz, \ \ \xi = \frac{\partial}{\partial s}, \ \ g = \eta \otimes \eta - (dx)^2 - (dy)^2 - (dz)^2 - (dt)^2$$

and

$$\phi(\frac{\partial}{\partial x}) = -\frac{\partial}{\partial x} - y\frac{\partial}{\partial s}, \quad \phi(\frac{\partial}{\partial y}) = -\frac{\partial}{\partial y},$$

$$\phi(\frac{\partial}{\partial z}) = -\frac{\partial}{\partial z} - t\frac{\partial}{\partial s}, \quad \phi(\frac{\partial}{\partial t}) = -\frac{\partial}{\partial t}, \quad \phi(\frac{\partial}{\partial s}) = 0,$$

the structure (ϕ, ξ, η, g) becomes an LP-Sasakian structure on $\mathbb{R}^5$. The metric tensor g can be expressed by the matrix

$$g = \begin{pmatrix} 1+y^2 & 0 & ty & 0 & -y \\ 0 & -1 & 0 & 0 & 0 \\ ty & 0 & -1+t^2 & 0 & -t \\ 0 & 0 & 0 & -1 & 0 \\ -y & 0 & -t & 0 & 1 \end{pmatrix}.$$

Example 2: Let $\mathbb{R}^4$ be the 4- dimensional real number space with a coordinate system (x, y, z, t). In $\mathbb{R}^4$ we define

$$\eta = dt - ydz - dx, \;\; \xi = \frac{\partial}{\partial t},$$

$$\begin{aligned} g &= e^{2t}(dx)^2 + e^{2t}(dy)^2 + (e^{2t} + y^2)(dz)^2 \\ &\quad + ydz \otimes dx + ydx \otimes dz - ydz \otimes dt - ydt \otimes dz - \eta \otimes \eta, \end{aligned}$$

and

$$\phi(\frac{\partial}{\partial x}) = \frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \quad \phi(\frac{\partial}{\partial y}) = \frac{\partial}{\partial y},$$
$$\phi(\frac{\partial}{\partial z}) = \frac{\partial}{\partial z}, \quad \phi(\frac{\partial}{\partial t}) = 0.$$

Then it can be seen that the structure (ϕ, ξ, η, g) becomes an LP-Sasakian structure on $\mathbb{R}^4$. The metric g can be expressed by

$$g = \begin{pmatrix} e^{2t}-1 & 0 & 0 & 1 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & e^{2t} & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

3 LP-Sasakian manifolds satisfying $P.R = 0$

In view of (1.1) the projective curvature tensor of an n-dimensional LP-Sasakian manifold is given by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y]. \tag{3.1}$$

Now from the above equation with the help of (2.6) and (2.9) we get

$$P(\xi, V)\xi = 0 = P(V, \xi)\xi. \tag{3.2}$$

In this section first we study LP-Sasakian manifolds satisfying

$$(P(X,Y).R)(U,V)W = 0. \tag{3.3}$$

Substituting $Y = \xi$ in (3.3) we have

$$\begin{aligned} (P(X,\xi).R)(U,V)W &= P(X,\xi)R(U,V)W - R(P(X,\xi)U,V)W \\ &\quad - R(U,P(X,\xi)V)W - R(U,V)P(X,\xi)W \end{aligned} \tag{3.4}$$

Putting $U = W = \xi$ in (3.4), we get

$$\begin{aligned}(P(X,\xi).R)(\xi,V)\xi &= P(X,\xi)R(\xi,V)\xi - R(P(X,\xi)\xi,V)\xi \\ &\quad -R(\xi,P(X,\xi)V)\xi - R(\xi,V)P(X,\xi)\xi \qquad (3.5)\end{aligned}$$

Now,

$$\begin{aligned}P(X,\xi)R(\xi,V)\xi &= P(X,\xi)(V - \eta(V)\xi) \\ &= P(X,\xi)V - \eta(V)P(X,\xi)\xi \\ &= P(X,\xi)V. \qquad (3.6)\end{aligned}$$

$$R(P(X,\xi)\xi,V)\xi = 0. \qquad (3.7)$$

$$\begin{aligned}R(\xi,P(X,\xi)V)\xi &= P(X,\xi)V - g(P(X,\xi)V,\xi)\xi \\ &= P(X,\xi)V - g(X,V)\xi + \frac{1}{n-1}S(X,V)\xi. \qquad (3.8)\end{aligned}$$

$$R(\xi,V)P(X,\xi)\xi = 0. \qquad (3.9)$$

Using (3.6), (3.7), (3.8)and (3.9) in (3.5) we have

$$P(X,\xi)V - P(X,\xi)V + g(X,V)\xi - \frac{1}{n-1}S(X,V)\xi = 0. \qquad (3.10)$$

Taking inner product of (3.10) by ξ we obtain

$$S(X,V) = (n-1)g(X,V). \qquad (3.11)$$

Therefore the manifold is an Einstein manifold. Thus we can state the following:

Theorem 3.1. *An LP-Sasakian manifold satisfying $P.R = 0$ is an Einstein manifold.*

Next we study LP-Sasakian manifolds satisfying

$$(R(X,Y).P)(U,V)W = 0 \qquad (3.12)$$

Now substituting $Y = \xi$ in (3.12) we have

$$\begin{aligned}(R(X,\xi).P)(U,V)W &= R(X,\xi)P(U,V)W - P(R(X,\xi)U,V)W \\ &\quad -P(U,R(X,\xi)V)W - P(U,V)R(X,\xi)W. \qquad (3.13)\end{aligned}$$

Putting $U = W = \xi$ in (3.13) we have

$$\begin{aligned}(R(X,\xi).P)(\xi,V)\xi &= R(X,\xi)P(\xi,V)\xi - P(R(X,\xi)\xi,V)\xi \\ &\quad -P(\xi,R(X,\xi)V)\xi - P(\xi,V)R(X,\xi)\xi. \qquad (3.14)\end{aligned}$$

From (3.2) we obtain

$$R(X,\xi)P(\xi,V)\xi = 0 = P(\xi,R(X,\xi)V)\xi. \tag{3.15}$$

Again

$$\begin{aligned} P(R(X,\xi)\xi,V)\xi &= P(\eta(X)\xi - X,V)\xi \\ &= -P(X,V)\xi + \eta(X)P(\xi,V)\xi \\ &= -P(X,V)\xi. \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} P(\xi,V)R(X,\xi)\xi &= P(\xi,V)(\eta(X)\xi - X) \\ &= \eta(X)P(\xi,V)\xi - P(\xi,V)X \\ &= -P(\xi,V)X. \end{aligned} \tag{3.17}$$

Using (3.15), (3.16), (3.17) in (3.14) we have

$$P(X,V)\xi + P(\xi,V)X = 0. \tag{3.18}$$

Taking the inner product of (3.18) by ξ we obtain

$$S(X,V) = (n-1)g(X,V). \tag{3.19}$$

Therefore the manifold is an Einstein manifold. Thus we can state the following:

Theorem 3.2. *An LP-Sasakian manifold satisfying $R.P = 0$ is an Einstein manifold.*

4 LP-Sasakian manifolds satisfying $P.S = 0$

In this section we study LP-Sasakian manifold satisfying $P.S = 0$. Therefore

$$(P(X,Y).S)(U,V) = 0. \tag{4.1}$$

This implies

$$S(P(X,Y)U,V) + S(U,P(X,Y)V) = 0. \tag{4.2}$$

Putting $Y = U = \xi$ in (4.2) we obtain

$$S(P(X,\xi)\xi,V) + S(\xi,P(X,\xi)V) = 0. \tag{4.3}$$

Using (3.2) in (4.3), we have

$$S(\xi,P(X,\xi)V) = 0. \tag{4.4}$$

This implies

$$(n-1)g(R(X,\xi)V - \frac{1}{n-1}[S(\xi,V)X - S(X,V)\xi],\xi) = 0. \tag{4.5}$$

It follows that

$$g(R(X,\xi)V,\xi) - \frac{1}{n-1}[(n-1)\eta(V)\eta(X) - S(X,V)] = 0. \tag{4.6}$$

Therefore

$$S(X,V) = (n-1)g(X,V). \tag{4.7}$$

Hence the manifold is an Einstein manifold.
Conversely, the manifold is an Einstein manifold, that is, $S(X,V) = \lambda g(X,V)$.

$$\begin{aligned}(P(X,Y).S)(U,V) &= S(P(X,Y)U,V) + S(U,P(X,Y)V) \\ &= \lambda[g(P(X,Y)U,V) + g(U,P(X,Y)V]. \end{aligned} \tag{4.8}$$

Since

$$g(P(X,Y)U,V) = -g(P(X,Y)V,U). \tag{4.9}$$

Using (4.9) in (4.8) we have

$$(P(X,Y).S)(U,V) = 0. \tag{4.10}$$

Thus we can state the following:

Theorem 4.1. *An LP-Sasakian manifold satisfies $P.S = 0$ if and only if it is an Einstein manifold.*

5 ϕ-Ricci symmetric LP-Sasakian manifolds

Proposition 5.1. *An n-dimensional ϕ-Ricci symmetric LP-Sasakian manifold is an Einstein manifold.*

Proof. Let us assume that the manifold is ϕ-Ricci symmetric. Then we have

$$\phi^2(\nabla_X Q)(Y) = 0.$$

Using (2.1) in the above, we get

$$(\nabla_X Q)(Y) + \eta((\nabla_X Q)(Y))\xi = 0. \tag{5.1}$$

From (5.1), it follows that

$$g((\nabla_X Q)(Y), Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0, \tag{5.2}$$

which on simplifying gives

$$g(\nabla_X Q(Y), Z) - S(\nabla_X Y, Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0. \tag{5.3}$$

Replacing Y by ξ in (5.3), we get

$$g(\nabla_X Q(\xi), Z) - S(\nabla_X \xi, Z) + \eta((\nabla_X Q)(\xi))\eta(Z) = 0. \tag{5.4}$$

By using (2.6) and (2.13)in (5.4), we obtain

$$(n-1)g(\phi X, Z) - S(\phi X, Z) + \eta((\nabla_X Q)(\xi))\eta(Z) = 0. \tag{5.5}$$

Replacing Z by ϕZ in (5.5), we have

$$S(\phi X, \phi Z) = (n-1)g(\phi X, \phi Z). \tag{5.6}$$

In view of (2.2) and (2.7), (5.6) becomes

$$S(X, Z) = (n-1)g(X, Z),$$

which implies that the manifold is an Einstein manifold.

□

Now, since a ϕ-symmetric manifold is ϕ-Ricci symmetric, we have

Corollary 5.1 *A ϕ-symmetric LP-Sasakian manifold is an Einstein manifold.*

Proposition 5.2. *If an n-dimensional LP-Sasakian manifold is an Einstein manifold, then it is ϕ-Ricci symmetric.*

Proof. Let us suppose that the manifold is an Einstein manifold. Then

$$S(X, Y) = \alpha g(X, Y),$$

where $S(X, Y) = g(QX, Y)$ and α is a constant. Hence $QX = \alpha X$. So, we have

$$\phi^2(\nabla_X Q)(Y) = 0.$$

This completes the proof. □

In view of Proposition 5.1 and Proposition 5.2, we have

Theorem 5.1. *An n-dimensional LP-Sasakian manifold is ϕ-Ricci symmetric if and only if it is an Einstein manifold.*

6 3-dimensional LP-Sasakian manifolds

Let us consider a 3-dimensional LP-Sasakian manifold. It is known that the conformal curvature tensor vanishes identically in the 3-dimensional Riemannian manifold. Thus we find

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &\quad -\frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{6.1}$$

where Q is the Ricci operator, that is, $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold.
Putting $Z = \xi$ in (6.1) and using (2.8) we have

$$\eta(Y)QX - \eta(X)QY = (\frac{r}{2} - 1)[\eta(Y)X - \eta(X)Y]. \tag{6.2}$$

Putting $Y = \xi$ in (6.2) and using (2.1) and (2.6), we get

$$QX = \frac{1}{2}[(r-2)X + (r-6)\eta(X)\xi], \tag{6.3}$$

that is,

$$S(X,Y) = \frac{1}{2}[(r-2)g(X,Y) + (r-6)\eta(X)\eta(Y)]. \tag{6.4}$$

An LP-Sasakian manifold is said to be a space form if the manifold is a space of constant curvature.
Lemma 6.1 *A 3-dimensional LP-Sasakian manifold is a space form if and only if the scalar curvature $r = 6$.*

Proof. Using (6.3) in (6.1), we get

$$\begin{aligned} R(X,Y)Z &= (\frac{r-4}{2})[g(Y,Z)X - g(X,Z)Y] + (\frac{r-6}{2})[g(Y,Z)\eta(X)\xi \\ &\quad - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{6.5}$$

From (6.5), the Lemma is obvious. □

Let M be a 3-dimensional LP-Sasakian manifold with conservative projective curvature tensor [5], that is, $divP = 0$. Then its Ricci tensor is given by

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z). \tag{6.6}$$

From this we obtain r =constant.
From (6.4) we have

$$\begin{aligned} (\nabla_X S)(Y,Z) &= \frac{1}{2}[dr(X)\{g(Y,Z) + \eta(Y)\eta(Z)\}. \\ &\quad +(r-6)\{\Omega(Y,X)\eta(Z) + \Omega(Z,X)\eta(Y)\}]. \end{aligned} \tag{6.7}$$

Using (6.7) we get from (6.6)

$$dr(X)[\frac{1}{2}g(Y,Z)+\eta(Y)\eta(Z)]-dr(Y)[\frac{1}{2}g(X,Z)+\eta(X)\eta(Z)]$$
$$+(r-6)\{\Omega(Z,X)\eta(Y)-\Omega(Z,Y)\eta(X)\}=0. \quad (6.8)$$

Taking a frame field and contracting over Y and Z, we get

$$dr(X)=(r-6)\psi\eta(X), \quad (6.9)$$

where $\psi=\sum_{i=1}^{3}\Omega(e_i,e_i)=trace\phi$.
If we assume that $\psi=trace\phi\neq 0$, that is, ξ is not harmonic, then $r=6$. So in view of Lemma 6.1 we state the following:

Theorem 6.1. *A 3-dimensional LP-Sasakian manifold satisfying* $divP=0$ *is a space form, provided the characteristic vector field* ξ *is not harmonic .*

7 Examples

Example 7.1: We consider the 3-dimensional manifold $M=\{(x,y,z)\in\mathbb{R}^3_1\}$, where (x,y,z) are standard coordinates of $\mathbb{R}^3_1$.

The vector fields

$$e_1=e^z\frac{\partial}{\partial y},\quad e_2=e^z(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}),\quad e_3=\frac{\partial}{\partial z}$$

are linearly independent at each point of M.

Let g be the Lorentzian metric defined by

$$g(e_1,e_1)=g(e_2,e_2)=1, g(e_3,e_3)=-1,$$

$$g(e_1,e_2)=g(e_1,e_3)=g(e_2,e_3)=0.$$

Let η be the 1-form defined by $\eta(Z)=g(Z,e_3)$ for any vector field $Z\varepsilon\chi(M)$.

Let ϕ be the $(1,1)$ tensor field defined by

$$\phi(e_1)=-e_1,\quad \phi(e_2)=-e_2,\quad \phi(e_3)=0.$$

Then using the linearity of ϕ and g we have

$$\eta(e_3)=-1,$$

$$\phi^2Z=Z+\eta(Z)e_3,$$

$$g(\phi Z,\phi W)=g(Z,W)+\eta(Z)\eta(W)$$

for any vector fields $Z,W\in\chi(M)$.

Then for $e_3=\xi$, the structure (ϕ,ξ,η,g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g. Then we have

$$[e_1, e_2] = 0 \quad , [e_1, e_3] = -e_1$$

and

$$[e_2, e_3] = -e_2.$$

Taking $e_3 = \xi$ and using Koszul's formula for the Lorentzian metric g, we can easily calculate

$$\nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -e_1,$$
$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_3 = -e_2,$$
$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

From the above it can be easily seen that $M^3(\phi, \xi, \eta, g)$ is an LP-Sasakian manifold. With the help of the above results it can be easily verified that

$$R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = -e_2, \quad R(e_1, e_3)e_3 = -e_1,$$
$$R(e_1, e_2)e_2 = e_1, \quad R(e_2, e_3)e_2 = -e_3, \quad R(e_1, e_3)e_2 = 0,$$
$$R(e_1, e_2)e_1 = -e_2, \quad R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e_1 = -e_3.$$

From the above expressions of the curvature tensor we obtain

$$\begin{aligned} S(e_1, e_1) &= g(R(e_1, e_2)e_2, e_1) - g(R(e_1, e_3)e_3, e_1) \\ &= 2. \end{aligned}$$

Similarly we have

$$S(e_2, e_2) = 2, S(e_3, e_3) = -2$$

and

$$S(e_i, e_j) = 0 (i \neq j).$$

Therefore,

$$r = S(e_1, e_1) + S(e_2, e_2) - S(e_3, e_3) = 6.$$

Therefore Theorem 6.1. is verified.

Example 7.2: Let us consider the 5-dimensional manifold $\tilde{M} = \{(x, y, z, u, v) \in \mathbb{R}^5 : (x, y, z, u, v) \neq (0, 0, 0, 0, 0)\}$, where (x, y, z, u, v) are the standard coordinates in $\mathbb{R}^5$. The vector fields

$$e_1 = e^z \frac{\partial}{\partial x}, \; e_2 = e^{z-ax} \frac{\partial}{\partial y}, \; e_3 = \frac{\partial}{\partial z}, \; e_4 = e^z \frac{\partial}{\partial u}, \; e_5 = e^{z-u} \frac{\partial}{\partial v}$$

are linearly independent at each point of $\tilde{M}$ where a is scalar. Let $\tilde{g}$ be the metric defined by

$$\tilde{g}(e_i, e_j) = 1, \quad for\ i = j \neq 3,$$
$$= 0, \quad for\ i \neq j,$$
$$= -1, \quad for\ i = j = 3.$$

Here i and j runs from 1 to 5. Let η be the 1-form defined by $\eta(Z) = \tilde{g}(Z, e_3)$, for any vector field Z tangent to $\tilde{M}$. Let φ be the $(1, 1)$ tensor field defined by

$$\varphi e_1 = -e_1,\ \varphi e_2 = -e_2,\ \varphi e_3 = 0,\ \varphi e_4 = -e_4,\ \varphi e_5 = -e_5.$$

Then using the linearity property of φ and $\tilde{g}$ we have

$$\eta(e_3) = -1,\ \varphi^2 Z = Z + \eta(Z)e_3$$

for any vector field Z tangent to $\tilde{M}$. Thus for $e_3 = \xi$, $\tilde{M}(\varphi, \xi, \eta, \tilde{g})$ defines an almost para-contact metric manifold. Let $\tilde{\nabla}$ be the Levi-Civita connection on $\tilde{M}$ with respect to the metric $\tilde{g}$. Then we have

$$[e_1, e_2] = -ae^z e_2, \quad [e_1, e_3] = -e_1, \quad [e_1, e_4] = 0, \quad [e_1, e_5] = 0,$$

$$[e_2, e_3] = -e_2, \quad [e_2, e_4] = 0, \quad [e_2, e_5] = o, \quad [e_3, e_4] = e_4,$$

$$[e_3, e_5] = e_5, \quad [e_4, e_5] = -e^z e_5.$$

Taking $e_3 = \xi$ and using Koszul's formula for $\tilde{g}$, it can be easily calculated that

$$\tilde{\nabla}_{e_1} e_1 = e_3, \quad \tilde{\nabla}_{e_1} e_2 = 0, \quad \tilde{\nabla}_{e_1} e_3 = -e_1, \quad \tilde{\nabla}_{e_1} e_4 = 0, \quad \tilde{\nabla}_{e_1} e_5 = 0,$$

$$\tilde{\nabla}_{e_2} e_1 = ae^z e_2, \quad \tilde{\nabla}_{e_2} e_2 = -ae^z e_1 e_3, \quad \tilde{\nabla}_{e_2} e_3 = -e_2, \quad \tilde{\nabla}_{e_2} e_4 = 0, \quad \tilde{\nabla}_{e_2} e_5 = 0,$$

$$\tilde{\nabla}_{e_3} e_1 = 0, \quad \tilde{\nabla}_{e_3} e_2 = 0, \quad \tilde{\nabla}_{e_3} e_3 = 0, \quad \tilde{\nabla}_{e_3} e_4 = 0, \quad \tilde{\nabla}_{e_3} e_5 = 0,$$

$$\tilde{\nabla}_{e_4} e_1 = 0, \quad \tilde{\nabla}_{e_4} e_2 = 0, \quad \tilde{\nabla}_{e_4} e_3 = -e_4, \quad \tilde{\nabla}_{e_4} e_4 = 0, \quad \tilde{\nabla}_{e_4} e_5 = 0,$$

$$\tilde{\nabla}_{e_5} e_1 = 0, \quad \tilde{\nabla}_{e_5} e_2 = 0, \quad \tilde{\nabla}_{e_5} e_3 = -e_5, \quad \tilde{\nabla}_{e_5} e_4 = e^z e_5, \quad \tilde{\nabla}_{e_5} e_5 = e_3 - e^z e_5.$$

From the above calculations, we see the manifold under consideration satisfies $\eta(\xi) = -1$ and $\tilde{\nabla}_X \xi = \varphi X$. Hence, $\tilde{M}$ is an *LP*-Sasakian manifold.

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