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PARALLEL AND PSEUDO-PARALLEL SUBMANIFOLDS IN GENERALIZED SASAKIAN SPACE–FORMS

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Abstract

In this article we study parallel submanifolds in generalized Sasakian space–forms and we find some conditions so that Legendre pseudo–parallel submanifolds of the generalized Sasakian space–forms be totally geodesic.

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1 Introduction

The study of parallel submanifolds of real space forms of constant sectional curvature was made by Ferus [12] and Takeuchi [17] and Kon [14], Nakagawa-Takagi [15] and others for complex space forms. In [16], two theorems concerning reduction of codimensions of parallel submanifolds in Sasakian space–forms were proved. We prove that such a theorem holds for parallel submanifolds of generalized Sasakian space–forms. Pseudo-parallel submanifolds are introduced in [6] and [7] as a generalization of semi-parallel submanifolds in the sense of [10]. The notion of pseudo–parallelism generalizes the notion of semi–parallelism in the same way as pseudo–symmetry (in the sense of [10]) generalizes semi–symmetry. In this article we prove that under certain conditions, Legendre pseudo–parallel submanifolds of generalized Sasakian space–forms are totally geodesic.

2 Preliminaries

We remember some necessary useful notions and results for our next considerations.

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Let $\widetilde{M}$ be a C^∞–differentiable, $2n+1$–dimensional almost contact manifold with the almost contact metric structure (F, ξ, η, g), where F is a $(1,1)$ tensor field, η is a 1-form, g is a Riemannian metric on $\widetilde{M}$, ξ is the Reeb vector field, $\chi(\widetilde{M})$ is the set of all vector fields on $\widetilde{M}$, all these tensors satisfying the following conditions :

$$F^2 = -I + \eta \otimes \xi; \quad \eta(\xi) = 1; \quad g(FX, FY) = g(X, Y) - \eta(X)\eta(Y) \tag{1}$$

for all X, Y in $\chi(\widetilde{M})$.

We consider the Sasaki form Ω on $\widetilde{M}$, given by $\Omega(X, Y) = g(X, FY)$. An integral submanifold M of the contact distribution $\mathcal{D} = \ker \eta$ is an *integral manifold* and such a submanifold is characterized by any of

1. $\eta = 0, \quad d\eta = 0;$
2. $FX \in \chi^\perp(M)$ for all X in $\chi(M)$.

Another property valid on these submanifolds and useful for our considerations is the following

Proposition 2.1. *Let M be an integral submanifold of the almost contact metric manifold $\widetilde{M}$. Then:*
i) $A_\xi = 0$*;*
ii) $A_{FX}Y = A_{FY}X$*;*
iii) $A_{FY}X = -[Fh(X, Y)]^T$*;*
iv) $\nabla^\perp_X(FY) = g(X, Y)\xi + F\nabla_X Y + [Fh(X, Y)]^\perp$*;*
v) $\nabla^\perp_X \xi = -FX$

for all X, Y in $\chi(M)$.

A maximal integral submanifold M of an almost contact metric manifold $\widetilde{M}$ is a *Legendre* submanifold.

Denote by

$$Osc_x M = T_x M \oplus N^1_x(M) \tag{2}$$

the first osculating space of the submanifold M at x, where M is a submanifold of a Riemannian manifold $\widetilde{M}$, x is a point of M, $N^1_x(M)$ is the subspace of $T_x(\widetilde{M})$ generated by $h(X, Y)$, $X, Y \in T_x M$.

If $\widetilde{M}$ is a symmetric Riemannian manifold (i.e $\widetilde{\nabla}\widetilde{R} = 0$) and $x \in \widetilde{M}$, then a subspace V of $T_x(M)$ is a *Lee triple system* if and only if $\widetilde{R}(X, Y)Z \in V$, for all $X, Y, Z \in V$. Here $\widetilde{R}$ represents the curvature tensor of $\widetilde{M}$. From [13] we have the following Theorem:

Theorem 2.2. *Let M be symmetric Riemannian manifold, x a point of M and $V \subset T_x M$ a Lee triple system. Then there exists an unique complete totally geodesic submanifold M^* of M with the property that $x \in M^*$ and $T_x M^* = V$.*

Given an almost contact metric manifold $\widetilde{M}$, we say that $\widetilde{M}$ is a *generalized Sasakian space–form* [2], if there exit three functions f_1, f_2, f_3 on $\widetilde{M}$ such that

$$\begin{aligned} \widetilde{R}(X,Y)Z &= f_1[g(Y,Z)X - g(X,Z)Y] \\ &+ f_2[g(X,FZ)FY - g(Y,FZ)FX + 2g(X,FY)FZ] \\ &+ f_3[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi] \end{aligned} \tag{3}$$

for any vector fields X, Y, Z on $\widetilde{M}$. In such a case, we will write $\widetilde{M}(f_1, f_2, f_3)$.

3 Parallel submanifolds of generalized Sasakian space–forms

A submanifold M of an almost contact metric manifold $\widetilde{M}$ is *parallel* if and only if

$$(\widetilde{\nabla}_X h)(Y,Z) = 0 \tag{4}$$

for X, Y, Z vector fields on M, where

$$(\widetilde{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

Proposition 3.1. *Let M be an n–dimensional complete connected orientated parallel submanifold of generalized Sasakian space–form $\widetilde{M}(f_1, f_2, f_3)$, with $n \geq 2$, $f_2 \neq 0$, $f_3 \neq 0$ and $3f_2 + f_3 \neq 0$. Then the Reeb vector field ξ is tangent or normal to M at any point of M.*

Proof. We consider $\xi = \xi^T \oplus \xi^{\perp}$, where ξ^T and $\xi^{\perp}$ represent the tangent, respectively, the normal component of ξ. From (3) we have

$$\begin{aligned} & f_2[g(X,FZ)FY - g(Y,FZ)FX + 2g(X,FY)FZ]^{\perp} \\ & \qquad + f_3[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\xi^{\perp} = 0 \end{aligned} \tag{5}$$

for all X, Y, Z in $\chi(\widetilde{M})$.
Suppose that both $\xi^T \neq 0$ and $\xi^{\perp} \neq 0$. Then both $F\xi^T \neq 0$ and $F\xi^{\perp} \neq 0$. Taking $Y = Z = \xi^T$ in (5), we obtain $3f_2 g(X, (F\xi^T)^T)(F\xi^T)^{\perp} = 0$, for all X in $\chi(M)$ and because $f_2 \neq 0$ we have two cases:

Case 1. $(F\xi^T)^{\perp} = 0$. Taking $X = \xi^T$ and $Y = Z = F\xi^T$ in (5) we have $(3f_2 + f_3)[-g(\xi^T, \xi^T) + \eta^2(\xi^T)]^T \eta(\xi^T)\xi^{\perp} = 0$ and then $\xi^{\perp} = 0$ (contradictorily with the fact that $\xi^{\perp} \neq 0$).

Case 2. $(F\xi^T)^T = 0$. Taking $X = Z = \xi^T$ in (5) we obtain $g(\xi^T, \xi^T)Y = g(Y, \xi^T)\xi^T$, that is $n \leq 2$ (contradictorily with the fact that $n \geq 2$) or $\xi^T = 0$ (contradictorily with the fact that $\xi^T \neq 0$). □

Proposition 3.2. *Let M be an n–dimensional connected orientated parallel submanifold of the generalized Sasakian space–form $\widetilde{M}(f_1, f_2, f_3)$, with $f_2 \neq 0$, $f_3 \neq 0$ and $3f_2 + f_3 \neq 0$. If M is tangent to the Reeb vector field ξ, then M is invariant or anti-invariant.*

Proof. Taking $\xi^{\perp} = 0$ and $Y = Z$ in (5) we have $3f_2 g(X, FY)(FY)^{\perp} = 0$, for all X, Y in $\chi(M)$. Because $f_2 \neq 0$ and M is connected , we obtain that $FX \in \chi^{\perp}(M)$ or $FY \in \chi(M)$, that is M is invariant or anti-invariant. □

If $\{X, FX\}$ is a 2–plane of $\widetilde{M}$ or M we denote by $K_{\widetilde{M}}(X, FX)$ and $K_M(X, FX)$ the F–sectional curvatures on $\widetilde{M}$, respectively on M. We have the following Theorem:

Proposition 3.3. *Let M be an invariant submanifold of generalized Sasakian space–form $\widetilde{M}(f_1, f_2, f_3)$. Then M is totally geodesic if and only if*

$$K_{\widetilde{M}}(X, FX) = K_M(X, FX)$$

for any $\{X, FX\}$ 2–plane of M. In this case

$$K_{\widetilde{M}}(X, FX) = K_M(X, FX) = f_1 + 3f_2[\eta^2(X) - 1]^2 - f_3\eta^2(X). \tag{6}$$

Proof. " $\Rightarrow$ " Because M is totally geodesic, using Gauss equation we have $K_{\widetilde{M}}(X, FX) = K_M(X, FX)$, for any $\{X, FX\}$ 2–plane on M. Then from (3) we obtain (6).

" $\Leftarrow$ " From $K_{\widetilde{M}}(X, FX) = K_M(X, FX)$ and Gauss equation we have $\|h(X, FX)\|^2 + \|h(X, X)\|^2 = 0$, for all $X \in \chi(M)$. Then $h(X, X) = 0$ and $h(X, Y) = 0$, for all $X, Y \in \chi(M)$. □

Theorem 3.4. *Let M be an n–dimensional orientated connected symmetric parallel submanifold of the generalized Sasakian space–form $\widetilde{M}(f_1, f_2, f_3)$, with $n \geq 2$, $f_2 \neq 0$, $f_3 \neq 0$ and $3f_2 + f_3 \neq 0$. Suppose that M is tangent to the Reeb vector field.*

i) If M is invariant then M is totally geodesic with F-sectional curvature given by (6)

ii) If M is anti–invariant then there exists an unique complete totally geodesic submanifold M^* of M so that

1. *$x \in M^*$ and $T_xM^* = Osc_xM$, for any $x \in M$;*
2. *M^* is invariant;*
3. $\dim(M^*) = 2\dim(M) - 1$.

Proof. i) results from Proposition 3.3.
ii) Because M is anti–invariant we have that $Fh(X, Y) \in \chi(M)$, for all $X, Y \in \chi(M)$. Now, from (2) and (3) we obtain that Osc_xM is a Lee triple system. Applying Theorem 2.2 we obtain ii) 1.
ii) 2. and ii) 3. result from the fact that, $FT_xM \subseteq N_x^1M$, $F(\xi) = 0$, $FN_x^1M \subseteq T_xM$ and M^* has odd dimension. □

4 Pseudo–parallel submanifolds of generalized Sasakian space–forms

Let M be a submanifold of the Riemannian manifold $\widetilde{M}$. We consider

$$\begin{aligned}(\widetilde{\nabla}_X\widetilde{\nabla}_Y h)(V,W) &= \nabla^{\perp}_X((\widetilde{\nabla}_Y h)(V,W)) - (\widetilde{\nabla}_Y h)(\nabla_X V,W)\\ &- (\widetilde{\nabla}_Y h)(V,\nabla_X W) \end{aligned} \tag{7}$$

and

$$\begin{aligned}(\widetilde{R}\cdot h)(X,Y,V,W) &= (\widetilde{\nabla}_X\widetilde{\nabla}_Y h)(V,W) - (\widetilde{\nabla}_Y\widetilde{\nabla}_X h)(V,W)\\ &- (\widetilde{\nabla}_{[X,Y]} h)(V,W) \end{aligned} \tag{8}$$

for all $X, Y, Z, W \in \chi(M)$.
From (7) and (8) we obtain that

$$\begin{aligned}(\widetilde{R}\cdot h)(X,Y,V,W) &= R^{\perp}(X,Y)h(V,W) - h(R(X,Y)V,W)\\ &- h(V,R(X,Y)W) \end{aligned} \tag{9}$$

where $R^{\perp}(X,Y)\vec{n} = \nabla^{\perp}_X\nabla^{\perp}_Y\vec{n} - \nabla^{\perp}_Y\nabla^{\perp}_X\vec{n} - \nabla^{\perp}_{[X,Y]}\vec{n}$, for any $X, Y \in \chi(M)$ and $\vec{n} \in \chi^{\perp}(M)$.
The submanifold M is *semi-parallel* [10] if

$$(\widetilde{R}\cdot h)(X,Y,V,W) = 0. \tag{10}$$

and M is *pseudo-parallel* [6] if

$$(\widetilde{R}\cdot h)(X,Y,V,W) + \Phi\cdot Q(g,h)(X,Y,V,W) = 0 \tag{11}$$

where Φ is a differential function on $\widetilde{M}$ and

$$\begin{aligned}Q(g,h)(X,Y,V,W) = h((X\wedge Y)V,W) + h(V,(X\wedge Y)W),\\ (X\wedge Y)V = g(Y,V)X - g(X,V)Y\end{aligned}$$

for all X, Y, V, W in $\chi(M)$.

Now, we consider M a Legendre submanifold of $2n+1$–dimensional generalized Sasakian space–form $\widetilde{M}(f_1, f_2, f_3)$. Let
$\{e_1, ..., e_n, e_{n+1} = e_{1^*} = Fe_1, ..., e_{2n} = e_{n^*} = Fe_n, e_{2n+1} = \xi\}$ be a local orthonormal basis on $\widetilde{M}$ so that $\{e_1, ..., e_n\}$ is a local orthonormal basis on M. We consider $i, j, k, l = \overline{1, n}$ and $\alpha, \beta = \overline{n+1, 2n+1}$ and the following decompositions and notations

$$h(e_i, e_j) = h^{\alpha}_{ij}e_{\alpha}; \quad \nabla_{e_i}e_j = \Gamma^l_{ij}e_l; \quad \widetilde{\nabla}_{e_k}e_{\alpha} = \Gamma^i_{k\alpha}e_i + \Gamma^{\beta}_{k\alpha}e_{\beta} \tag{12}$$

$$(\widetilde{\nabla}_{e_k}h)(e_i, e_j) = h^{\alpha}_{ijk}e_{\alpha}; \quad (\widetilde{\nabla}_{e_l}\widetilde{\nabla}_{e_k}h)(e_i, e_j) = h^{\alpha}_{ijkl}e_{\alpha} \tag{13}$$

where Γ^m_{rp} are connection coeficients with $m, r, p = \overline{1, 2n+1}$.

Proposition 4.1. *Let M be a minimal pseudo–parallel Legendre submanifold of the generalized Sasakian space–form $\widetilde{M}(f_1, f_2, f_3)$. Then:*

$$\frac{1}{2}\Delta \|h\|^2 = \Phi n \|h\|^2 + \left\|\widetilde{\nabla} h\right\|^2. \tag{14}$$

Proof. The second fundamental form h is a tensor field of type (1, 2) and $\widetilde{\nabla} h$ is a tensor field of type (1, 3). From (12), (13) and the local expresion of the covariant derivative of a tensor field of type (r, s), we obtain that:

$$\begin{aligned} h^{\alpha}_{ijk} &= g((\widetilde{\nabla}_{e_k} h)(e_i, e_j), e_\alpha) \\ &= e_k(h^{\alpha}_{ij}) + h^{\beta}_{ij}\Gamma^{\alpha}_{k\beta} - \Gamma^{l}_{ki}h^{\alpha}_{lj} - \Gamma^{l}_{kj}h^{\alpha}_{il} \\ &= \widetilde{\nabla}_{e_k} h^{\alpha}_{ij} \end{aligned} \tag{15}$$

and

$$\begin{aligned} h^{\alpha}_{ijkl} &= g((\widetilde{\nabla}_{e_l}\widetilde{\nabla}_{e_k} h)(e_i, e_j), e_\alpha) \\ &= e_l(h^{\beta}_{ijk}) + \Gamma^{\alpha}_{l\beta}h^{\beta}_{ijk} - \Gamma^{r}_{li}h^{\alpha}_{rjk} \\ &- \Gamma^{r}_{lj}h^{\alpha}_{irk} - \Gamma^{r}_{lk}h^{\alpha}_{ijr} \\ &= \widetilde{\nabla}_{e_l} h^{\alpha}_{ijk} \\ &= \widetilde{\nabla}_{e_l}\widetilde{\nabla}_{e_k} h^{\alpha}_{ij}. \end{aligned} \tag{16}$$

Moreover,

$$\|h\|^2 = \sum_{i,j=1}^{n}\sum_{\alpha=n+1}^{2n+1}(h^{\alpha}_{ij})^2; \quad \left\|\widetilde{\nabla} h\right\|^2 = \sum_{i,j,k,l}^{n}\sum_{\alpha=n+1}^{2n+1}(h^{\alpha}_{ijkl})^2. \tag{17}$$

From the properties of the Laplacian of a differential function we have $\Delta h^{\alpha}_{ij} = \sum_{k=1}^{n} h^{\alpha}_{ijkk}$ and

$$\frac{1}{2}\Delta \|h\|^2 = \sum_{i,j,k,l}^{n} g((\widetilde{\nabla}_{e_k}\widetilde{\nabla}_{e_k} h)(e_i, e_j), h(e_i, e_j)) + \left\|\widetilde{\nabla} h\right\|^2. \tag{18}$$

Because M is a Legendre submanifold, from (3) we have $R^{\perp}(X, Y)Z = 0$, for all $X, Y, Z \in \chi(M)$ and then using Codazzi equation, we have that $\widetilde{\nabla} h$ is totally symmetric. Now, from the fact that $\widetilde{\nabla} h$ is totally symmetric, we obtain that:

$$(\widetilde{\nabla}_{e_k}\widetilde{\nabla}_{e_k} h)(e_i, e_j) = (\widetilde{\nabla}_{e_k}\widetilde{\nabla}_{e_i} h)(e_k, e_j)$$

$$(\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_k} h)(e_k, e_j) = (\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_k} h)(e_j, e_k).$$

Taking into account these last relations and the fact that M is pseudo–parallel we obtain that

$$\begin{aligned} g((\widetilde{\nabla}_{e_k}\widetilde{\nabla}_{e_k} h)(e_i, e_j), h(e_i, e_j)) &= g((\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_j} h)(e_k, e_k), h(e_i, e_j)) \\ &- \Phi[\delta_{ik} g(h(e_k, e_j), h(e_i, e_j)) \\ &- \delta_{kk} g(h(e_i, e_j), h(e_i, e_j)) \\ &+ \delta_{ij} g(h(e_k, e_k), h(e_i, e_j) \\ &- \delta_{kj} g(h(e_k, e_i), h(e_i, e_j))], \end{aligned} \tag{19}$$

where δ_{ij} are Kronecker symbols. We consider the decomposition $H = \sum_{\alpha=n+1}^{2n+1} H^\alpha e_\alpha$. Then $\|H\|^2 = \sum_{\alpha=n+1}^{2n+1}(H^\alpha)^2$, $H^\alpha = \frac{1}{n}\sum_{k=1}^{n} h_{kk}^\alpha$ and

$$\sum_{i.j.k=1}^{n} g((\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_j}h)(e_k,e_k),h(e_i,e_j)) = n\sum_{i,j=1}^{n}\sum_{\alpha=n+1}^{2n+1} h_{ij}^\alpha \widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_j}H^\alpha. \tag{20}$$

From (18), (19), (20) we have that

$$\begin{aligned}\frac{1}{2}\Delta\|h\|^2 &= \sum_{i,j=1}^{n}\sum_{\alpha=n+1}^{2n+1} h_{ij}^\alpha(\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_j}H^\alpha) \\ &- \Phi[n^2\|H\|^2 - n\|h\|^2] + \left\|\widetilde{\nabla}h\right\|^2 \end{aligned}\tag{21}$$

Finally, because M is minimal we obtain (14). □

In [9], Chern, do Carmo and Kobayashi obtained a formula for the Laplacian of the square of the length of the second fundamental form of a minimal immersion into a locally symmetric space. The same formula also holds for an integral submanifold of a generalized Sasakian space–form, that is:

$$\begin{aligned}\frac{1}{2}\Delta\|h\|^2 &= \left\|\widetilde{\nabla}h\right\|^2 + \sum_{\alpha,\beta} tr(A_{e_\alpha}A_{e_\beta} - A_{e_\beta}A_{e_\alpha})^2 \\ &- \sum_{\alpha,\beta}(trA_{e_\alpha}A_{e_\beta})^2 - 4\sum_{\alpha,\beta,i,j,k}\widetilde{R}_{\beta ij}^\alpha h_{jk}^\alpha h_{ik}^\beta \\ &+ \sum_{\alpha,\beta,i,j,k}\widetilde{R}_{k\beta k}^\alpha h_{ij}^\alpha h_{ij}^\beta \\ &- 2\sum_{\alpha,i,j,k,l}\widetilde{R}_{jkj}^i h_{il}^\alpha h_{kl}^\alpha - 2\sum_{\alpha,i,j,k,l}\widetilde{R}_{jkl}^i h_{il}^\alpha h_{jk}^\alpha \end{aligned}\tag{22}$$

where $\widetilde{R}_{ABC} = \widetilde{R}(e_C,e_B)e_A$, $A,B,C = \overline{1,2n+1}$ and $trA_{\vec{n}}$ represents the trace of the Weingarten operator $A_{\vec{n}}$.

Proposition 4.2. *Let M be a minimal Legendre submanifold of the generalized Sasakian space–form $\widetilde{M}(f_1,f_2,f_3)$. Then:*

$$\begin{aligned}\frac{1}{2}\Delta\|h\|^2 &= \left\|\widetilde{\nabla}h\right\|^2 + (f_2 + nf_1)\|h\|^2 \\ &- \left\{\sum_{\alpha,\beta}\left\|[A_{e_\alpha},A_{e_\beta}]\right\|^2 + \sum_{\alpha,\beta}(trA_{e_\alpha}A_{e_\beta})^2\right\}.\end{aligned}\tag{23}$$

Proof. From Proposition 2.1 we have $h_{jk}^{n+i} = h_{ik}^{n+j}$.
Because M is Legendre and minimal, taking into account (3) we have

$$\sum_{\alpha,\beta,i,j,k}\widetilde{R}_{\beta ij}^\alpha h_{jk}^\alpha h_{ik}^\beta = -f_2\|h\|^2;$$

$$\sum_{\alpha,\beta,i,j,k} \widetilde{R}^{\alpha}_{k\beta k} h^{\alpha}_{ij} h^{\beta}_{ij} = (-3f_2 - nf_1)\,\|h\|^2\,;$$

$$\sum_{\alpha,i,j,k,l} \widetilde{R}^{i}_{jkj} h^{\alpha}_{il} h^{\alpha}_{kl} = f_1(1-n)\,\|h\|^2\,; \sum_{\alpha,i,j,k,l} \widetilde{R}^{i}_{jkl} h^{\alpha}_{il} h^{\alpha}_{jk} = -f_1\,\|h\|^2\,.$$

Moreover, because $g(A_{\vec{n}}X, Y) = g(A_{\vec{n}}Y, X)$ we obtain that

$$tr(A_{e_\alpha}A_{e_\beta} - A_{e_\beta}A_{e_\alpha})^2 = -\left\|[A_{e_\alpha}, A_{e_\beta}]\right\|^2.$$

From these relations and (22) we obtain (23). □

Now, from Proposition 4.1 and Proposition 4.2 it is easy to prove the following Theorem:

Theorem 4.3. *Let M be a minimal pseudo-parallel Legendre submanifold of the generalized Sasakian space–form $\widetilde{M}(f_1, f_2, f_3)$ so that*

$$\Phi n - f_2 - nf_1 \geq 0. \tag{24}$$

Then M is totally geodesic.

We observe that for $f_1 = \frac{c-3}{4}$, $f_2 = f_3 = \frac{c+1}{4}$, $\widetilde{M}(f_1, f_2, f_3)$ is a Kenmotsu space–form and (24) is equivalent with $\Phi - \frac{n(c-3)+c+1}{4} \geq 0$ and for $f_1 = \frac{c+3}{4}$, $f_2 = f_3 = \frac{c-1}{4}$, $\Phi - \frac{n(c+3)+c-1}{4} \geq 0$ we are in the case of Sasakian space–forms [18].

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