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ON MASTROIANNI OPERATORS AND THEIR DURRMEYER TYPE GENERALIZATION

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Abstract

In this paper, we define and study some approximation properties of a Durrmeyer type operators associated with Mastroianni operators.

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1 Introduction

In [5], [6] G. Mastroianni defined and studied a general class of linear positive approximation operators, which was generalized by O. Agratini , B. Della Vechia [1]. Briefly we recall this construction. Let $(\Phi_n)_{n\geq 1}$ be a sequence of real functions defined on $[0,\infty) := \mathbb{R}_+$ which are infinitely differentiable on $\mathbb{R}_+$ and satisfy the following conditions:

(i). $\Phi_n(0) = 1$, $n \in \mathbb{N}$;

(ii). $(-1)^k \Phi_n^{(k)}(x) \geq 0$, for every $n \in \mathbb{N}$, $x \in \mathbb{R}_+$ and $k \in \mathbb{N} \cup \{0\} := \mathbb{N}_0$;

(iii). for each $(n,k) \in \mathbb{N} \times \mathbb{N}_0$ there exists a number $p(n,k) \in \mathbb{N}$ and a function $\alpha_{n,k} \in \mathbb{R}^{\mathbb{R}}$ such that

$$\Phi_n^{(i+k)}(x) = (-1)^k \Phi_{p(n,k)}^{(i)}(x)\alpha_{n,k}(x),\ i \in \mathbb{N}_0,\ x \in \mathbb{R}_+ \tag{1}$$

and

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(iv). $\lim\limits_{n\to\infty}\dfrac{n}{p(n,k)}=\lim\limits_{n\to\infty}\dfrac{\alpha_{n,k}(x)}{n^k}=1.$

Remark 1. *It is easy to see that the next relation is true*

$$\lim_{n\to\infty}\frac{\Phi_n^{(k)}(0)}{n^k}=\lim_{n\to\infty}\frac{(-1)^k\alpha_{n,k}(0)}{n^k}=(-1)^k. \tag{2}$$

The Mastroianni operators $M_n : C_2(\mathbb{R}_+) \longrightarrow C(\mathbb{R}_+)$ are defined by the following formula

$$M_n(f,x)=\sum_{k=0}^{\infty}m_{n,k}(x)f\left(\frac{k}{n}\right) \tag{3}$$

with the basis functions,

$$m_{n,k}(x)=\frac{(-x)^k\Phi_n^{(k)}(x)}{k!} \tag{4}$$

and $C_2(\mathbb{R}_+)=\left\{f\in C(\mathbb{R}_+)|(\exists)\lim\limits_{x\to\infty}\dfrac{|f(x)|}{1+x^2}<\infty\right\}$. The space $C_2(\mathbb{R}_+)$ endowed with the norm $\|f\|_*=\sup\left\{\dfrac{|f(x)|}{1+x^2},\ x\geq 0\right\}$ is a Banach space. For these operators and for the test functions $e_r(x)=x^r$, $r=0,\,1,\,2$ we have

$$\begin{aligned}M_n\left(e_0;x\right)&=\Phi_n(0),\\ M_n\left(e_1;x\right)&=-\frac{\Phi_n'(0)}{n}x,\\ M_n\left(e_2;x\right)&=\frac{\Phi_n''(0)x^2-\Phi_n'(0)x}{n^2}.\end{aligned} \tag{5}$$

Recent results about Durrmeyer type operators [2], [3], [4], [8], in terms of the hypergeometric and confluent hypergeometric functions, have considered the definition of the basis functions, the family's functions defined as:

$$\Phi_{n,c}(x)=\begin{cases}e^{-nx} & ,c=0,\ x\geq 0\\ (1+cx)^{-\frac{n}{c}} & ,c\in\mathbb{N},\ x\geq 0\end{cases}$$

For these functions we have
$\Phi_{n,c}^{(k+1)}(x)=-n\Phi_{n+c,c}^{(k)}(x)$, $n>\max\{0,-c\}$ respectively
$\Phi_{n,c}^{(i+k)}(x)=(-1)^k n_{[k,-c]}\Phi_{n+kc,c}^{(i)}(x)$
where $n_{[k,-c]}=n(n+c)(n+2c)\cdots(n+\overline{k-1}c)$ is the factorial power of the k-th order of n with the increment $-c$ and $n_{[0,-c]}=1$.

So, conditions (iii)-(iv) are true, for $p(n,k)=n+kc$ and $\alpha_{n,k}(x)=n_{[k,-c]}$.

In the next section we propose a Mastroianni –Durrmeyer operator, when the sequence of functions $(\Phi_n)_{n\geq 1}$ satisfy the conditions (i)-(iv) and other supplementary conditions.

2 Main results

We consider the sequence of functions $(\Phi_n)_{n\geq 1}$ which satisfy the conditions (i)-(iv) and the next supplementary conditions, for any $n \in \mathbb{N}$ and $r,\, k \in \mathbb{N}_0$:

(v). $\lim\limits_{x\to\infty} x^r \Phi_n^{(k)}(x) = 0$

(vi). $(\exists) J_{n,k,r} := \int\limits_0^\infty x^r \Phi_n^{(k)}(x)dx < \infty$, $(\exists) J_{n,0,0} := \int\limits_0^\infty \Phi_n(x)dx \neq 0.$

We define the operators of Durrmeyer type associated with Mastroianni operators (3)-(4) for each real value function $f \in \mathbb{R}^{\mathbb{R}_+}$ for which the series exists:

$$DM_n(f;x) = \sum_{k=0}^{\infty} m_{n,k}(x) \frac{\int\limits_0^\infty m_{n,k}(t)f(t)dt}{\int\limits_0^\infty m_{n,k}(t)dt} = \int\limits_0^\infty K_n(t,x)f(t)dt$$

with the kernel $K_n(t,x) = \dfrac{1}{I_{n,0,0}} \sum\limits_{k=0}^{\infty} m_{n,k}(x)m_{n,k}(t)$, $I_{n,0,0} = \int\limits_0^\infty \Phi_n(t)dt \neq 0.$

Lemma 1. *The next identity is true for any $n \in \mathbb{N}$ and $r,\, k \in \mathbb{N}_0$*

$$I_{n,k,r} = \frac{(r+1)_k}{k!} I_{n,0,r},$$

where $I_{n,k,r} := \int\limits_0^\infty t^r m_{n,k}(t)dt = \dfrac{(-1)^k}{k!} J_{n,k,r+k}$ and $(n)_k = n(n+1)(n+2)\cdots(n+k-1) = n_{[k,-1]}$, $(n)_0 = 1$ is the Pochhammer symbol (or the factorial power of the k-th order of n and the increment -1).
So, $(1)_k = k!$, $(2)_k = (k+1)!$.

The proof presupposes an easy computation using the identities:
$I_{n,0,r} = J_{n,0,r} = \int\limits_0^\infty t^r \Phi_n(t)dt$, $r \geq 0$, (the moments of the r-th order reported to Φ_n)
$I_{n,k,0} = \int\limits_0^\infty m_{n,k}(t)dt = \dfrac{(-1)^k}{k!} J_{n,k,k}$,
$J_{n,k,k} = (-1)^k k! J_{n,0,0}$, $k \geq 0$,
$I_{n,k,0} = I_{n,0,0} = J_{n,0,0} = \int\limits_0^\infty \Phi_n(t)dt$,
$I_{n,k,r} = \int\limits_0^\infty t^r m_{n,k}(t)dt = \dfrac{(-1)^k}{k!} \int\limits_0^\infty t^{r+k} \Phi_n^{(k)}(t)dt = \dfrac{(r+1)_k}{k!} \int\limits_0^\infty t^r \Phi_n(t)dt$
$= \dfrac{(r+1)_k}{k!} J_{n,0,r} = \dfrac{(r+1)_k}{k!} I_{n,0,r}$, (the moments of the r-th order reported to $m_{n,k}$).

Lemma 2. *The moments of the operators $DM_n(f;x)$ are given for $e_r(x) = x^r$, $r \in \mathbb{N}_0$ as*

$$DM_n\left(e_r;x\right) = \frac{I_{n,0,r}}{I_{n,0,0}} \sum_{k=0}^{\infty} \frac{(r+1)_k}{k!} m_{n,k}(x).$$

Futher, we have

$$\begin{aligned} DM_n\left(e_0;x\right) &= 1, \\ DM_n\left(e_1;x\right) &= \frac{I_{n,0,1}}{I_{n,0,0}} \left(1 - x\Phi_n'(0)\right), \\ DM_n\left(e_2;x\right) &= \frac{I_{n,0,2}}{2I_{n,0,0}} \left(x^2\Phi_n''(0) - 4x\Phi_n'(0) + 2\right). \end{aligned}$$

Proof. Using lemma (1) we obtain

$$\begin{aligned} DM_n\left(e_r;x\right) &= \frac{1}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) I_{n,k,r} = \frac{I_{n,0,r}}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) \frac{(r+1)_k}{k!}. \\ DM_n\left(e_0;x\right) &= \frac{1}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) I_{n,k,0} = \frac{I_{n,0,0}}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) \frac{(1)_k}{k!} = 1, \\ DM_n\left(e_1;x\right) &= \frac{1}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) I_{n,k,1} = \frac{I_{n,0,1}}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) \frac{(2)_k}{k!} \\ &= \frac{I_{n,0,1}}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) \frac{(k+1)!}{k!} = \frac{I_{n,0,1}}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x)(k+1) \\ &= \frac{nI_{n,0,1}}{I_{n,0,0}} \left(M_n\left(e_1;x\right) + \frac{1}{n}\right) = \frac{nI_{n,0,1}}{I_{n,0,0}} \left(-\frac{\Phi_n'(0)}{n} x + \frac{1}{n}\right) \\ DM_n\left(e_2;x\right) &= \frac{1}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) I_{n,k,2} = \frac{I_{n,0,2}}{I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) \frac{(3)_k}{k!} \\ &= \frac{I_{n,0,2}}{2I_{n,0,0}} \sum_{k=0}^{\infty} m_{n,k}(x) \frac{(k+2)!}{k!} \\ &= \frac{n^2 I_{n,0,2}}{2I_{n,0,0}} \left(M_n\left(e_2;x\right) + \frac{3}{n} M_n\left(e_1;x\right) + \frac{2}{n^2}\right) \\ &= \frac{n^2 I_{n,0,2}}{2I_{n,0,0}} \left(\frac{\Phi_n''(0)x^2 - \Phi_n'(0)x}{n^2} - \frac{3\Phi_n'(0)x}{n^2} + \frac{2}{n^2}\right). \end{aligned}$$

□

Theorem 1. *If $\lim_{n\to\infty} \frac{n^r I_{n,0,r}}{r! I_{n,0,0}} = 1$, $r = 0, 1, 2$, then $\lim_{n\to\infty} DM_n(f;x) = f(x)$, $(\forall) f \in C_2(\mathbb{R}_+)$. The convergence is uniform on each compact $[0,b]$, $b > 0$ and*

$$|DM_n(f;x) - f(x)| \leq 2\omega\left(f, \delta_n(x)\right), \text{ with}$$

$$\delta_n(x) = \left\{ \frac{I_{n,0,2}}{2I_{n,0,0}} \left(x^2 \Phi_n''(0) - 4x\Phi_n'(0) + 2 \right) - 2x \frac{I_{n,0,1}}{I_{n,0,0}} \left(1 - x\Phi_n'(0)\right) + x^2 \right\}^{\frac{1}{2}}.$$

Proof. Because $\lim\limits_{n\to\infty} \dfrac{\Phi_n^{(k)}(0)}{n^k} = \lim\limits_{n\to\infty} \dfrac{(-1)^k \alpha_{n,k}(0)}{n^k} = (-1)^k$ and $\lim\limits_{n\to\infty} \dfrac{n^r I_{n,0,r}}{r! I_{n,0,0}} = 1$, $r = 0, 1, 2$ we have $\lim\limits_{n\to\infty} DM_n(e_r; x) = e_r(x)$, $r = 0, 1, 2$ and so, the Bohmann-Korovkin assure a part of the conclusions of the theorem.

On the other hand, using a result of O. Shisha, B. Mond [7] with the continuity modulus of f, we obtain a quantitative estimation of the remainder of the approximation formula:

$$|DM_n(f; x) - f(x)| \le \left(1 + \delta_n^{-1}(x) \sqrt{DM_n\left((e_1 - xe_0)^2; x \right)} \right) \omega\left(f, \delta_n(x)\right),$$

with $DM_n\left((e_1 - xe_0)^2; x \right) =$
$= \dfrac{I_{n,0,2}}{2I_{n,0,0}} \left(x^2 \Phi_n''(0) - 4x\Phi_n'(0) + 2 \right) - 2x \dfrac{I_{n,0,1}}{I_{n,0,0}} \left(1 - x\Phi_n'(0)\right) + x^2.$ □

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