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$\mathbb{R}$ - COMPLEX HERMITIAN (α, β) -METRICS

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Abstract

In this paper we extend the known results on $\mathbb{R}$ - complex Hermitian Finsler spaces from [4], by the study of three special $\mathbb{R}$ - complex Hermitian Finsler spaces with (α, β)-metrics. We characterize the $\mathbb{R}$ - complex Hermitian Finsler versions of the Kropina metric, Matsumoto metric and another special (α, β)-metric. Moreover, we find the conditions for two of these three $\mathbb{R}$ - complex Hermitian Finsler spaces to be Berwald. Based on [4,5,6,7,8], we write some explicit examples.

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1 Preliminaries

The notion of complex Finsler space appeared for the first time in a paper written by Rizza in 1963, [21], as a generalization of the similar notion from the real case, requiring the homogeneity of the fundamental function with respect to the fibre variables, for any complex scalars λ. The first example comes from the complex hyperbolic geometry and was given by S. Kobayashi in 1975, [14]. The Kobayashi metric has given an impulse to the study of complex Finsler geometry.

A complex Finsler geometry, which contains many interesting results, has been developed in the papers [1, 2, 15, 18, 3, 5, 6, 7, 13, etc.].

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In paper [19], the well-known definition of a complex Finsler space [1, 2,13] was extended, reducing the scalars to $\lambda \in \mathbb{R}$. The outcome was a new class of Finsler space called the $\mathbb{R}$- complex Finsler spaces [19].

The interest for this class of Finsler spaces was stimulated by the fact that the Finsler geometry means, first of all, distance and this refers to curves depending on the real parameter.

In the present paper, following the ideas from real Finsler spaces with (α,β)-metric [10, 16, 17, 22] we introduce the similar notions on $\mathbb{R}-$ complex Finsler spaces.

In this section we keep the general setting from [4,18,19] and subsequently we recall only some needed notions (for more details, see [18]).

Let M be an $n-$ dimensional complex manifold and $z=(z^k)_{k=\overline{1,n}}$ be the complex coordinates in a local chart. The complexified of the real tangent bundle $T_C M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$. The bundle $T'M$ is itself a complex manifold and the local coordinates in a local chart will be denoted by $u=(z^k,\eta^k)_{k=\overline{1,n}}$. These are changed into $(z'^k,\eta'^k)_{k=\overline{1,n}}$ by the rules $z'^k=z'^k(z)$ and $\eta'^k=\frac{\partial z'^k}{\partial z^l}\eta^l$.

A $\mathbb{R}-$ complex Finsler space is a pair (M,F), where F is a continuous function $F:T'M\longrightarrow\mathbb{R}_+$ satisfying the conditions:

i) $L=F^2$ is smooth on $\widetilde{T'M}=T'M\backslash\{0\}$;

ii) $F(z,\eta)\geq 0$ the equality holds if and only if $\eta=0$;

iii) $F(z,\lambda\eta,\bar{z},\lambda\bar{\eta})=|\lambda|\ F(z,\eta,\bar{z},\bar{\eta}),\forall\lambda\in\mathbb{R}$.

The fundamental function L of a $\mathbb{R}-$ complex Finsler space, induces the following tensors:

$$g_{ij}=\frac{\partial^2 L}{\partial\eta^i\partial\eta^j}\ ;\ g_{i\bar{j}}=\frac{\partial^2 L}{\partial\eta^i\partial\bar{\eta}^j}\ ;\ g_{\bar{i}\bar{j}}=\frac{\partial^2 L}{\partial\bar{\eta}^i\partial\bar{\eta}^j}, \tag{1}$$

which satisfy interesting properties, obtained as consequences of the homogeneity condition *iii)*, [19],

$$\frac{\partial L}{\partial\eta^i}\eta^i+\frac{\partial L}{\partial\bar{\eta}^i}\bar{\eta}^i=2L\ ;\ g_{ij}\eta^i+g_{j\bar{i}}\bar{\eta}^i=\frac{\partial L}{\partial\eta^j}; \tag{2}$$

$$2L=g_{ij}\eta^i\eta^j+2g_{i\bar{j}}\eta^i\bar{\eta}^j+g_{\bar{i}\bar{j}}\bar{\eta}^i\bar{\eta}^j;$$

$$\frac{\partial g_{ik}}{\partial\eta^j}\eta^j+\frac{\partial g_{ik}}{\partial\bar{\eta}^j}\bar{\eta}^j=0\ ;\ \frac{\partial g_{i\bar{k}}}{\partial\eta^j}\eta^j+\frac{\partial g_{i\bar{k}}}{\partial\bar{\eta}^j}\bar{\eta}^j=0.$$

An $\mathbb{R}-$ complex Finsler space with $g_{i\bar{j}}(z)$ (or $g_{ij}(z)$) will be called purely Hermitian [4].

Having an $\mathbb{R}-$ complex Finsler space, if we suppose that F satisfies the regularity conditions: $g_{i\bar{j}}$ is nondegenerated, (i.e., $\det(g_{i\bar{j}})\neq 0$, in any $u\in\widetilde{T'M}$), and it defines a positive definite Levi-form for all $z\in M$, then such a class of spaces is called $\mathbb{R}-$ complex Hermitian Finsler space, [19].

Consider the sections of the complexified tangent bundle of $T'M$. Let $VT'M\subset T'(T'M)$ be the vertical bundle, locally spanned by $\{\frac{\partial}{\partial\eta^k}\}$, and $VT''M$ its conjugate. The idea of complex nonlinear connection, briefly (*c.n.c.*), is an instrument in 'linearization' of the geometry of the manifold $T'M$. A (*c.n.c.*) is a supplementary complex subbundle to $VT'M$ in $T'(T'M)$, i.e. $T'(T'M) = HT'M \oplus VT'M$. The horizontal distribution $H_uT'M$ is locally spanned by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z, \eta)$ are the coefficients of (*c.n.c.*).

The pair $\{\delta_k = \frac{\delta}{\delta z^k}, \dot{\partial}_k = \frac{\partial}{\partial \eta^k}\}$ will be called the adapted frame of (*c.n.c.*) which obeys the change rules $\delta_k = \frac{\partial z'^j}{\partial z^k}\delta'_j$ and $\dot{\partial}_k = \frac{\partial z'^j}{\partial z^k}\dot{\partial}'_j$. By conjugation everywhere we have obtained an adapted frame $\{\delta_{\bar{k}}, \dot{\partial}_{\bar{k}}\}$ on $T''_u(T'M)$. The dual adapted bases are $\{dz^k, \delta\eta^k\}$ and $\{d\bar{z}^k, \delta\bar{\eta}^k\}$.

A (*c.n.c.*) related only to the fundamental function of the $\mathbb{R}-$ complex Hermitian Finsler space (M, F), (called Chern-Finsler (*c.n.c.*)), has the following local coefficients

$$N_k^i = g^{\bar{m}i}\frac{\partial^2 L}{\partial z^k \partial \bar{\eta}^m} = g^{\bar{m}i}(\frac{\partial g_{\bar{r}\bar{m}}}{\partial z^k}\bar{\eta}^r + \frac{\partial g_{s\bar{m}}}{\partial z^k}\eta^s). \tag{3}$$

Also, in an $\mathbb{R}-$ complex Hermitian Finsler space, we have recovered the Chern-Finsler connection, [19], which is metrical, of $(1, 0)-$ type, and it is given by

$$L_{jk}^i = g^{\bar{m}i}(\delta_j g_{k\bar{m}}) \ ; \ C_{jk}^i = g^{\bar{m}i}(\dot{\partial}_j g_{k\bar{m}}) \ ; \ L_{j\bar{k}}^i = C_{j\bar{k}}^i = 0, \tag{4}$$

where here and further on δ_k is the frame corresponding to the Chern-Finsler (*c.n.c.*).

Finally we recall from [4] the definition of an $\mathbb{R}-$ complex Hermitian Finsler space with Berwald property: (M, F) is Berwald if the local coefficients L_{jk}^i depend only on the position z. In this case, the local coefficients of the Chern-Finsler (*c.n.c.*) have the particular form

$$N_k^i = L_{jk}^i(z)\eta^j + (\dot{\partial}_{\bar{h}} N_j^i)(z)\bar{\eta}^h.$$

2 $\mathbb{R}-$ complex Hermitian Kropina spaces

We consider $z \in M$, $\eta \in T'_z M$, $\eta = \eta^i \frac{\partial}{\partial z^i}$. An $\mathbb{R}-$ complex Finsler space (M, F) is called $\mathbb{R}-$ complex Kropina space if

$$F = \frac{\alpha^2}{\beta}, \ \beta \neq 0 \tag{5}$$

where

$$\begin{aligned} \alpha^2(z, \eta, \bar{z}, \bar{\eta}) &= Re\{a_{ij}\eta^i\eta^j\} + a_{i\bar{j}}\eta^i\bar{\eta}^j; \\ \beta(z, \eta, \bar{z}, \bar{\eta}) &= Re\{b_i\eta^i\}. \end{aligned} \tag{6}$$

with $a_{ij} = a_{ij}(z)$, $a_{i\bar{j}} = a_{i\bar{j}}(z)$, and $b = b_i(z)dz^i$ is a differential $(1, 0)-$ form. The Kropina function (5) produces two tensor fields g_{ij} and $g_{i\bar{j}}$.

In order to study the $\mathbb{R}-$ complex Hermitian Finsler spaces with Kropina metric, we suppose that $a_{ij} = 0$. Thus, only the tensor field $g_{i\bar{j}}$ is invertible and it is characterized by the following properties:

Proposition 1. *[20] For the $\mathbb{R}-$ complex Hermitian Kropina space, with $a_{ij}=0$, we have*

i)

$$g_{i\bar{j}}=\frac{2F}{\beta}a_{i\bar{j}}+\frac{2}{\beta^2}l_il_{\bar{j}}+\frac{3F^2}{2\beta^2}b_ib_{\bar{j}}+\frac{-2F}{\beta^2}\left(b_{\bar{j}}l_i+b_il_{\bar{j}}\right) \tag{8}$$

or, in the equivalent form

$$g_{i\bar{j}}=\frac{2F}{\beta}a_{i\bar{j}}-\frac{2}{\beta^2}l_il_{\bar{j}}+\frac{F^2}{2\beta^2}b_ib_{\bar{j}}+\frac{1}{F^2}\eta_i\eta_{\bar{j}}; \tag{9}$$

ii)

$$\begin{aligned} g^{\bar{j}k}&=\tfrac{\beta}{2F}a^{\bar{j}k}+\tfrac{\beta(-4\beta+F\omega)}{2F^2N}\eta^{\bar{j}}\eta^k+\tfrac{\beta(-3F\beta+\alpha^2)}{2FN}b^{\bar{j}}b^k+\\ &+\tfrac{\beta(4\beta-\bar{\varepsilon})}{2FN}b^{\bar{j}}\eta^k+\tfrac{\beta(4\beta-\varepsilon)}{2FN}\eta^{\bar{j}}b^k, \end{aligned} \tag{10}$$

where

$$\begin{aligned} &N=|\varepsilon|^2-\alpha^2\omega+3F\beta\omega+8\beta^2-8\beta Re(\varepsilon),\\ &\alpha^2=a_{i\bar{j}}\eta^i\eta^{\bar{j}}=l_{\bar{i}}\eta^{\bar{i}},\ l^j=a^{\bar{i}j}l_{\bar{i}}=\eta^j,\\ &b^k=a^{\bar{j}k}b_{\bar{j}},\ \varepsilon=l_{\bar{i}}b^{\bar{i}},\ \bar{\varepsilon}=b_{\bar{i}}\eta^{\bar{i}},\ \omega=b_{\bar{i}}b^{\bar{i}}. \end{aligned} \tag{11}$$

Proposition 2. *Let (M,F) be an $\mathbb{R}-$ complex Hermitian Kropina space with $a_{ij}=0$. Then we have the following expressions of Chern-Finsler(c.n.c.):*

$$\begin{aligned} N_j^i&=\overset{a}{N_j^i}+\left(\tfrac{2\varepsilon^2-\alpha^2\omega+12\beta^2-10\beta\varepsilon}{FN\beta}\eta^i+\tfrac{-2\varepsilon^2-12\beta^2+11\beta\varepsilon}{N\beta}b^i\right)\tfrac{\partial a_{l\bar{m}}}{\partial z^j}\eta^l\bar{\eta}^m+\\ &+2\left[\tfrac{-2\varepsilon^2-10\beta^2+9\beta\varepsilon}{N\beta}\eta^i+\tfrac{F(3\varepsilon^2+16\beta^2-15\beta\varepsilon)}{N\beta}b^i\right]\left(\overset{a}{\delta_j}\beta\right)+\\ &+\left[\tfrac{F^2}{2\beta}\left(g^{\bar{m}i}-a^{\bar{m}i}\right)\right]\tfrac{\partial l_{\bar{m}}}{\partial z^j}-\tfrac{F^2}{2\beta}g^{\bar{m}i}\tfrac{\partial b_{\bar{m}}}{\partial z^j} \end{aligned} \tag{12}$$

where

$$\overset{a}{N_j^k}:=a^{\bar{m}i}\frac{\partial a_{s\bar{m}}}{\partial z^k}\eta^s;\ \overset{a}{\delta_j}\beta=\frac{1}{2}(\frac{\partial\bar{b}^r}{\partial z^j}l_{\bar{r}}+\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r) \tag{13}$$

Lemma 1. *Let (M,F) be an $\mathbb{R}-$ complex Hermitian Kropina space with $a_{ij}=0$. If (M,F) is Berwald then*

$$\begin{aligned} &2FN\beta\left(N_j^i-\overset{a}{N_j^i}\right)l_i=(26\varepsilon^2\alpha^2-44\alpha^2\beta\varepsilon-2\alpha^4\omega+24\alpha^2\beta^2-\\ &-4F\varepsilon^3)\tfrac{\partial a_{l\bar{m}}}{\partial z^j}\eta^l\bar{\eta}^m+2(-19F\alpha^2\varepsilon^2-20\alpha^4\beta+18\alpha^4\varepsilon+3F^2\varepsilon^3+\\ &+16\alpha^4\beta\varepsilon)\left(\overset{a}{\delta_j}\beta\right)+\left(-8\alpha^4\beta+2\alpha^4\beta\omega+8\alpha^2\beta\varepsilon-\varepsilon^2\alpha^2\right)\bar{\eta}^m+\\ &+\left(-6\alpha^2\beta\varepsilon+\alpha^4\varepsilon+4\alpha^4\beta\right)b^{\bar{m}}\left(\tfrac{\partial l_{\bar{m}}}{\partial z^j}-\tfrac{F}{2}\tfrac{\partial b_{\bar{m}}}{\partial z^j}\right)--\tfrac{F}{2}a^{\bar{m}i}\tfrac{\partial b_{\bar{m}}}{\partial z^j} \end{aligned} \tag{14}$$

$$
\begin{gathered}
2FN\beta\left(N_j^i - \overset{a}{N_j^i}\right) l_i = (28\varepsilon^2\beta - 4\varepsilon^3 - 20\alpha^2\beta\omega + 24\alpha^2\omega\varepsilon + \\
+48\beta^3 - 64\beta^2\varepsilon - -4F\varepsilon^2\omega)\tfrac{\partial a_{l\bar{m}}}{\partial z^j}\eta^l\bar{\eta}^m + \\
2(-26F\beta\varepsilon^2 + 4F\varepsilon^3 - 40F\beta^3 + 56F\beta^2\varepsilon + 3F^2\varepsilon^2\omega - 16\alpha^4\omega - \\
-15F^2\beta\varepsilon\omega)\left(\overset{a}{\delta}_j\beta\right) + [(-16\alpha^2\beta^2 + 8F\beta^2\varepsilon + 4F^2\omega\beta^2 - \\
2F^2\omega\beta\varepsilon = -2\alpha^2\varepsilon\omega)\bar{\eta}^m + (-6F\omega\beta^2 + 2\alpha^4\omega + 16\alpha^2\beta^2 \\
-8F\beta^2\varepsilon - 4F\beta^2 + 2\alpha^2\varepsilon)]\left(\tfrac{\partial l_{\bar{m}}}{\partial z^j} - \tfrac{F}{2}\tfrac{\partial b_{\bar{m}}}{\partial z^j}\right) - -\tfrac{F}{2}a^{\bar{m}i}\tfrac{\partial b_{\bar{m}}}{\partial z^j}
\end{gathered}
$$

1.

At this moment we want to emphasize the difference between this class of spaces and $\mathbb{R}-$ complex Hermitian Randers spaces. In our case we have only rational terms, and for this reason we have the next Corollary

Corollary 1. *Let (M, F) be an $\mathbb{R}-$ complex Hermitian Kropina space with $a_{ij} = 0$. If $\overset{a}{\delta}_j\beta = 0$ and*

$$
\begin{aligned}
&2FN\beta\left(N_j^i - \overset{a}{N_j^i}\right) l_i = (26\varepsilon^2\alpha^2 - 44\alpha^2\beta\varepsilon - 2\alpha^4\omega + 24\alpha^2\beta^2 - \\
&-4F\varepsilon^3)\tfrac{\partial a_{l\bar{m}}}{\partial z^j}\eta^l\bar{\eta}^m \\
&2FN\beta\left(N_j^i - \overset{a}{N_j^i}\right) b_i = (28\varepsilon^2\beta - 4\varepsilon^3 - 20\alpha^2\beta\omega + 24\alpha^2\omega\varepsilon + \\
&48\beta^3 - -64\beta^2\varepsilon - 4F\varepsilon^2\omega)\tfrac{\partial a_{l\bar{m}}}{\partial z^j}\eta^l\bar{\eta}^m
\end{aligned} \tag{15}
$$

then (M,F) is a Berwald space and $N_j^i = \overset{a}{N_j^i}$.

With α, β from [4] and using the same technique, as in the Randers case (exhaustively studied in [4]), we can write a first example of Kropina metric.

Example 1. As in [4], on $M = \mathbb{C}^3$ we set the metric

$$\alpha^2 = e^{z^1+\bar{z}^1}\left|\eta^1\right|^2 + e^{z^2+\bar{z}^2}\left|\eta^2\right|^2 + e^{z^1+\bar{z}^1+z^3+\bar{z}^3}\left|\eta^3\right|^2. \tag{16}$$

and the $(1, 0)$ - differential form

$$\varepsilon = e^{z^2}\eta^2. \tag{17}$$

Then, $2\beta = e^{z^2}\eta^2 + e^{\bar{z}^2}\bar{\eta}^2$ and so, $a_{ij} = 0$, $b_i = b^i = 0$, $(i, j = 1, 3)$, $b_2 = e^{z^2}$, $b^2 = e^{-z^2}$ and $\omega = 1$.

Using (16) and (17), we have

$$F = \frac{e^{z^1+\bar{z}^1}\left|\eta^1\right|^2 + e^{z^2+\bar{z}^2}\left|\eta^2\right|^2 + e^{z^1+\bar{z}^1+z^3+\bar{z}^3}\left|\eta^3\right|^2}{\frac{1}{2}(e^{z^2}\eta^2 + e^{\bar{z}^2}\bar{\eta}^2)}, \tag{18}$$

which is a Hermitian Kropina metric.

3 A special class of $\mathbb{R}-$ complex Hermitian Finsler space with (α, β)-metric

Following the ideas from real case [22], we shall introduce a new class of $\mathbb{R}-$ complex Finsler metrics.

We consider $z \in M$, $\eta \in T'_z M$, $\eta = \eta^i \frac{\partial}{\partial z^i}$. We define the function F on $T'M$,

$$F = \alpha^2 + \varepsilon\beta^2, \;\; \varepsilon = \pm 1, \tag{19}$$

where

$$\begin{aligned} &\alpha^2(z, \eta, \bar{z}, \bar{\eta}) = Re\{a_{ij}\eta^i\eta^j\} + a_{i\bar{j}}\eta^i\bar{\eta}^j; \\ &\beta(z, \eta, \bar{z}, \bar{\eta}) = Re\{b_i\eta^i\}, \end{aligned} \tag{20}$$

with $a_{ij} = a_{ij}(z)$, $a_{i\bar{j}} = a_{i\bar{j}}(z)$, and $b = b_i(z)dz^i$ is a differential $(1,0)-$ form. The function (19) produces two tensor fields g_{ij} and $g_{i\bar{j}}$.

In order to study the $\mathbb{R}-$ complex Hermitian Finsler space with this metric, we suppose that $a_{ij} = 0$. Thus, only tensor field $g_{i\bar{j}}$ is invertible.

Proposition 3. *The fundamental metric tensor of the $\mathbb{R}-$ complex Hermitian Finsler space with the (α, β)-metric: $F(\alpha, \beta) = \sqrt{\alpha^2 + \varepsilon\beta^2}$, $\varepsilon = \pm 1$ is given by*

$$g_{i\bar{j}} = a_{i\bar{j}} + \frac{\varepsilon}{2} b_i b_{\bar{j}} \tag{21}$$

Proof. The invariants of an $\mathbb{R}-$ complex Finsler space (α, β)-metric for this class are: $\delta_0, \delta_1 = \varepsilon\beta, \delta_{-2} = \delta_{-1} = 0, \mu = \frac{\varepsilon}{2}, \varepsilon = \pm 1$. Using them in Theorem 2.1. [7] by direct calculation we have the result. □

The next aim is to find the formulas for the determinant and the inverse of tensor field $g_{i\bar{j}}$. The solution is obtained by the followin Lemma from [6], for an arbitrary non-singular Hermitian matrix $(Q_{i\bar{j}})$.

Lemma 2. *Suppose:*

- *$(Q_{i\bar{j}})$ is a non-singular $n \times n$ complex matrix with inverse $(Q^{\bar{j}i})$;*
- *C_i and $C_{\bar{i}} = \overline{C_i}$, $i = 1, .., n$, are complex numbers;*
- *$C^i = Q^{\bar{j}i} C_{\bar{j}}$ and its conjugates; $C^2 = C^i C_i = \bar{C}^i C_{\bar{i}}$; $H_{i\bar{j}} = Q_{i\bar{j}} \pm C_i C_{\bar{j}}$*

Then

i) $\det(H_{i\bar{j}}) = (1 \pm C^2)\det(Q_{i\bar{j}})$

ii) Whenever $1 \pm C^2 \neq 0$, the matrix $(H_{i\bar{j}})$ is invertible and in this case its inverse is $H^{\bar{j}i} = Q^{\bar{j}i} \mp \frac{1}{1 \pm C^2} C^i C^{\bar{j}}$.

Proposition 4. *For the $\mathbb{R}-$ complex Hermitian Finsler space with the metric $F = \sqrt{\alpha^2 + \varepsilon\beta^2}, \varepsilon = \pm 1$ the determinant and the inverse of the fundamental metric tensor $g_{i\bar{j}}$ are given by*

$$i) \qquad g^{\bar{j}i} = a^{\bar{j}i} + \frac{1}{2 + \bar{\omega}} a^{\bar{k}i} a^{\bar{j}l} b_l b_{\bar{k}}, \tag{22}$$

$$ii) \qquad \det\left(g_{i\bar{j}}\right) = \frac{2 + \bar{\omega}}{2} det(a_{i\bar{j}}), \tag{23}$$

where

$$\omega = b_i b^i,\ b_i = b^{\bar{j}} a_{i\bar{j}},\ b^i = a^{\bar{j}i} b_{\bar{j}}.$$

Proof. Applying Lemma 2 we set $Q_{i\bar{j}} = a_{ij}$ and $C_i = \frac{1}{\sqrt{2}} b_i$. We obtain $Q^{\bar{j}i} = a^{\bar{j}i}$ $C^i = a^{\bar{j}i}, \frac{1}{\sqrt{2}} b_{\bar{j}},\ C^2 = \frac{1}{2}\bar{\omega},\ 1 + C^2 = \frac{2+\bar{\omega}}{2} \neq 0$, We have $H_{i\bar{j}} = g_{i\bar{j}}$. $H^{\bar{j}i} = a^{\bar{j}i} + \frac{2}{2+\bar{\omega}}(a^{\bar{k}i}\frac{1}{\sqrt{2}} b_{\bar{k}})(a^{\bar{j}l}\frac{1}{\sqrt{2}} b_l)$, $det(H_{i\bar{j}}) = (1+C^2) det(a_{i\bar{j}})$. From here, immediately results i) and ii). □

Proposition 5. *Let (M,F) be an* $\mathbb{R}-$ *complex Hermitian space with* $F = \sqrt{\alpha^2 + \varepsilon\beta^2}$ *and* $a_{ij} = 0, \varepsilon = \pm 1$. *Then we have the following expressions of Chern-Finsler (c.n.c.)*

$$N^i_j = \overset{a}{N}{}^i_j + \frac{1+\omega}{2+\omega}\left(\overset{a}{\delta}_j \beta\right) b^i + \frac{1}{2+\omega}\frac{\partial l_{\bar{m}}}{\partial z^j} b^i b^{\bar{m}} + 2\beta g^{\bar{m}i}\frac{\partial b_{\bar{m}}}{\partial z^j}, \tag{24}$$

where

$$\overset{a}{N}{}^i_j = a^{\bar{m}i}\frac{\partial a_{l\bar{m}}}{\partial z^j}\eta^l \ \text{ and } \ \overset{a}{\delta}_j \beta = \frac{1}{2}\left(\frac{\partial \bar{b}^r}{\partial z^j} l_{\bar{r}} + \frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r\right). \tag{25}$$

After a direct calculus, we can prove that,

Proposition 6. *Let* (M,F) *be an* $\mathbb{R}-$ *complex Hermitian space with* $F = \sqrt{\alpha^2 + \varepsilon\beta^2}$ *and* $a_{ij} = 0, \varepsilon = \pm 1$. *If* (M,F) *is Berwald then*

$$\begin{aligned}
(2+\omega)\left(N^i_j - \overset{a}{N}{}^i_j\right) l_i &= (2+\omega)\left(\overset{a}{\delta}_j \beta\right)\bar{\varepsilon} + \bar{\varepsilon}\tfrac{\partial l_{\bar{m}}}{\partial z^j} b^{\bar{m}} + 2\beta\,(2+\omega)\, g^{\bar{m}i}\tfrac{\partial b_{\bar{m}}}{\partial z^j} l_i \\
(2+\omega)\left(N^i_j - \overset{a}{N}{}^i_j\right) b_i &= (2+\omega)\left(\overset{a}{\delta}_j \beta\right)\omega + \omega\tfrac{\partial l_{\bar{m}}}{\partial z^j} b^{\bar{m}} + 2\beta\,(2+\omega)\, g^{\bar{m}i}\tfrac{\partial b_{\bar{m}}}{\partial z^j} b_i.
\end{aligned} \tag{26}$$

Theorem 1. *Let* (M,F) *be an* $\mathbb{R}-$ *complex Hermitian Finsler space, with* $a_{ij} = 0$ *and* $F = \sqrt{\alpha^2 + \varepsilon\beta^2}$, $\varepsilon = \pm 1$. *If* (M,F) *is a Berwald space and* $(N^i_j - \overset{a}{N}{}^i_j) b_i = 0$, *then* $\overset{a}{\delta}_j \beta = 0$ *and* $N^i_j = \overset{a}{N}{}^i_j$.

Example 2. We consider α as in [8], given by

$$\alpha^2(z,\eta) = \frac{|\eta|^2 + \varepsilon\left(|z|^2|\eta|^2 - |<z,\eta>|^2\right)}{(1+\varepsilon|z|^2)^2}, \tag{27}$$

defined over the disk $\Delta^n_r = \left\{z \in \mathbf{C}^n,\ |z| < r,\ \ r = \sqrt{\frac{1}{|\varepsilon|}}\right\}$ if $\varepsilon < 0$, on $\mathbf{C}^n$ if $\varepsilon = 0$ and on the complex projective space $P^n(\mathbf{C})$ if $\varepsilon > 0$, where $|<z,\eta>|^2 = <z,\eta> \overline{<z,\eta>}$. By computation, we obtain $a_{ij} = 0$ and $a_{i\bar{j}} = \frac{1}{1+\varepsilon|z|^2}\left(\delta_{i\bar{j}} - \varepsilon\frac{\bar{z}^i z^j}{1+\varepsilon|z|^2}\right)$ and so, $\alpha^2(z,\eta) = a_{i\bar{j}}(z)\eta^i\bar{\eta}^j$. Thus purely Hermitian metrics which have special properties are determined. They are Kähler with constant holomorphic curvature $\mathcal{K}_\alpha = 4\varepsilon$. Particularly, for $\varepsilon = -1$ we obtain the Bergman metric on the unit disk

$\Delta^n = \Delta_1^n$; for $\varepsilon = 0$ the Euclidean metric on $\mathbf{C}^n$, and for $\varepsilon = 1$ the Fubini-Study metric on $P^n(\mathbf{C})$. Setting $\beta(z,\eta) = Re\frac{<z,\eta>}{1+\varepsilon|z|^2}$, where $b_i = \frac{\bar{z}^i}{1+\varepsilon|z|^2}$,, we obtain some examples of this class of $\mathbb{R}$ - complex Hermitian metrics

$$F_\varepsilon = \frac{|\eta|^2+\varepsilon\left(|z|^2|\eta|^2-|<z,\eta>|^2\right)}{(1+\varepsilon|z|^2)^2} \pm \left(Re\frac{<z,\eta>}{1+\varepsilon|z|^2}\right)^2. \tag{28}$$

4 $\mathbb{R}-$ complex Hermitian Matsumoto spaces

Following the ideas from the real case [10, 16, 17, 22], we shall introduce a new class of $\mathbb{R}-$ complex Finsler space with (α,β)-metric.

We consider $z \in M$, $\eta \in T'_zM$, $\eta = \eta^i\frac{\partial}{\partial z^i}$. An $\mathbb{R}-$ complex Finsler space (M,F) is called $\mathbb{R}-$ complex Matsumoto space if

$$F(\alpha,\beta) = \frac{\alpha^2}{\alpha-\beta}, \quad \alpha \neq \beta, \tag{29}$$

where

$$\begin{aligned} &\alpha^2(z,\eta,\bar{z},\bar{\eta}) = Re\{a_{ij}\eta^i\eta^j\} + a_{i\bar{j}}\eta^i\eta^{\bar{j}};\\ &\beta(z,\eta,\bar{z},\bar{\eta}) = Re\{b_i\eta^i\}, \end{aligned} \tag{30}$$

with $a_{ij} = a_{ij}(z)$, $a_{i\bar{j}} = a_{i\bar{j}}(z)$, and $b = b_i(z)dz^i$ is a $(1,0)-$ differential form.

Taking into account the $2-$homogeneity condition of L :

$$L(\alpha(z,\lambda\eta,\bar{z},\lambda\bar{\eta}),\beta(z,\lambda\eta,\bar{z},\lambda\bar{\eta})) = \lambda^2 L(\alpha(z,\eta,\bar{z},\bar{\eta}),\beta(z,\eta,\bar{z},\bar{\eta})), \lambda \in R_+, \tag{31}$$

we have,

Proposition 7. *In an $\mathbb{R}-$complex Hermitian Matsumoto space the following equalities hold*

$$\begin{aligned} &\alpha L_\alpha + \beta L_\beta = 2L,\ \alpha L_{\alpha\alpha} + \beta L_{\alpha\beta} = L_\alpha,\\ &\alpha L_{\alpha\beta} + \beta L_{\beta\beta} = L_\beta,\ \alpha^2 L_{\alpha\alpha} + 2\alpha\beta L_{\alpha\beta} + \beta^2 L_{\beta\beta} = 2L, \end{aligned} \tag{32}$$

where

$$L_\alpha = \frac{\partial L}{\partial\alpha},\ L_\beta = \frac{\partial L}{\partial\beta},\ L_{\alpha\beta} = \frac{\partial^2 L}{\partial\alpha\partial\beta},\ L_{\alpha\alpha} = \frac{\partial^2 L}{\partial\alpha^2},\ \ L_{\beta\beta} = \frac{\partial^2 L}{\partial\beta^2}. \tag{33}$$

The Matsumoto function (29) produces two tensor fields g_{ij} and $g_{i\bar{j}}$.

In order to study the $\mathbb{R}-$ complex Hermitian Matsumoto spaces, we suppose that $a_{ij} = 0$. Thus, only the tensor field $g_{i\bar{j}}$ is invertible.

Proposition 8. *The fundamental metric tensor field of a $\mathbb{R}-$ complex Hermitian Matsumoto space is given by*

$$g_{i\bar{j}} = \frac{\alpha^3-2\alpha^2\beta}{(\alpha-\beta)^3}a_{i\bar{j}} + \frac{-\alpha\beta+4\beta^2}{2(\alpha-\beta)^4}l_il_{\bar{j}} + \frac{3\alpha^4}{2(\alpha-\beta)^4}b_ib_{\bar{j}} + \frac{\alpha^2(\alpha-4\beta)}{2(\alpha-\beta)^4}(b_{\bar{j}}l_i + b_il_{\bar{j}}) \tag{34}$$

or, in the equivalent form

$$g_{i\bar{j}} = \frac{\alpha^3-2\alpha^2\beta}{(\alpha-\beta)^3}a_{i\bar{j}} + \frac{4\beta-\alpha}{2(\alpha-\beta)^3}l_il_{\bar{j}} + \frac{F}{(\alpha-\beta)(\alpha-2\beta)}b_ib_{\bar{j}} + \frac{\alpha-4\beta}{2F(\alpha-2\beta)}\eta_i\eta_{\bar{j}}. \tag{35}$$

Proof. The invariants of an $\mathbb{R}-$ complex Hermitian Matsumoto space are: $\rho_0 = \frac{\alpha^2(\alpha-2\beta)}{(\alpha-\beta)^3}$, $\rho_1 = \frac{F}{\alpha-\beta}$, $\rho_{-2} = \frac{-\beta(\alpha-4\beta)}{2(\alpha-\beta)^4}$, $\rho_{-1} = \frac{F(\alpha-4\beta)}{2\alpha^2(\alpha-\beta)^2}$, $\mu_0 = \frac{3F}{2(\alpha-\beta)^2}$. Using them in Theorem 2.1. [7] by direct calculation we have the result. □

Proposition 9. *For an* $\mathbb{R}-$ *complex Hermitian Matsumoto space the determinant and the inverse of the fundamental metric tensor* $g_{i\bar{j}}$ *are given by:*

i) $g^{\bar{j}i} = \frac{(\alpha-\beta)^3}{\alpha^2(\alpha-2\beta)} H^{\bar{j}i}$, *with*

$$\begin{aligned} H^{\bar{j}i} = a^{\bar{j}i} + \frac{(\alpha-4\beta)D}{M\bar{M}}\eta^i\eta^{\bar{j}} + \frac{\alpha^2 M}{A} b^i b^{\bar{j}} + \frac{\alpha^2(\alpha-4\beta)\bar{\varepsilon}}{A}\eta^i b^{\bar{j}} + \\ \frac{\alpha^2(\alpha-4\beta)M\varepsilon}{A\bar{M}} b^i\eta^{\bar{j}} + \frac{(\alpha-\beta)^5(\alpha-4\beta)}{L}(a^{\bar{j}i}\eta_{\bar{j}} + \frac{(\alpha-4\beta)P\bar{\mu}}{M\bar{M}}\eta^i + \\ + \frac{\alpha^2 M\bar{\nu}}{A} b^i + \frac{\alpha^2(\alpha-4\beta)\bar{\varepsilon}\bar{\nu}}{A}\eta^i + \frac{\alpha^2(\alpha-4\beta)M\varepsilon\bar{\mu}}{A\bar{M}} b^i)(a^{\bar{j}l}\eta_l + \\ + \frac{(\alpha-4\beta)P\mu}{M\bar{M}}\eta^{\bar{j}} + \frac{\alpha^2 M\nu}{A} b^{\bar{j}} + \frac{\alpha^2(\alpha-4\beta)\bar{\varepsilon}\nu}{A}\eta^{\bar{j}} + \frac{\alpha^2(\alpha-4\beta)M\varepsilon\mu}{A\bar{M}} b^{\bar{j}}), \end{aligned} \tag{36}$$

where

$$\begin{aligned} & M = 2\alpha^3 - 4\alpha^2\beta - \alpha\gamma + 4\beta\gamma, \\ & D = \bar{M} + M[\alpha^2|\varepsilon|^2(\alpha-4\beta)], \\ & L = 2\alpha^6(\alpha-2\beta)^2 + B(\alpha-\beta)^5(\alpha-4\beta), \\ & \mu = \eta^i\eta_i, \ \ \nu = b^i\eta_i, \\ & P = \bar{M} + M\alpha^3|\varepsilon|^2 - 4M\alpha^2|\varepsilon|^2\beta, \end{aligned} \tag{37}$$

$$\begin{aligned} A = (\alpha^2 - 4\alpha\beta + 4\beta^2)(2\alpha^2 - 4\alpha^2\beta - \alpha\gamma + 4\beta\gamma) + \alpha^2(2\omega\alpha^2 - 4\alpha^2\beta\omega - \\ - \alpha\gamma\omega + 4\beta\gamma\omega + \alpha|\varepsilon|^2 - 4\beta|\varepsilon|^2, \end{aligned}$$

$$B = a^{\bar{j}i}\eta_{\bar{j}}\eta_i +$$

$$+ \frac{(\alpha-4\beta)(2\alpha^3 - 4\alpha^2\beta - \alpha\bar{\gamma} + 4\beta\bar{\gamma} + \alpha^2|\varepsilon|^2(\alpha-4\beta)(2\alpha^3 - 4\alpha^2\beta - \alpha\gamma + 4\beta\gamma)}{(2\alpha^3 - 4\alpha^2\beta - \alpha\gamma + 4\beta\gamma)(2\alpha^3 - 4\alpha^2\beta - \alpha\bar{\gamma} + 4\beta\bar{\gamma})} \times$$

$$\times\eta^i\eta_i\eta^{\bar{j}}\eta_{\bar{j}} +$$
$$+ \frac{\alpha^2(2\alpha^3 - 4\alpha^2\beta - \alpha\gamma + 4\beta\gamma)}{A} b^i b^{\bar{j}}\eta_i\eta_{\bar{j}} + \frac{\alpha^2(\alpha-4\beta)(2\alpha^3 - 4\alpha^2\beta - \alpha\gamma + 4\beta\gamma)}{A}$$

$$+ \left(\frac{\bar{\varepsilon}}{2\alpha^3 - 4\alpha^2\beta - \alpha\gamma + 4\beta\gamma}\eta^i b^{\bar{j}}\eta_i\eta_{\bar{j}} + \frac{\varepsilon}{(2\alpha^3 - 4\alpha^2\beta - \alpha\bar{\gamma} + 4\beta\bar{\gamma})} b^i\eta^{\bar{j}}\eta_{\bar{j}}\eta_i\right)$$

ii) $\det\left(g_{i\bar{j}}\right) = \left(\frac{\alpha^2(\alpha-2\beta)}{(\alpha-\beta)^3}\right)^n det(H_{i\bar{j}})$, *with*

$$det(H_{i\bar{j}}) = \frac{2\alpha^6(\alpha-2\beta)^2 + B(\alpha-\beta)^5(\alpha-4\beta)}{4\alpha^8(\alpha-2\beta)^7 M} \cdot A^2 \cdot det(a_{i\bar{j}}),$$

$$\omega = b_i b^i, \ b_i = b^{\bar{j}} a_{i\bar{j}}, \ b^i = a^{\bar{j}i} b_{\bar{j}}.$$

Proof. I. We set: $Q_{\bar{j}i} = a_{\bar{j}i}$ and $C_i = \frac{1}{\alpha}\sqrt{\frac{\alpha-4\beta}{2(\alpha-2\beta)}}l_i$. By applying Lemma 2 we obtain: $Q^{\bar{j}i} = a^{\bar{j}i}$, $C^2 = \frac{\alpha-4\beta}{2\alpha^2(\alpha-2\beta)}\gamma$, $C^i = \frac{1}{\alpha}\sqrt{\frac{\alpha-4\beta}{2(\alpha-2\beta)}}\eta^i$. So the matrix $H_{i\bar{j}} = a_{i\bar{j}} - \frac{\alpha-4\beta}{2\alpha^2(\alpha-2\beta)}l_i l_{\bar{j}}$ is invertible with: $H_{\bar{j}i} = a_{\bar{j}i} + \frac{\alpha-4\beta}{2\alpha^3-4\alpha^2\beta-\alpha\gamma+4\beta\gamma}\eta^i\eta^{\bar{j}}$. II. Now, we consider: $Q_{i\bar{j}} = a_{i\bar{j}} - \frac{\alpha-4\beta}{2\alpha^2(\alpha-2\beta)}l_i l_{\bar{j}}$ and $C_i = \frac{\alpha}{\alpha-2\beta}b_i$. By applying Lemma 2 we obtain this time:

$$C^2 = \frac{\alpha^2}{(\alpha-2\beta)^2}\left(\omega + \frac{(\alpha-4\beta)|\varepsilon|^2}{2\alpha^3-4\alpha^2\beta-\alpha\gamma+4\beta\gamma}\right)$$

It results that the the inverse of:

$$H^{i\bar{j}} = a^{i\bar{j}} - \frac{\alpha-4\beta}{2\alpha^2(\alpha-2\beta)}l_i l_{\bar{j}} + \frac{\alpha^2}{(\alpha-2\beta)^2}b_i b_{\bar{j}}$$

exists and it is:

$$H_{\bar{j}i} = a_{\bar{j}i} + (\alpha-4\beta)\left(\frac{1}{2\alpha^3-4\alpha^2\beta-\alpha\gamma+4\beta\gamma} + \frac{\alpha^2|\varepsilon|^2(\alpha-4\beta)}{2\alpha^3-4\alpha^2\beta-\alpha\bar{\gamma}+4\beta\bar{\gamma}}\right)\eta^i\eta^{\bar{j}} +$$

$$+\frac{\alpha^2(2\alpha^3-4\alpha^2\beta-\alpha\gamma+4\beta\gamma}{A}b^i b^{\bar{j}} +$$

$$+\frac{\alpha^2(\alpha-4\beta)(2\alpha^3-4\alpha^2\beta-\alpha\gamma+4\beta\gamma)}{A}\left(\frac{\bar{\varepsilon}}{2\alpha^3-4\alpha^2\beta-\alpha\gamma+4\beta\gamma}\eta^i b^{\bar{j}} +\right.$$

$$\left.+\frac{\varepsilon}{2\alpha^3-4\alpha^2\beta-\alpha\bar{\gamma}+4\beta\bar{\gamma}}b^i\eta^{\bar{j}}\right).$$

III. Finally we put $Q_{i\bar{j}} = a_{i\bar{j}} - \frac{\alpha-4\beta}{2\alpha^2(\alpha-2\beta)}l_i l_{\bar{j}} + \frac{\alpha^2}{(\alpha-2\beta)^2}b_i b_{\bar{j}}$ and $C_i = \frac{(\alpha-\beta)^2}{\alpha^3(\alpha-2\beta)\sqrt{2}}\sqrt{(\alpha-\beta)(\alpha-4\beta)}\eta_i$. From here we obtain:

$$det(H_{i\bar{j}}) = det\left(a_{i\bar{j}} - \frac{\alpha-4\beta}{2\alpha^2(\alpha-2\beta)}l_i l_{\bar{j}} + \frac{\alpha^2}{(\alpha-2\beta)^2}b_i b_{\bar{j}} + \frac{(\alpha-\beta)^5(\alpha-4\beta)}{2\alpha^6(\alpha-2\beta)^2}\eta_i\eta_{\bar{j}}\right) =$$

$$= \frac{2\alpha^6(\alpha-2\beta)^2 + \beta(\alpha-\beta)^5(\alpha-4\beta)}{2\alpha^6(\alpha-2\beta)^2}\cdot\frac{A}{2\alpha^2(\alpha-2\beta)^3}\cdot\frac{A}{(\alpha-2\beta)^2 M}\cdot det(a_{i\bar{j}})$$

But $g_{i\bar{j}} = \frac{\alpha^2(\alpha-2\beta)}{(\alpha-\beta)^3}H_{i\bar{j}}$, with $H_{i\bar{j}}$ from III. Thus, $g^{\bar{j}i} = \frac{(\alpha-\beta)^3}{\alpha^2(\alpha-2\beta)}H^{\bar{j}i}$. From here, immediately results i) and ii). □

Proposition 10. *Let (M,F) be an* $\mathbb{R}-$ *complex Hermitian Matsumoto space with*

$a_{ij}=0$. *Then we have the following expressions of Chern-Finsler (c.n.c.),*

$$\begin{aligned}N^i_j &= \overset{a}{N}{}^i_j + \frac{1}{2\alpha^2(\alpha-\beta)(\alpha-2\beta)}\{[(\alpha-2\alpha\beta+2\beta^2)+\alpha(Q+U+W)\cdot\\ &\cdot(\alpha(\alpha-2\alpha\beta+2\beta^2)+\bar{\varepsilon})+(S+V)(\bar{\varepsilon}(\alpha-2\alpha\beta+2\beta^2)+\omega\alpha)+\\ &+\alpha(\alpha-\beta)^6(\alpha-4\beta)(\alpha-2\alpha\beta+2\beta^2+\bar{\varepsilon})]\eta^i+[(\alpha-2\alpha\beta+2\beta^2)(R\bar{\varepsilon}+\\ &+T\alpha^2+Z\alpha^2)+\alpha(\alpha-2\alpha\beta+2\beta^2)(R\omega+T\bar{\varepsilon}+Z\bar{\varepsilon})]b^i\}\frac{\partial a_{l\bar{m}}}{\partial z^j}\eta^l\eta^{\bar{m}}+\\ &+\frac{1}{2\alpha^2(\alpha-\beta)(\alpha-2\beta)}\{(2\alpha-2\beta-1+3\alpha^2\bar{\varepsilon})[\alpha^2(S+V+\alpha^2(U+V+Q)+\\ &+(\alpha-\beta)^6(\alpha-4\beta)]\eta^i+\alpha^2[(Z\alpha^2+T\alpha^2+R)(2\alpha-2\beta-1+3\alpha^2\bar{\varepsilon})+\\ &+3\alpha^2]b^i\}(\overset{a}{\delta}_j\beta)+[Q+U+W+\frac{(\alpha-\beta)^7(\alpha-4\beta)}{\alpha^4}]\frac{\partial l_{\bar{m}}}{\partial z^j}\eta^{\bar{m}}\eta^i+\\ &+(S+V)\frac{\partial l_{\bar{m}}}{\partial z^j}b^{\bar{m}}\eta^i+(T+Z)\frac{\partial l_{\bar{m}}}{\partial z^j}\eta^{\bar{m}}b^i+R\frac{\partial l_{\bar{m}}}{\partial z^j}b^{\bar{m}}b^i+\frac{\alpha^2}{(\alpha-2\beta)}g^{\bar{m}i}\frac{\partial b_{\bar{m}}}{\partial z^j},\end{aligned}$$

where

$Q=\frac{(\alpha-4\beta)D}{M\bar{M}},\ R=\frac{\alpha^2M}{A},\ S=\frac{\alpha^2(\alpha-4\beta)\bar{\varepsilon}}{A},\ T=\frac{\alpha^2(\alpha-4\beta)M\bar{\varepsilon}}{AM},$

$U=(\alpha-4\beta)\left(\frac{P^2\mu^2}{M^2}+\frac{\alpha^2\bar{\varepsilon}\bar{\nu}}{A}\right)+\frac{\alpha^2(\alpha-4\beta)^2P\bar{\varepsilon}\mu(\bar{\nu}+\nu)}{A\bar{M}M},$

$V=\frac{\alpha^2(\alpha-4\beta)M}{A}\left(\frac{P\mu}{M\bar{M}}+\frac{\alpha\bar{\varepsilon}\bar{\nu}}{A}\right)\left(\nu+\frac{(\alpha-4\beta)\varepsilon\mu}{\bar{M}}\right),$

$W=(\alpha-4\beta)\left(\frac{P\mu}{M\bar{M}}+\frac{\alpha^2\bar{\varepsilon}\nu}{A}\right)+(\alpha-4\beta)\left(\frac{P\bar{\mu}}{M\bar{M}}+\frac{\alpha^2\bar{\varepsilon}\bar{\nu}}{A}\right),$

$Z=\frac{\alpha^2M}{A}\left(\frac{\bar{\nu}}{A}+\frac{(\alpha-4\beta)\varepsilon\bar{\mu}}{\bar{M}}\right).$

Also,with α and β from [4] we have an example,of Matsumoto metric.

Example 3. As in [4], on $M=\mathbb{C}^2$ we consider the metric

$$\alpha^2=e^{z^1+\bar{z}^1}\left|\eta^1\right|^2+e^{z^2+\bar{z}^2}\left|\eta^2\right|^2$$

and $\varepsilon=e^{z^2}\eta^2$. These imply $a_{ij}=0$, $(i,j=1,2)$, $2\beta=e^{z^2}\eta^2+e^{\bar{z}^2}\bar{\eta}^2$, $b_1=b^1=0$, $b_2=e^{z^2}$, $b^2=e^{-z^2}$ and $\omega=1$. With the above tools we obtain an $\mathbb{R}-$ complex Matsumoto metric

$$F=\frac{e^{z^1+\bar{z}^1}\left|\eta^1\right|^2+e^{z^2+\bar{z}^2}\left|\eta^2\right|^2}{\sqrt{e^{z^1+\bar{z}^1}\left|\eta^1\right|^2+e^{z^2+\bar{z}^2}\left|\eta^2\right|^2}-\frac{1}{2}(e^{z^2}\eta^2+e^{\bar{z}^2}\bar{\eta}^2)},\tag{39}$$

which is an $\mathbb{R}-$ complex Hermitian Matsumoto metric.

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