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GEOMETRIC INTERPRETATION OF THE CURVATURE TENSOR IN MODEL SPACE UNIFIED THEORY OF GRAVITATIONAL AND ELECTROMAGNETIC INTERACTIONS

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Abstract

The authors of [1] suggest that the space velocities of the particles is a four nonholonomic distribution on the manifold of higher dimension. This distribution is given 4-potential of the electromagnetic field. The equation of admissible (horizontal) geodesic for this distribution coincides with the equations of motion of a charged particle of the general theory of relativity. The metric tensor of the Lorentzian signature $(+,-,-,-)$ is defined on the distribution, which allows us to determine causality, as in the general theory of relativity. The authors introduced the covariant derivative (linear connection) and the curvature tensor for distribution. In [2], for any distribution of intrinsic connection, we construct its extension - extended connection. To ask continuation connectivity means to identify some vector field on corresponding distribution. Using convenient coordinate system this field has the form: $\vec{u} = \partial_n + G^a_n \partial_{n+a}$. The purpose of this paper is to find an explicit expression vector field $\vec{u}$ for which the curvature tensor of extended connectivity coincides with the tensor obtained in [1].

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1 Interior and extended connections

Let X be a smooth manifold with contact structure [2]. A coordinate chart $K(x^\alpha)$ $(\alpha, \beta, \gamma = 1, ..., n;\ a, b, c = 1, ..., n-1)$ on the manifold X is called adapted to distribution D if $D^\perp = \text{span}\left(\frac{\partial}{\partial x^n}\right)$.

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Let $P : TX \to D$ be the projection map defined by the decomposition $TX = D \oplus D^{\perp}$ and let $K(x^{\alpha})$ be an adapted coordinate map. Vector fields

$$P(\partial_a) = \vec{e}_a = \partial_a - \Gamma^n_a \partial_n$$

are linearly independent, and linearly generate the system D over the domain of the definition of the coordinate map:

$$D = \mathrm{span}(\vec{e}_a).$$

Thus we have on X the non-holonomic field of bases $(\vec{e}_a, \partial_n)$ and the corresponding field of cobases

$$(dx^a, \theta^n = dx^n + \Gamma^n_a dx^a).$$

It can be checked directly that

$$[\vec{e}_a, \vec{e}_b] = M^n_{ab} \partial_n,$$

where the components M^n_{ab} form the so-called tensor of non-holonomicity [2]. Under the assumption that for all adapted coordinate systems it holds $\vec{\xi} = \partial_n$, the following equality takes place

$$[\vec{e}_a, \vec{e}_b] = 2\omega_{ba} \partial_n,$$

where $\omega = d\eta$. We say also that the basis $\vec{e}_a = \partial_a - \Gamma^n_a \partial_n$ is adapted, as the basis defined by an adapted coordinate map. Note that $\partial_n \Gamma^n_a = 0$.

We call a tensor field defined on an almost contact metric manifold admissible (to the distribution D) if the coordinate form of an admissible tensor field of type (p, q) in an adapted coordinate map looks like

$$t = t^{a_1,\dots,a_p}_{b_1,\dots,b_q} \vec{e}_{a_1} \otimes \dots \otimes \vec{e}_{a_p} \otimes dx^{b_1} \otimes \dots \otimes dx^{b_q}.$$

An intrinsic linear connection on a non-holonomic manifold D is defined in [2] as a map

$$\nabla : \Gamma D \times \Gamma D \to \Gamma D$$

that satisfies the following conditions:

$$\begin{aligned} &1) \quad \nabla_{f_1 \vec{u}_1 + f_2 \vec{u}_2} = f_1 \nabla_{\vec{u}_1} + f_2 \nabla_{\vec{u}_2}; \\ &2) \quad \nabla_{\vec{u}} f \vec{v} = f \nabla_{\vec{u}} \vec{v} + (\vec{u} f) \vec{v}, \end{aligned}$$

where ΓD is the module of admissible vector fields. The Christoffel symbols are defined by the relation

$$\nabla_{\vec{e}_a} \vec{e}_b = \Gamma^c_{ab} \vec{e}_c.$$

The torsion S of the intrinsic linear connection is defined by the formula

$$S(\vec{x}, \vec{y}) = \nabla_{\vec{x}} \vec{y} - \nabla_{\vec{y}} \vec{x} - p[\vec{x}, \vec{y}].$$

Thus with respect to an adapted coordinate system it holds $S^c_{ab} = \Gamma^c_{ab} - \Gamma^c_{ba}$.

The action of an intrinsic linear connection can be extended in a natural way to admissible tensor fields. An important example of an intrinsic linear connection is the intrinsic metric connection that is uniquely defined by the conditions $\nabla g = 0$ and $S = 0$, [2]. With respect to the adapted coordinates it holds $\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\vec{e}_b g_{cd} - \vec{e}_c g_{bd} - \vec{e}_d g_{bc})$.

In the same way as a linear connection on a smooth manifold, an intrinsic connection can be defined by giving a horizontal distribution over the total space of some vector bundle. The role of such a bundle is played by the distribution D. The notion of *a connection over a distribution* was applied to non-holonomic manifolds with admissible Finsler metrics in [2]. It could be said that over a distribution D a connection is given if the distribution $\tilde{D} = \pi_*^{-1}(D)$, where $\pi : D \to X$ is the natural projection, could be decomposed into a direct some of the form

$$\tilde{D} = HD \oplus VD,$$

where VD is the vertical distribution on the total space D.

Let us introduce a structure of a smooth manifold on D. This structure is defined in the following way. To each adapted coordinate map $K(x^\alpha)$ on the manifold X we put in correspondence the coordinate map $\tilde{K}(x^\alpha, x^{n+\alpha})$ on the manifold D, where $x^{n+\alpha}$ are the coordinates of an admissible vector with respect to the basis

$$\vec{e}_a = \partial_a - \Gamma^n_a \partial_n.$$

The constructed over-coordinate map will be called adapted. Thus the assignment of a connection over a distribution is equivalent to the assignment of the object $G^a_b(X^a, X^{n+a})$ such that

$$HD = \mathrm{span}(\vec{\varepsilon}_a),$$

where $\vec{\varepsilon}_a = \partial_a - \Gamma^n_a \partial_n - G^b_a \partial_{n+b}$. If it holds

$$G^a_b(x^a, x^{n+a}) = \Gamma^a_{bc}(x^a) x^{n+c},$$

then the connection over the distribution D is defined by the intrinsic linear connection. In [2] the notion of the prolonged connection was introduced. The prolonged connection can be obtained from an intrinsic connection by the equality

$$TD = \tilde{HD} \oplus VD,$$

where $HD \subset \tilde{HD}$. Essentially, the prolonged connection is a connection in a vector bundle. As it follows from the definition of the extended connection, for its assignment (under the condition that a connection on the distribution is already defined) it is enough to define a vector field on the manifold D that has the following coordinate form: $\vec{u} = \partial_n + G^a_n \partial_{n+a}$. The components of the object G^a_n are transformed as the components of a vector on the base. Setting $G^a_n = 0$, we get an extended connection denoted by ∇^1. The admissible tensor field

$$R(\vec{u}, \vec{v})\vec{w} = \nabla_{\vec{u}} \nabla_{\vec{v}} \vec{w} - \nabla_{\vec{v}} \nabla_{\vec{u}} \vec{w} - \nabla_{p[\vec{u},\vec{v}]} \vec{w} - p[q[\vec{u}, \vec{v}], \vec{w}]$$

is called by Wagner the first Schouten curvature tensor. With respect to the adapted coordinates it holds

$$R^a_{bcd} = 2\vec{e}_{[a}\Gamma^d_{b]c} + 2\Gamma^d_{[a||e||}\Gamma^e_{b]c}.$$

If distribution D does not contain any integrable subdistribution of dimension $n-2$, then the Schouten curvature tensor is zero if and only if the parallel transport of admissible vectors does not depend on the curve. We consider the case where the object G^a_n is not equal to zero. On the basis of physical considerations the connection is constructed in [1], its curvature tensor in adapted coordinates takes the form $R^a_{bcl} = \partial_c \Gamma^a_{lb} - \partial_l \Gamma^a_{cb} + (\Gamma^s_{lb}\Gamma^a_{cs} - \Gamma^s_{cs}\Gamma^a_{ls}) + \frac{\varepsilon_0 k}{2c^2} F^a_b F_{cl}$. To calculate the object G^a_n we use the curvature tensor R^a_{bcd} [1] which can be obtained from the decomposition $[\varepsilon_a, \varepsilon_b] = \omega_{ba}\vec{u} + R^c_{abd}x^{n+d}\partial_{n+c}$ where $\vec{\varepsilon_a} = \partial_a - \Gamma^n_a \partial_n - G^b_a \partial_{n+b}$.

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